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ON LINEAR COMBINATIONS OF ORTHOGONAL POLYNOMIALS

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Abstract. In this expository paper, linear combinations of orthogonal polynomials are considered. Properties like orthogonality and interlacing of zeros are presented.

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1. Introduction

Given a linear functional u on the linear space of polynomials with real coefficients, a sequence of monic polynomials $\{P_n\}_{n\geq 0}$ with deg $P_n = n$ is said to be orthogonal with respect to u if $\langle u, P_n P_m\rangle = 0$ for every $n \neq m$ and $\langle u, P_n^2\rangle \neq 0$ for every $n = 0, 1, \dots$.

A very well known result (Favard's theorem, see [7] for instance) gives a characterization of a quasi–definite (respectively positive definite) linear functional in terms of the three–term recurrence relation that the sequence $\{P_n\}_{n\geq 0}$ satisfies, i.e.

$$\begin{aligned} xP_n(x) &= P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \\ P_0(x) &= 1, \quad P_1(x) = x - \beta_0, \end{aligned} \tag{1}$$

whith $\gamma_n \neq 0$ (respectively $\gamma_n > 0$).

In particular, if u is a positive definite linear functional then there exists a positive Borel measure μ supported on an infinite subset of $\mathbb{R}$ such that $\langle u, q\rangle = \int_{\mathbb{R}} q\, d\mu$ for every polynomial q. In such a situation, the zeros of P_n are real, simple, and they are located in the convex hull of the support of the measure μ, supp(μ). Furthermore, the zeros of P_{n-1} interlace with those of P_n, and this is a relevant fact in numerical quadrature i.e. in the discrete representation

$$\int_{\mathbb{R}} q\, d\mu \sim \sum_{j=1}^{n} \lambda_j q(c_j)\,. \tag{2}$$

If we choose the nodes $(c_j)_{j=1}^n$ as the zeros of P_n then (2) has degree of precision $2n-1$, i.e. it is exact for every polynomial of degree at most $2n-1$ but not for all polynomials of bigger

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degree. As a consequence of the interlacing property aforementioned, the Christoffel-Cotes numbers $(\lambda_j)_{j=1}^n$ are positive.

>From a long time, linear combinations of orthogonal polynomials have attracted the interest of many authors because of its connections with several topics in mathematics, like quasi–orthogonality, moment problem, mechanical quadrature formulas and so on. Among others, see [1], [2], [5], [12], [13] and the references therein.

In a remarkable paper ([14]), Shohat studying mechanical quadrature formulas with positive coefficients considered such linear combinations. More precisely, he proved that (2) has degree of precision $2n-1-k$ if and only if we choose $(c_j)_{j=1}^n$ as the zeros of Q_n where $Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + \cdots + a_k P_{n-k}(x)$ with $a_k \neq 0$. Moreover, properties of their zeros were studied and also a question about orthogonality was mentioned.

Grinshpun, in [9], studied the orthogonality of special linear combinations of polynomials orthogonal with respect to a weight function supported on an interval of the real line. Such families of orthogonal polynomials come up in some extremal problems of Zolotarev–Markov type as well as in problems of least deviating from zero. A special feature of this representation is that the coefficients do not depend on n. The relevant fact proved in this paper is that the Bernstein–Szegő polynomials and just them can be represented as a linear combination of Chebyshev polynomials with coefficients independent of n and fixed length.

Linear combinations of orthogonal polynomials appears not only in a mathematical context but also as solutions of several physical problems. For example, related with the isotonic oscillator see [6].

In this note, we expose some results on linear combinations of orthogonal polynomials concerning orthogonality and properties of their zeros.

2. Orthogonality and Jacobi matrices

In the sequel, $\{P_n\}_{n\geq 0}$ denotes a sequence of monic polynomials orthogonal (SMOP) with respect to a quasi–definite linear functional u.

Let $\{Q_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg Q_n = n$ such that,

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + \cdots + a_k P_{n-k}(x),\ n \geq k+1 \tag{3}$$

with $k \geq 1$ fixed and where the coefficients are independent of n and $a_k \neq 0$.

