Chapter 11
The inverse problem for linearly related orthogonal polynomials: General case

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Abstract We study the inverse problem in the theory of (standard) orthogonal polynomials involving two polynomials families $(P_n)_n$ and $(Q_n)_n$ which are connected by a linear algebraic structure such as

$$P_n(x) + \sum_{i=1}^{N} r_{i,n} P_{n-i}(x) = Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x),$$

for all $n = 0, 1, \ldots$ where N and M are arbitrary nonnegative integer numbers.

11.1 Introduction

We had the honor and pleasure of collaborating with Zé Carlos. He was distinguished by his bonhomie, good work and knowledge and we will never forget how wonderful it was to work with him. His memory will remain with us.

The idea to address the general case in the framework of the inverse problem for linearly related orthogonal polynomials arose after our joint work [3] . We share with him how to do it. Unfortunately for us, he was unable to participate in its development, although he was aware of everything we achieved and encouraged us to continue.

The analysis of linear structure relations involving two monic orthogonal polynomial sequences (MOPS), $(P_n)_n$ and $(Q_n)_n$, such as

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$$P_n(x) + \sum_{i=1}^{N} r_{i,n} P_{n-i}(x) = Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x), \quad n \geq 0, \tag{11.1}$$

where N and M are fixed nonnegative integer numbers, and $(r_{i,n})_n$ and $(s_{i,n})_n$ are sequences of complex numbers (and empty sum equals zero), has been a subject of research interest in the last decades. In the literature, many works can be found where this type of relations is studied from different points of view. About the interest and importance of the study of these structure relations we refer to the introduction given in [10] and [3], as well as the references therein.

In many of these works the main problem stated and solved therein was the following inverse problem: assuming that $(P_n)_n$ is a MOPS and $(Q_n)_n$ only a simple set of polynomials (Q_n is a polynomial of degree n), verifying (11.1), to find necessary and sufficient conditions so that $(Q_n)_n$ is also a MOPS and to obtain the relation between the corresponding regular linear functionals. We want to notice that most of these papers deal with relations considering concrete values for N and M (see [1], [2], [3], [4], [5], [7], [8]). In this contribution we analyze the inverse problem for any values of N and M.

A classical tool for working with algebraic properties of orthogonal sequences of polynomials is the use of recurrence relations. Any MOPS $(P_n)_n$ is characterized by a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n = 0, 1, \ldots$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, where $(\beta_n)_n$ and $(\gamma_n)_n$ are sequences of complex numbers such that $\gamma_n \neq 0$ for all $n = 1, 2, \ldots$. This is known as Favard's theorem (see, e.g., [6]).

However this tool can be replaced by the use of dual basis which produces a natural way for studying the algebraic properties of sequences of orthogonal polynomials. Any simple set of polynomials $(P_n)_n$ has a dual basis $(\mathbf{a}_n)_n$, that is

$$\langle \mathbf{a}_n, P_j \rangle := \delta_{n,j}, \quad n, j = 0, 1, \ldots$$

being $\delta_{n,j}$ the usual Kronecker symbol. Moreover, if $(P_n)_n$ is a MOPS with respect to the linear functional $\mathbf{u}$, then the associated dual basis is

$$\mathbf{a}_n = \frac{P_n}{\langle \mathbf{u}, P_n^2 \rangle} \mathbf{u}, \quad n = 0, 1, \ldots$$

see [9]. A more detailed description can be seen in Section 2 of [10].

This is the main tool used by Petronilho in [10] to solve part of the inverse problem for general relations as (11.1) assuming the orthogonality of both sequences $(P_n)_n$ and $(Q_n)_n$ with respect to $\mathbf{u}$ and $\mathbf{v}$, as well as some additional assumptions which we will call initial conditions. More precisely, he obtains that the rational transformation between the linear functionals $\mathbf{u}$ and $\mathbf{v}$ is

$$\Phi_M \mathbf{u} = \Psi_N \mathbf{v},$$

where Φ_M and Ψ_N are polynomials of degrees M and N, respectively. In Section 2, we make an exhaustive study about these initial conditions. We obtain, in Theorem 11.2, that these initial conditions characterize the existence of such rational transformation and the non existence of another relationship with degrees less than or equal to M and N, respectively. Moreover these initial conditions allow us to prove important facts like:

(a) All the coefficients $r_{N,n}$ and $s_{M,n}$ in (11.1) are not zero for $n \geq N+M$.

(b) There exist constant sequences whose values are the coefficients of the polynomials Φ_M and Ψ_N. It is important to note that the existence of such constant sequences was already obtained in [7] for $N=1, M=0$, in [1] for $N=1, M=1$, in [4] for $N=2, M=0$ and in [3] for $N=2, M=1$, in terms of the recurrence coefficients. Here, for arbitrary values of N and M, we obtain this property in a compact form by using linear functionals and determinants.