An immediate consequence of the representation (3) is that $\{Q_n\}_{n\geq 0}$ constitutes a sequence of quasi–orthogonal polynomials of order k.

Next, we show a necessary and sufficient condition in order for the sequence $\{Q_n\}_{n\geq 0}$ to be orthogonal with respect to a quasi–definite linear functional v and we give the relation between the linear functionals u and v, via Jacobi matrices.

If the coefficients in (3) depend on n, a necessary and sufficient condition can be deduced using similar techniques to those employed in the case of constant coefficients. For the sake of simplicity we omit the statement of the corresponding theorem.

Theorem 2.1 ([1]). *Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials with recurrence coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ ($\gamma_n \neq 0$) and let $\{Q_n\}_{n\geq 0}$ be a sequence of monic polynomials such that, for $n \geq k+1$,*

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + \cdots + a_k P_{n-k}(x)$$

where $\{a_j\}_{j=1}^k$ *are constant coefficients and* $a_k \neq 0$. *Then* $\{Q_n\}_{n\geq 0}$ *is orthogonal with respect to a quasi–definite linear functional if and only if the following conditions hold*

(i) For each j, $1 \leq j \leq k$, *the polynomials* Q_j *satisfy a three term recurrence relation* $xQ_j(x) = Q_{j+1}(x) + \widetilde{\beta}_j Q_j(x) + \widetilde{\gamma}_j Q_{j-1}(x)$, *with* $\widetilde{\gamma}_j \neq 0$.

(ii) For $n \geq k+2$

$$\begin{aligned} &\gamma_n + a_1(\beta_{n-1} - \beta_n) = \gamma_{n-k}\,, \\ &a_{j-1}(\gamma_{n-k} - \gamma_{n-j+1}) = a_j(\beta_{n-j} - \beta_n)\,, \quad 2 \leq j \leq k\,. \end{aligned}$$

(iii)

$$\begin{aligned} &\gamma_{k+1} + a_1(\beta_k - \beta_{k+1}) \neq 0\,, \\ &a_j\gamma_{k-j+1} + a_{j+1}(\beta_{k-j} - \beta_{k+1}) = a_j^{(k)}[\gamma_{k+1} + a_1(\beta_k - \beta_{k+1})]\,,\ 1 \leq j \leq k-1\,, \\ &a_k\gamma_1 = a_k^{(k)}[\gamma_{k+1} + a_1(\beta_k - \beta_{k+1})]\,, \end{aligned}$$

where $a_j^{(k)}$, $j = 1, \ldots, k$, *denotes the coefficient of* P_{k-j} *in the Fourier expansion of* Q_k *in terms of the orthogonal system* $\{P_j\}_{j=0}^k$.

Moreover, denoting by $\widetilde{\beta_n}$ *and* $\widetilde{\gamma_n}$ *the coefficients of the three-term recurrence relation for the polynomials* Q_n, *we have for* $n \geq k+1$

$$\widetilde{\beta_n} = \beta_n, \quad \widetilde{\gamma_n} = \gamma_n + a_1(\beta_{n-1} - \beta_n)\,.$$

Now, we perform the relation between the orthogonality linear functionals in terms of the Jacobi matrices. Consider two families of monic polynomials $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ orthogonal with respect to the quasi-definite linear functionals u and v, respectively, satisfying the condition (3). It is well known (see, e.g., [11]) that the relation between both linear functionals is $u = h_k v$ where h_k is a polynomial of degree k. Writing $\mathbf{P} = (P_0, P_1, \ldots, P_n, \ldots)^T$ and $\mathbf{Q} = (Q_0, Q_1, \ldots, Q_n, \ldots)^T$ for the column vectors associated with these orthogonal families, and $\mathbf{J}_P$ and $\mathbf{J}_Q$ for the corresponding Jacobi matrices, we get

$$x\,\mathbf{P} = \mathbf{J}_P\,\mathbf{P}\,, \quad x\,\mathbf{Q} = \mathbf{J}_Q\,\mathbf{Q}\,. \tag{4}$$

If $\mathbf{M}$ denotes the matrix associated with the change of bases $\mathbf{Q} = \mathbf{MP}$, then $\mathbf{M}$ is a lower triangular matrix with diagonal entries equal to 1 and zero subdiagonals from the $(k+1)$–th one. >From (4) it follows $\mathbf{M}\,\mathbf{J}_P\,\mathbf{P} = x\,\mathbf{M}\,\mathbf{P} = \mathbf{J}_Q\,\mathbf{M}\,\mathbf{P}$ and, therefore,

$$\mathbf{M}\,\mathbf{J}_P = \mathbf{J}_Q\,\mathbf{M}\,. \tag{5}$$

Next, we describe *a simple algorithm to compute the polynomial* h_k.