On the other hand, in Section 3 we get necessary and sufficient conditions in order to the sequence $(Q_n)_n$ defined recursively by (11.1) becomes also a MOPS. The main advance in this line is to introduce some auxiliary polynomials R_n, which are precisely the linear combinations of the polynomials P_n that appear in the relation (11.1). These polynomials do not necessarily have to be orthogonal, but they are interesting in two senses. First, they allow to simplify the computations in the problem of characterizing the orthogonality of the sequence $(Q_n)_n$. And second and more important, is that the conditions which characterize the orthogonality of $(Q_n)_n$ correspond to most of the conditions ($n \geq N+M+1$) that characterize the orthogonality of two simpler problems. More precisely, assuming that $(P_n)_n$ is a MOPS, to characterize the orthogonality of $(R_n)_n$ and then, assuming that $(R_n)_n$ is a MOPS, to characterize the orthogonality of $(Q_n)_n$. In some way the problem $N-M$ can be divided in two simpler problems $N-0$ and $0-M$ but always keeping in mind that not all the conditions of regularity (orthogonality) appear, the first ones are different. This is because, as we have already mentioned, these auxiliary polynomials $(R_n)_n$ do not have to be orthogonal.

11.2 Relation between the regular functionals and consequences

Let $(P_n)_n$ and $(Q_n)_n$ be two sequences of monic polynomials orthogonal with respect to the regular functionals $\mathbf{u}$ and $\mathbf{v}$, respectively, normalized by $\langle \mathbf{u}, 1\rangle = 1 = \langle \mathbf{v}, 1\rangle$. Suppose that these families of polynomials are linearly related by (11.1), that is

$$P_n(x) + \sum_{i=1}^{N} r_{i,n} P_{n-i}(x) = Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x), \quad n \geq 0.$$

Consider the following auxiliary polynomials, namely R_n,

$$R_n(x) := P_n(x) + \sum_{i=1}^{N} r_{i,n} P_{n-i}(x) = Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x).$$

Denote by $(\mathbf{c}_n)_n$, $(\mathbf{a}_n)_n$ and $(\mathbf{b}_n)_n$ the dual basis corresponding to $(R_n)_n$, $(P_n)_n$ and $(Q_n)_n$, respectively. Expanding $(\mathbf{a}_n)_{n=0}^{M-1}$ and $(\mathbf{b}_n)_{n=0}^{N-1}$ in terms of $(\mathbf{c}_n)_n$, we can write

$$\mathbf{A}(\mathbf{c}_0, \ldots, \mathbf{c}_{M+N-1})^T = (\mathbf{a}_0, \ldots, \mathbf{a}_{M-1}, \mathbf{b}_0, \ldots, \mathbf{b}_{N-1})^T, \tag{11.2}$$

where $\mathbf{A}$ is a $(M+N) \times (M+N)$ matrix whose elements are the coefficients that appear in the relation (11.1). Its explicit expression can be seen in [10, Theorem 1.1]. There, it was proved that the initial conditions

$$\det \mathbf{A} \neq 0, \quad r_{N,N+M} \neq 0, \quad \text{and} \quad \mathrm{s_{M,N+M}} \neq 0, \tag{11.3}$$

yield the following relation between the linear functionals $\mathbf{u}$ and $\mathbf{v}$

$$\Phi_M \mathbf{u} = \Psi_N \mathbf{v}, \tag{11.4}$$

where Φ_M and Ψ_N are polynomials of (exact) degrees M and N, respectively.

In the sequel we will use the following notations:

$$\overline{P}_n(x) = \frac{P_n(x)}{\langle \mathbf{u}, {P_n}^2 \rangle}, \quad \text{and} \quad \overline{Q}_n(x) = \frac{Q_n(x)}{\langle \mathbf{v}, {Q_n}^2 \rangle}.$$

For $n \geq N-1$, we introduce the $N \times N$ matrices $\mathbf{B_n}$ and $\mathbf{B}_n^i$, $i = 0, \ldots, N-1$, where

$$\mathbf{B}_n := \begin{pmatrix} \langle \overline{Q}_{N-1}\mathbf{v}, P_n \rangle & \cdots & \langle \overline{Q}_0 \mathbf{v}, P_n \rangle \\ \langle \overline{Q}_{N-1}\mathbf{v}, P_{n-1} \rangle & \cdots & \langle \overline{Q}_0 \mathbf{v}, P_{n-1} \rangle \\ \vdots & \ddots & \vdots \\ \langle \overline{Q}_{N-1}\mathbf{v}, P_{n-(N-1)} \rangle & \cdots & \langle \overline{Q}_0 \mathbf{v}, P_{n-(N-1)} \rangle \end{pmatrix}_{N \times N}$$

and $\mathbf{B}_n^i$:= the matrix obtained replacing in $\mathbf{B}_n$ the i-th column by the vector $(\langle \overline{Q}_N \mathbf{v}, P_n \rangle, \ldots, \langle \overline{Q}_N \mathbf{v}, P_{n-(N-1)} \rangle)^T$.