(1) From the data $\mathbf{M}$ and $\mathbf{J}_P$, we have (5) and we can deduce $\mathbf{J}_Q$.

(2) From $\mathbf{J}_P$ and $\mathbf{J}_Q$ we deduce $\mathbf{D}_P$ and $\mathbf{D}_Q$, respectively, where $\mathbf{D}_P = \langle u, \mathbf{P}\mathbf{P}^T\rangle$ and $\mathbf{D}_Q = \langle v, \mathbf{Q}\mathbf{Q}^T\rangle$.

(3) If we write $h_k(x) = c_0 + c_1 x + \cdots + c_k x^k$, we get

$$h_k(\mathbf{J}_P) = c_0 I + c_1 \mathbf{J}_P + \cdots + c_k \mathbf{J}_P^k = \mathbf{D}_P \mathbf{M}^T \mathbf{D}_Q^{-1} \mathbf{M}\,,$$

which is a system of linear equations with $k+1$ unknowns. Notice that the matrices of the first and second terms are $2k+1$ diagonal.

If the monic polynomials $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ were replaced by the corresponding orthonormal polynomials $\{\widetilde{P_n}\}_{n\geq 0}$ and $\{\widetilde{Q_n}\}_{n\geq 0}$, similar computations would have led to

$$h_k(\mathbf{J}_{\widetilde{P}}) = \widetilde{\mathbf{M}}^T \widetilde{\mathbf{M}}\,, \qquad h_k(\mathbf{J}_{\widetilde{Q}}) = \widetilde{\mathbf{M}}\widetilde{\mathbf{M}}^T\,,$$

where $\widetilde{\mathbf{M}}$ denotes the matrix of the change of bases, that is $\widetilde{\mathbf{Q}}\widetilde{\mathbf{M}}\widetilde{\mathbf{P}}$. This gives us an interesting interpretation of the matrix operation involving the linear combination of the orthogonal polynomials $Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + \cdots + a_k P_{n-k}(x), \quad n \geq k+1$.

On the other hand, if the symbol $(\mathbf{A})_n$ denotes the truncation of any infinite matrix $\mathbf{A}$ at level $n+1$, it can be seen that

$$(\mathbf{J}_Q)_n = (\mathbf{M})_n\,[(\mathbf{J}_P)_n - \mathbf{L}_n]\,(\mathbf{M})_n^{-1}\,,$$

where

$$\mathbf{L}_n = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ . & \dots & . & . & \dots & . \\ 0 & \dots & 0 & 0 & \dots & 0 \\ . & \dots & . & . & \dots & . \\ 0 & \dots & 0 & a_k & \dots & a_1 \end{pmatrix} \in \mathbb{R}^{(n+1,n+1)}\,.$$

This expression means that $(\mathbf{J}_Q)_n$ is similar to a rank–one perturbation of the matrix $(\mathbf{J}_P)_n$ and this perturbation is given by the matrix $\mathbf{L}_n$. In particular, the zeros of the polynomial Q_n are the zeros of the characteristic polynomial of the matrix $(\mathbf{J}_P)_n - \mathbf{L}_n$.

2.1. The Case $k = 2$

Among the classical orthogonal polynomial families, the Chebyshev polynomials are unique families such that the sequence of polynomials $\{Q_n\}_{n\geq 0}$ defined by (3) is orthogonal (see for example [3]). But, *what happens if the sequence $\{P_n\}_{n\geq 0}$ is not a classical one?*

Next Theorem describes, for the case $k = 2$, all the families of monic polynomials $\{P_n\}_{n\geq 0}$ orthogonal with respect to a quasi–definite linear functional such that the new families $\{Q_n\}_{n\geq 0}$ are also orthogonal.