In a similar way for $n \geq M-1$, we introduce the $M \times M$ matrices $\widetilde{\mathbf{B}}_n$ and $\widetilde{\mathbf{B}}_n^i$, $i = 0, \ldots, M-1$, where

$$\widetilde{\mathbf{B}}_n := \begin{pmatrix} \langle \overline{P}_{M-1}\mathbf{u}, Q_n \rangle & \cdots & \langle \overline{P}_0 \mathbf{u}, Q_n \rangle \\ \langle \overline{P}_{M-1}\mathbf{u}, Q_{n-1} \rangle & \cdots & \langle \overline{P}_0 \mathbf{u}, Q_{n-1} \rangle \\ \vdots & \ddots & \vdots \\ \langle \overline{P}_{M-1}\mathbf{u}, Q_{n-(M-1)} \rangle & \cdots & \langle \overline{P}_0 \mathbf{u}, Q_{n-(M-1)} \rangle \end{pmatrix}_{M \times M}$$

and $\widetilde{\mathbf{B}}_n^i$:= the matrix obtained replacing in $\widetilde{\mathbf{B}}_n$ the i-th column by the vector $\left(\langle \overline{P}_M \mathbf{u}, Q_n \rangle, \ldots, \langle \overline{P}_M \mathbf{u}, Q_{n-(M-1)} \rangle \right)^T$.

Next, we show a property about the determinants of the matrices $\mathbf{B_n}$ and $\widetilde{\mathbf{B}}_n$ which plays an important role in this work.

Lemma 11.1 *Let $(P_n)_n$ and $(Q_n)_n$ be two sequences of monic polynomials linearly related by (11.1).*
(a) If the polynomials Q_n are orthogonal with respect to the functional $\mathbf{v}$, then

$$det\, \mathbf{B}_n = (-1)^N\, r_{N,n}\, det\, \mathbf{B}_{n-1}, \quad for \quad n \geq M+N.$$

(b) If the polynomials P_n are orthogonal with respect to the functional $\mathbf{u}$, then

$$det\, \widetilde{\mathbf{B}}_n = (-1)^M\, s_{M,n}\, det\, \widetilde{\mathbf{B}}_{n-1}, \quad for \quad n \geq M+N.$$

Proof (a) First we get that $\det \mathbf{B}_{N+M} = (-1)^N\, r_{N,N+M} \det \mathbf{B}_{N+M-1}$. Indeed, taking into account that

$$P_{N+M}(x) = Q_{N+M}(x) + \sum_{i=1}^{M} s_{i,N+M} Q_{N+M-i}(x) - \sum_{i=1}^{N} r_{i,N+M} P_{N+M-i}(x),$$

by (11.1) for $n = N+M$, using the orthogonality of the polynomials Q_n with respect to $\mathbf{v}$ and developing the determinant of $\mathbf{B}_{N+M}$ by the first row, it can be derived that

$$\det \mathbf{B}_{N+M} = -\,(-1)^{N-1}\, r_{N,N+M} \det \mathbf{B}_{N+M-1}.$$

To conclude the proof of (a) it suffices to observe that the same argument works for any fixed $n > N+M$.

(b) This property can be similarly derived with the appropriate changes. □

Now, we will see that the polynomials Ψ_N and Φ_M which satisfy the relation (11.4) can be written as:

$$\Psi_N(x) = r_{N,M+N}\, \overline{Q}_N(x) + \sum_{i=0}^{N-1} \lambda_i\, \overline{Q}_i(x), \tag{11.5}$$

and

$$\Phi_M(x) = s_{M,M+N}\, \overline{P}_M(x) + \sum_{i=0}^{M-1} \mu_i\, \overline{P}_i(x). \tag{11.6}$$

Indeed, if Ψ_N is a polynomial of degree N it can be written as

$$\Psi_N(x) = \lambda_N\, \overline{Q}_N(x) + \lambda_{N-1}\, \overline{Q}_{N-1}(x) + \cdots + \lambda_1\, \overline{Q}_1 + \lambda_0,$$

with $\lambda_N \neq 0$. Then, using the dual basis of $(P_n)_n$ we have

$$\Psi_N \mathbf{v} = \sum_{j=0}^{M} \langle \Psi_N \mathbf{v}, P_j \rangle\, \overline{P}_j \mathbf{u}, \tag{11.7}$$

because $\langle \Psi_N \mathbf{v}, P_j \rangle = \langle \Phi_M \mathbf{u}, P_j \rangle = 0, \quad \text{for} \quad j \geq M + 1$. Moreover, by (11.1) $r_{N,N+M} \langle \Psi_N \mathbf{v}, P_M \rangle = s_{M,N+M} \langle \Psi_N \mathbf{v}, Q_N \rangle = \lambda_N \, s_{M,N+M}$, and from (11.7) we get the expressions (11.5) and (11.6) for the polynomials Ψ_N and Φ_M, respectively.

Next, we obtain a characterization of the initial conditions (11.3) in terms of the relations between the linear functionals $\mathbf{u}$ and $\mathbf{v}$.

Theorem 11.2 *Let $(P_n)_n$ and $(Q_n)_n$ be two MOPSs with respect to the regular functionals $\mathbf{u}$ and $\mathbf{v}$, respectively, normalized by $\langle \mathbf{u}, 1 \rangle = 1 = \langle \mathbf{v}, 1 \rangle$. Assume that they are related by (11.1) where $N, M \geq 1$.*

Then, the following statements are equivalent:

(a)

$$det\, \mathbf{A} \neq 0, \quad r_{N,N+M} \neq 0, \quad \text{and} \quad \mathrm{s_{M,N+M}} \neq 0.$$

(b) There exist polynomials Φ_M, Ψ_N of degrees M, N, respectively, such that $\Phi_M \, \mathbf{u} = \Psi_N \, \mathbf{v}$, and there is no other relationship $\widetilde{\Phi}_M \, \mathbf{u} = \widetilde{\Psi}_N \, \mathbf{v}$, with degrees of polynomials $\widetilde{\Phi}_M$ and $\widetilde{\Psi}_N$ less than or equal to M and N respectively.