Theorem 2.2 ([1]). *Let $\{P_n\}_{n\geq 0}$ be an SMOP with respect to a quasi–definite linear functional. Assume that a_1 and a_2 are real numbers with $a_2 \neq 0$ and Q_n the monic polynomials defined by*

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + a_2 P_{n-2}(x)\,, \quad n \geq 3\,. \tag{6}$$

Then the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$ depends on the choice of a_1 and a_2. More precisely, $\{Q_n\}_{n\geq 0}$ is an SMOP if and only if $\gamma_3 + a_1(\beta_2 - \beta_3) \neq 0$, and

(i) if $a_1 = 0$, for $n \geq 4$, $\beta_n = \beta_{n-2}$ and $\gamma_n = \gamma_{n-2}$.

(ii) if $a_1 \neq 0$ *and* $a_1^2 = 4a_2$*, then for* $n \geq 2$ *,*

$$\beta_n = A + Bn + Cn^2\,, \qquad \gamma_n = D + En + Fn^2,$$

with $a_1 C = 2F\,, \quad a_1 B = 2E - 2F\,, \quad (A,B,C,D,E,F \in \mathbb{R})$.

(iii) if $a_1 \neq 0$ *and* $a_1^2 > 4a_2$*, then for* $n \geq 2$ *,*

$$\beta_n = A + B\lambda^n + C\lambda^{-n}\,, \qquad \gamma_n = D + E\lambda^n + F\lambda^{-n}\,,$$

with $a_1 C = (1+\lambda)F\,, \quad a_1\lambda B = (1+\lambda)E\,, \quad (A,B,C,D,E,F \in \mathbb{R})\,,$
where λ *is the unique solution in* $(-1,1)$ *of the equation* $a_1^2\lambda = a_2(1+\lambda)^2$.

(iv) if $a_1 \neq 0$ *and* $a_1^2 < 4a_2$*, and let* $\lambda = e^{i\theta}$ *be the unique solution of the equation* $a_1^2\lambda = a_2(1+\lambda)^2$ *with* $\theta \in (0,\pi)$*, then for* $n \geq 2$

$$\beta_n = A + Be^{in\theta} + \overline{B}e^{-in\theta}\,, \qquad \gamma_n = D + Ee^{in\theta} + \overline{E}e^{-in\theta}\,,$$

with $a_1\lambda\, B = (1+\lambda)\, E, \quad (A, D \in \mathbb{R}\,, B, E \in \mathbb{C})\,.$

Notice that, when $a_1 = 0$ the SMOP $\{P_n\}_{n\geq 0}$ such that the sequence $\{Q_n\}_{n\geq 0}$ defined by (6) is also an SMOP can be explicitly given in terms of the Chebyshev polynomials, see [10, p. 109]. However, in general the explicit description of all sequences $\{P_n\}_{n\geq 0}$, as well as their orthogonality measure, is an open problem.

We point out a difference between the cases $k = 1$ and $k = 2$. Let Q_n be the monic polynomials defined by

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x), \quad n \geq 2\,,$$

with $a_1 \neq 0$. From Theorem 2.1 written for $k = 1$, it follows that $\{Q_n\}_{n\geq 0}$ is an SMOP if and only if

$$\begin{aligned} &\gamma_2 + a_1(\beta_1 - \beta_2) \neq 0 \\ &\gamma_n - \gamma_2 = a_1(\beta_n - \beta_2)\,, \quad n \geq 3\,. \end{aligned} \tag{7}$$

Thus, in the case $k = 1$, for any sequence of $\{\gamma_n\}_{n\geq 1}$ with $\gamma_n \neq 0$, if we take $\beta_0, \beta_1 \in \mathbb{R}$, and β_n $(n \geq 2)$ satisfying (7), we obtain all the SMOP $\{P_n\}_{n\geq 0}$ such that $\{Q_n\}_{n\geq 0}$ is also an SMOP. However, in the case $k = 2$ with $a_1 \neq 0$, it can be seen that the recurrence coefficients γ_n and β_n have to be solutions of the following difference equation with constant coefficients

$$y_n + \left(1 - \frac{a_1^2}{a_2}\right) y_{n-1} - \left(1 - \frac{a_1^2}{a_2}\right) y_{n-2} - y_{n-3} = 0\,, \quad n \geq 5\,.$$

Therefore, although in both cases we get that β_n and γ_n have a similar asymptotic behaviour, roughly speaking, for $k = 2$ there are fewer families $\{P_n\}_{n\geq 0}$.