Proof $(a) \Longrightarrow (b)$

First we prove that the polynomials Ψ_N and Φ_M defined by (11.5) and (11.6) such that $\Phi_M \, \mathbf{u} = \Psi_N \, \mathbf{v}$ are unique. Indeed, using the dual basis $(\mathbf{a}_n)_n$ and $(\mathbf{b}_n)_n$ corresponding to $(P_n)_n$ and $(Q_n)_n$, respectively, we have

$$\Psi_N \, \mathbf{v} = r_{N,M+N} \, \mathbf{b}_N + \lambda_{N-1} \, \mathbf{b}_{N-1} + \cdots + \lambda_1 \, \mathbf{b}_1 + \lambda_0 \, \mathbf{b}_0,$$

and

$$\Phi_M \, \mathbf{u} = s_{M,M+N} \, \mathbf{a}_M + \mu_{M-1} \, \mathbf{a}_{M-1} + \cdots + \mu_1 \, \mathbf{a}_1 + \mu_0 \, \mathbf{a}_0,$$

since $\mathbf{a}_n = \overline{P}_n \, \mathbf{u}$ and $\mathbf{b}_n = \overline{Q}_n \, \mathbf{v}$ for $n \geq 0$.

So, by the relation between the functionals (11.4) and the definition (11.2) of the matrix $\mathbf{A}$, we get

$$\begin{aligned}
&r_{N,M+N} \, \mathbf{b}_N - s_{M,M+N} \, \mathbf{a}_M = \\
&(\mu_0, \ldots, \mu_{M-1}, -\lambda_0, \ldots, -\lambda_{N-1})(\mathbf{a}_0, \ldots, \mathbf{a}_{M-1}, \mathbf{b}_0, \ldots, \mathbf{b}_{N-1})^T = \\
&(\mu_0, \ldots, \mu_{M-1}, -\lambda_0, \ldots, -\lambda_{N-1}) \, \mathbf{A} \, (\mathbf{c}_0, \ldots, \mathbf{c}_{M+N-1})^T.
\end{aligned}$$

Thus, since $(\mathbf{c}_n)_n$ is a basis and there exists $\mathbf{A}^{-1}$ the inverse matrix of $\mathbf{A}$, the vector $(\mu_0, \ldots, \mu_{M-1}, -\lambda_0, \ldots, -\lambda_{N-1})$ is unique. Therefore there exist unique polynomials Ψ_N and Φ_M of degrees respectively N, M such that $\Phi_M \, \mathbf{u} = \Psi_N \, \mathbf{v}$.

Now, we suppose that there exist another polynomials $\widetilde{\Psi}_N$ and $\widetilde{\Phi}_M$ satisfying $\widetilde{\Phi}_M \, \mathbf{u} = \widetilde{\Psi}_N \, \mathbf{v}$, with $\deg \widetilde{\Psi}_N \leq N$ and $\deg \widetilde{\Phi}_M \leq M$. Thus, we have

$$(\Phi_M - \widetilde{\Phi}_M) \, \mathbf{u} = (\Psi_N - \widetilde{\Psi}_N) \, \mathbf{v},$$

so it is not possible that $\deg \widetilde{\Psi}_N < N$ and $\deg \widetilde{\Phi}_M < M$ hold simultaneously.

To conclude, it suffices to observe that from the two relations between the functionals $\mathbf{u}$ and $\mathbf{v}$, we also obtain the following relation

$$\widetilde{\Psi}_N\, \Phi_M\, \mathbf{v} = \widetilde{\Phi}_M\, \Psi_N\, \mathbf{v},$$

which yields a contradiction if either deg $\widetilde{\Phi}_M = M$ and deg $\widetilde{\Psi}_N < N$ or deg $\widetilde{\Phi}_M < M$ and deg $\widetilde{\Psi}_N = N$.

$(b) \Longrightarrow (a)$

Consider the systems

$$\langle \Psi_N \mathbf{v}, P_n \rangle = 0, \quad n = M+1, \dots, N+M, \tag{11.8}$$

and

$$\langle \Phi_M \mathbf{u}, Q_n \rangle = 0, \quad n = N+1, \dots, N+M,$$

where the unknowns are, respectively, $(\lambda_i)_{i=0}^{N-1}$ and $(\mu_i)_{i=0}^{M-1}$. Notice that, by hypothesis, the solutions of these systems are unique. Thus, the respective coefficient matrices $\mathbf{B}_{N+M}$ and $\widetilde{\mathbf{B}}_{N+M}$ have maximum rank that is

$$\mathrm{d}et\ \mathbf{B}_{N+M} \neq 0, \quad \text{and} \quad \mathrm{d}et\ \widetilde{\mathbf{B}}_{N+M} \neq 0.$$

So, by Lemma 11.1 we obtain $r_{N,M+N} \neq 0$ and $s_{M,M+N} \neq 0$.

Now, it remains only to prove that $\det \mathbf{A} \neq 0$. To do this, we consider the system with two equations, one of which is the expansion of $\mathbf{b}_N$ as a linear combination of $(\mathbf{c}_i)_{i=0}^{M+N}$ and the other one is the corresponding expansion of $\mathbf{a}_M$. Then, multiplying the first of these equations by $r_{N,M+N}$ and the second one by $s_{M,M+N}$, and subtracting the resulting equations (this will eliminate $\mathbf{c}_{M+N}$) we get

$$r_{N,M+N}\, \mathbf{b}_N - s_{M,M+N}\, \mathbf{a}_M = \sum_{i=0}^{M+N-1} X_i\, \mathbf{c}_i,$$

(see the proof of Theorem 1.1 in [10], for instance). On the other hand, we know

$$r_{N,M+N}\, \mathbf{b}_N - s_{M,M+N}\, \mathbf{a}_M = (\mu_0, \dots, \mu_{M-1}, -\lambda_0, \dots, -\lambda_{N-1})\mathbf{A}(\mathbf{c}_0, \dots, \mathbf{c}_{M+N-1})^T.$$

Thus,

$$\mathbf{A}^T\, (\mu_0, \dots, \mu_{M-1}, -\lambda_0, \dots, -\lambda_{N-1})^T = (X_0, \dots X_{M+N-1})^T,$$

and then $\det \mathbf{A} = \det \mathbf{A}^T \neq 0$ because the system has a unique solution. □

In the following theorem we prove that the initial conditions (11.3) allow us to assure that the lengths of the linear combinations of $(P_n)_n$ and $(Q_n)_n$ in (11.1) are exactly $N+1$ and $M+1$ respectively. Moreover, we can find constant sequences whose values are precisely the coefficients of the polynomials Ψ_N and Φ_M.

Theorem 11.3 *Let $(P_n)_n$ and $(Q_n)_n$ be two MOPSs with respect to the regular functionals $\mathbf{u}$ and $\mathbf{v}$, respectively, normalized by $\langle \mathbf{u}, 1\rangle = 1 = \langle \mathbf{v}, 1\rangle$. Assume that they are related by (11.1) where $N, M \geq 1$ and the coefficients satisfy the initial conditions (11.3). Then, the following properties hold:*
(a) All the coefficients $r_{N,n}$ and $s_{M,n}$ in (11.1) are not zero, for every $n \geq N+M$.

(b) There exist constant sequences $(\lambda_{i,n})_{n\geq N+M}$ and $(\mu_{i,n})_{n\geq N+M}$ such that $\lambda_{i,n} = \lambda_i$, $i = 0, \dots, N-1$ and $\mu_{i,n} = \mu_i$, $i = 0, \dots, M-1$, where $(\lambda_i)_{i=0}^{N-1}$ and $(\mu_i)_{i=0}^{M-1}$ are the coefficients which appear in the expressions (11.5) and (11.6) of the polynomials Ψ_N and Φ_M.

Proof (a) First we observe that for $n \geq N + M$ we have

$$\frac{s_{M,N+M}}{\langle \mathbf{u}, P_M^2\rangle} r_{N,n} \langle \mathbf{u}, P_{n-N}^2\rangle = \langle \Phi_M \mathbf{u}, (P_n + \sum_{i=1}^{N} r_{i,n} P_{n-i}) \, Q_{n-(N+M)}\rangle$$
$$= \langle \Psi_N \mathbf{v}, (Q_n + \sum_{i=1}^{M} s_{i,n} Q_{n-i}) \, Q_{n-(N+M)}\rangle = \frac{r_{N,N+M}}{\langle \mathbf{v}, Q_N^2\rangle} s_{M,n} \langle \mathbf{v}, Q_{n-M}^2\rangle,$$

taking into account the hypothesis (11.1), (11.4), the expressions (11.5)-(11.6) and the orthogonality of the polynomials P_n and Q_n with respect to the functionals $\mathbf{u}$ and $\mathbf{v}$, respectively. Thus, since $r_{N,N+M} \, s_{M,N+M} \neq 0$, it is enough to prove that either $r_{N,n} \neq 0$ or $s_{M,n} \neq 0$, for $n \geq N + M + 1$.

Here, we will prove that $r_{N,n} \neq 0$, for $n \geq N + M + 1$.

Assume that there exists $n \geq N+M+1$ such that $r_{N,n} = 0$ and let $n_0 := \min\{n; n \geq N + M + 1, r_{N,n} = 0\}$. Then Lemma 11.1 gives $\det \mathbf{B}_{n_0} = 0$, so the coefficient matrix associated with the system

$$\langle \Psi_N \mathbf{v}, P_n\rangle = 0, \quad n = n_0 - (N-1), \dots, n_0,$$

has not maximum rank and therefore there exists another solution, namely, $(\tilde{\lambda}_i)_{i=0}^{N-1}$ such that

$$\langle \widetilde{\Psi}_N \mathbf{v}, P_n\rangle = 0, \quad n = n_0 - (N-1), \dots, n_0,$$

where

$$\widetilde{\Psi}_N(x) = r_{N,M+N} \, \overline{Q}_N(x) + \tilde{\lambda}_{N-1} \, \overline{Q}_{N-1}(x) + \cdots + \tilde{\lambda}_1 \, \overline{Q}_1 + \tilde{\lambda}_0.$$

Moreover $\langle \widetilde{\Psi}_N \mathbf{v}, P_{n_0-N}\rangle \neq 0$. Indeed, if $\langle \widetilde{\Psi}_N \mathbf{v}, P_{n_0-N}\rangle = 0$ then the system

$$\langle \widetilde{\Psi}_N \mathbf{v}, P_n\rangle = 0, \quad n = n_0 - N, \dots, n_0 - 1,$$

has two solutions which yields a contradiction since the coefficient matrix of this system, $\mathbf{B}_{n_0-1}$, has maximum rank by definition of n_0.