When more than three polynomials P_n are involved in the definition of Q_n, the description of all families $\{P_n\}_{n\geq 0}$ in terms of the recurrence coefficients remains open.

3. Interlacing of zeros

Let μ be a positive Borel measure on the real line. Recall that if $\{P_n\}_{n\geq 0}$ is the sequence of monic orthogonal polynomials with respect to μ, the n zeros of P_n are real and distinct and lie in the convex hull of supp(μ). Further, if $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ are the zeros of P_n and $x_{1,n+1} < x_{2,n+1} < \cdots < x_{n+1,n+1}$ are the zeros of P_{n+1}, then the interlacing of the zeros

$$x_{1,n+1} < x_{1,n} < x_{2,n+1} < x_{2,n} < \cdots < x_{n,n+1} < x_{n,n} < x_{n+1,n+1}\,, \tag{8}$$

holds.

Stieltjes proved a stronger result than this one, namely that the zeros of P_m and P_n are interlaced whenever $m < n$ in the sense that there are m disjoint intervals of the form $(x_{j,n}, x_{j+1,n})$ such that each contains a zero of P_m (see [15]).

In [16], it is shown that a necessary and sufficient condition for two real polynomials, of consecutive degrees, to be embedded in an orthogonal sequence is that their zeros interlaced. Recently Driver, see [8], gives an example to illustrate that the Wendroff's result cannot be extended to the case of three polynomials of consecutive degrees. Likewise, in [4], the authors show that the zeros of Van Vleck polynomials corresponding to Stieltjes polynomials of successive degrees interlace but the spectral polynomials formed from the Van Vleck zeros are not orthogonal with respect to any measure. Then, *the interlacing property of the zeros of polynomials of adjacent degrees in a sequence is a weaker property than the orthogonality of such a sequence*.

Concerning the linear combinations of polynomials orthogonal with respect to μ, given by (3), the quasi–orthogonality condition yields the polynomial Q_n has at least $n - k$ distinct odd–order zeros in the interior of the convex hull of supp(μ). This result is the best possible (see [2], [14]).

An interlacing of the nodes of a quadrature formula (2) and the zeros of a linear combination of the corresponding orthogonal polynomials is deduced in the main theorem in [2]. Its statement suggests that the property of interlacing of zeros is inherited from the positivity of the Cotes numbers in a quadrature formula rather than directly from orthogonality. As a consequence, in the same paper, the authors obtain the following generalization of the Stieltjes' result above quoted:

Theorem 3.1 ([2]). *Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials and suppose that $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ are the zeros of P_n. Let $Q_m(x) = a_m P_m(x) + \cdots + a_s P_s(x)$ where $a_s a_m \neq 0$, $1 \leq s \leq m \leq n$ and $s < n$. Then there are at least s disjoint intervals $(x_{j,n}, x_{j+1,n})$ that contains at least one zero of Q_m.*

Different properties related to the interlacing of the zeros of particular linear combinations of orthogonal polynomials are given in several papers: [2], [5], [8] and [14]. Most of them correspond to the cases $k = 1$ and $k = 2$ in formula (3). We only point out Theorem VIII in [14] where when $k = 1$, that is $Q_n = P_n + a_1 P_{n-1}$, interlacing results for the zeros of Q_n, Q_{n-1}, P_n and P_{n-1} are obtained depending on the sign of the coefficient a_1.

4. Conclusion

Linear combinations of orthogonal polynomials are an old topic in the field of orthogonal polynomials. In the last years, they have been object of an increasing study. Special attention has been given to orthogonality and properties of their zeros.

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