Hence, using the relation (11.1), we have

$$\langle (\Psi_N - \widetilde{\Psi}_N)\mathbf{v}, P_n\rangle = -\langle \widetilde{\Psi}_N \mathbf{v}, P_n\rangle = 0, \quad n \geq n_0 - (N-1),$$

and

$$\langle (\Psi_N - \widetilde{\Psi}_N)\mathbf{v}, P_{n_0-N}\rangle = -\langle \widetilde{\Psi}_N \mathbf{v}, P_{n_0-N}\rangle \neq 0.$$

Denote by h_{N-1} the polynomial $\Psi_N - \widetilde{\Psi}_N$ of degree less than or equal to $N-1$. Then, writting $h_{N-1}\mathbf{v}$ in the dual basis of $(P_n)_n$, we have

$$h_{N-1}\mathbf{v} = \sum_{j=0}^{n_0-N} \frac{\langle h_{N-1}\mathbf{v}, P_j\rangle}{\langle \mathbf{u}, P_j^2\rangle} P_j\mathbf{u},$$

so there exists a polynomial φ_{n_0-N} of degree $n_0 - N$ such that $h_{N-1}\mathbf{v} = \varphi_{n_0-N}\mathbf{u}$.

Finally observe that since the functionals $\mathbf{u}$ and $\mathbf{v}$ satisfy the two relations

$$h_{N-1}\mathbf{v} = \varphi_{n_0-N}\,\mathbf{u} \quad \text{and} \quad \Psi_N\,\mathbf{v} = \Phi_M\mathbf{u},$$

it can be obtained

$$h_{N-1}\Phi_M\mathbf{v} = \varphi_{n_0-N}\Psi_N\mathbf{v},$$

which leads to a contradiction, taking into account the degrees of the polynomials $h_{N-1}\Phi_M$ and $\varphi_{n_0-N}\Psi_N$.

(b) From the previous theorem, we already know that the system (11.8)

$$\langle \Psi_N\mathbf{v}, P_n\rangle = 0,\ n = M+1, \ldots, N+M,$$

has a unique solution, namely $(\lambda_i)_{i=0}^{N-1}$, which are the coefficients of the polynomial Ψ_N. So, since $\det \mathbf{B}_{N+M} \neq 0$ by Cramer's rule

$$\lambda_i = -r_{N,N+M}\,\frac{\det\,\mathbf{B}^i_{N+M}}{\det\,\mathbf{B}_{N+M}}, \quad \text{for} \quad i = 0, \ldots, N-1.$$

On the other hand, since $\Phi_M\mathbf{u} = \Psi_N\mathbf{v}$, it is obvious that

$$\langle \Psi_N\mathbf{v}, P_n\rangle = 0,\ n \geq M+1.$$

Now, for each n fixed, $n \geq N+M$, we can consider the system

$$\langle \Psi_N\mathbf{v}, P_i\rangle = 0,\ i = n-(N-1), \ldots, n,$$

whose matrix of coefficients $\mathbf{B}_n$ has maximum rank, because we know

$$\det\,\mathbf{B}_n = (-1)^N\,r_{N,n}\ldots(-1)^N\,r_{N,N+M+1}\det\,\mathbf{B}_{N+M} \neq 0,$$

by (a) and Lemma 11.1. Then, for every $n \geq N+M$, this system has a unique solution for $(\lambda_i)_{i=0}^{N-1}$, namely $(\lambda_{i,n})_{i=0}^{N-1}$, and again by Cramer's rule we have

$$\lambda_{i,n} = -r_{N,N+M}\,\frac{\det\,\mathbf{B}^i_n}{\det\,\mathbf{B}_n}, \quad \text{for} \quad i = 0, \ldots, N-1,$$

for all $n \geq N+M$.

Thus, we can assure there exist N constant sequences, namely $(\lambda_{i,n})_{i=0}^{N-1}$, such that for every $n \geq N+M$ we have $\lambda_{i,n} = \lambda_i$ that is the value of these constants coincide with the coefficients of the polynomial Ψ_N.

To conclude the proof of (b) , we work in the same way with the polynomial Φ_M. So, we have

$$\langle \Phi_M \mathbf{u}, Q_n \rangle = 0, \; n \geq N + 1.$$

Thus, for each n fixed, $n \geq N + M$ we can consider the system

$$\langle \Phi_M \mathbf{u}, Q_i \rangle = 0, \; i = n - (M - 1), \ldots, n,$$

where the unknowns are $(\mu_i)_{i=0}^{M-1}$, that is the coefficients of the polynomial Φ_M, see (11.6). The uniqueness of the polynomial Φ_M obtained in the previous theorem leads us to state that $\det \widetilde{\mathbf{B}}_n \neq 0$, for all $n \geq N + M$. Then, in the same way as before, we obtain that there exist M constant sequences, namely $(\mu_{i,n})_{i=0}^{M-1}$, such that for every $n \geq N + M$, we have $\mu_{i,n} = \mu_i$, where

$$\mu_{i,n} = -s_{M,N+M} \frac{\det \widetilde{\mathbf{B}}_n^i}{\det \widetilde{\mathbf{B}}_n}.$$

Note that the values of these constants coincide with the coefficients of the polynomial Φ_M. □

Observe that the notation used before Lemma 11.1 does not work for either $N = 0$ or $M = 0$. So we conclude this section, for the sake of completeness, showing the analogous results for these particular situations. We will show only the case $M = 0$ and $N \geq 1$, that is

$$P_n(x) + \sum_{i=1}^{N} r_{i,n} P_{n-i}(x) = Q_n(x), \quad n \geq 0, \tag{11.9}$$

where $N \geq 1$ and $r_{i,n}$ $(i = 1, \ldots, N)$ are complex numbers. The other case is totally analogous with the appropriate changes.

Theorem 11.4 *Let $(P_n)_n$ and $(Q_n)_n$ be two sequences of monic polynomials with respect to the regular functionals $\mathbf{u}$ and $\mathbf{v}$, respectively, normalized by $\langle \mathbf{u}, 1 \rangle = 1 = \langle \mathbf{v}, 1 \rangle$. Suppose that these families of polynomials are linearly related by (11.9) with the initial condition $r_{N,N} \neq 0$. Then, the following properties hold:*

(a) The exists a unique polynomial Ψ_N of degree N such that $\mathbf{u} = \Psi_N \, \mathbf{v}$, where Ψ_N is defined by (11.5) with $M = 0$.

(b) All the coefficients $r_{N,n}$ in (11.9) are not zero, for every $n \geq N$.

(c) There exist constant sequences $(\lambda_{i,n})_{n \geq N}$ such that $\lambda_{i,n} = \lambda_i$, $i = 0, \ldots, N-1$, where $(\lambda_i)_{i=0}^{N-1}$ are the coefficients which appear in the expression (11.5) of the polynomial Ψ_N.

Proof The proof follows the same ideas of the previous theorems. In any case, for the sake of completeness, we briefly expose the arguments used.

Observe that $\langle \mathbf{u}, Q_n \rangle = \langle \mathbf{u}, P_n + \sum_{i=1}^{N} r_{i,n} P_{n-i} \rangle = 0$, for $n \geq N + 1$. Then, using the dual basis of $(Q_n)_n$ we get $\mathbf{u} = \sum_{n=0}^{N} \langle \mathbf{u}, Q_n \rangle \, \overline{Q}_n \, \mathbf{v}$, where $\langle \mathbf{u}, Q_N \rangle = r_{N,N} \neq 0$ and therefore there exists a unique polynomial of degree N such that

$$\mathbf{u} = \Psi_N \, \mathbf{v},$$

where Ψ_N is defined by (11.5).

On the other hand, we can check that $det\ \mathbf{B}_{N-1} = 1$ and that (a) of the Lemma 11.1 holds with $M = 0$. Thus, the coefficient matrix $\mathbf{B}_N$ of the system

$$\langle \Psi_N\ \mathbf{v}, P_n \rangle = 0,\ n = 1, \ldots, N,$$

has maximum rank and the coefficients $(\lambda_i)_{i=0}^{N-1}$ of the polynomial Ψ_N are

$$\lambda_i = -r_{N,N} \frac{\det \mathbf{B}_N^i}{\det \mathbf{B}_N}, \quad \text{for} \quad i = 0, \ldots, N-1.$$

Moreover, in the same way as in the previous theorem, it can be proved that

$$r_{N,N} \neq 0 \Longrightarrow r_{N,n} \neq 0, \quad n \geq N.$$

Thus, $\det \mathbf{B}_n \neq 0$, for $n \geq N$ and

$$\lambda_i = \lambda_{i,n} = -r_{N,N} \frac{\det \mathbf{B}_n^i}{\det \mathbf{B}_n}, \quad \text{for} \quad i = 0, \ldots, N-1, \quad n \geq N.$$

□

11.3 Orthogonality characterizations

Let $(P_n)_n$ and $(Q_n)_n$ be two sequences of monic polynomials linked by a structure relation as (11.1), with the conventions $r_{N,n} s_{M,n} \neq 0$, for all $n \geq N + M$.

In this section, we want to find necessary and sufficient conditions in order to $(Q_n)_n$ be a MOPS if $(P_n)_n$ is a MOPS. As we have mentioned in the introduction, to get this general case, we introduce some auxiliary polynomials R_n and so, in some way, the problem $N - M$ can be divided in two simpler problems $N - 0$ and $0 - M$. So, we previously study these particular situations that correspond to consider in the relation (11.1) either $M = 0$ or $N = 0$.

From now on, $(P_n)_n$ denotes a MOPS with respect to a regular functional $\mathbf{u}$ and $(\beta_n)_n$ and $(\gamma_n)_n$ the corresponding sequences of recurrence coefficients, that is

$$\begin{aligned} &P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 0, \\ &P_0(x) = 1, \quad P_{-1}(x) = 0, \end{aligned} \tag{11.10}$$

with $\gamma_n \neq 0$ for all $n \geq 1$.

Proposition 11.5 *Let $(P_n)_n$ be a MOPS with recurrence coefficients $(\beta_n)_n$ and $(\gamma_n)_n$. Fixed $N \geq 1$, we define a sequence $(R_n)_n$ of monic polynomials by*

$$R_n(x) := P_n(x) + \sum_{i=1}^{N} r_{i,n} P_{n-i}(x), \quad n \geq 0, \tag{11.11}$$

where $r_{i,n}$ are complex numbers such that $r_{i,n} = 0$, if $i > n$ and $r_{N,n} \neq 0$, for all $n \geq N$.

Then $(R_n)_n$ is a MOPS with recurrence coefficients $(\beta_n^*)_n$ and $(\gamma_n^*)_n$ where

$$\beta_n^* = \beta_n + r_{1,n} - r_{1,n+1}, \quad n \geq 0, \tag{11.12}$$

$$\gamma_n^* = \gamma_n + r_{1,n}(\beta_{n-1} - \beta_n^*) + r_{2,n} - r_{2,n+1}, \quad n \geq 1, \tag{11.13}$$

if and only if $\gamma_i^* \neq 0$, $i = 1, \ldots, N$, and the following formulas hold:

$$A_{i,n} = 0, \quad n \geq i, \quad 2 \leq i \leq N-1, \quad N \geq 3, \tag{11.14}$$

$$A_{N,n} = 0, \quad n \geq N, \quad N \geq 2, \tag{11.15}$$

$$A_{N+1,n} = 0, \quad n \geq N+1, \quad N \geq 1, \tag{11.16}$$

where

$$A_{i,n} := r_{i+1,n+1} - r_{i+1,n} + r_{i,n}(\beta_n^* - \beta_{n-i}) + r_{i-1,n-1}\gamma_n^* - r_{i-1,n}\gamma_{n+1-i},$$

$$A_{N,n} := r_{N,n}(\beta_n^* - \beta_{n-N}) + r_{N-1,n-1}\gamma_n^* - r_{N-1,n}\gamma_{n+1-N},$$

$$A_{N+1,n} := r_{N,n-1}\gamma_n^* - r_{N,n}\gamma_{n-N}.$$

Proof We will characterize when the sequence $(R_n)_n$ is a MOPS, that is when it satisfies a three-term recurrence relation as

$$R_{n+1}(x) = (x - \beta_n^*)R_n(x) - \gamma_n^* R_{n-1}(x), \quad n \geq 0, \tag{11.17}$$

with $\gamma_n^* \neq 0, \quad n \geq 1$.

Inserting formula (11.10) in (11.11) and applying (11.11) to $xP_n(x)$ and again (11.10) to $xP_{n-i}, i = 1, \ldots, N$, and (11.11) to $P_n(x)$ and next to $P_{n-1}(x)$ we have for $n \geq 1$

$$\begin{aligned}
&R_{n+1}(x) = (x - \beta_n^*)R_n(x) - \gamma_n^* R_{n-1}(x) \\
&+ \sum_{i=2}^{N-1} [r_{i+1,n+1} - r_{i+1,n} - r_{i,n}(\beta_{n-i} - \beta_n^*) - r_{i-1,n}\gamma_{n+1-i} + r_{i-1,n-1}\gamma_n^*] P_{n-i}(x) \\
&+ [r_{N,n}(\beta_n^* - \beta_{n-N}) - r_{N-1,n}\gamma_{n+1-N} + r_{N-1,n-1}\gamma_n^*] P_{n-N}(x) \\
&+ [-r_{N,n}\gamma_{n-N} + r_{N,n-1}\gamma_n^*] P_{n-1-N}(x),
\end{aligned}$$

using (11.12) and (11.13).

As a first consequence for $n = 1$ and any $N \geq 1$, we realize that (11.17) is true with $\gamma_1^* \neq 0$ if and only if $\gamma_1^* \neq 0$.

Now, taking into account that the sequence $(P_n)_n$ is a basis, we have that (11.17) with $\gamma_n^* \neq 0$, for $n \geq 2$ holds if and only if $\gamma_i^* \neq 0$, $2 \leq i \leq N$, and the conditions (11.14), (11.15) and (11.16) are satisfied. □

Remark. Observe that the condition (11.16) assures that $\gamma_n^* \neq 0$, for $n \geq N+1$.

In the following proposition, we change the hypothesis and assume now that $(R_n)_n$ is a MOPS and characterize the orthogonality of the others polynomials which appear in the linear combination.

Proposition 11.6 *Let $(R_n)_n$ be a MOPS with recurrence coefficients $(\beta_n^*)_n$ and $(\gamma_n^*)_n$. Fixed $M \geq 1$, we define a sequence $(Q_n)_n$ of monic polynomials recursively by*

$$R_n(x) := Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x), \quad n \geq 0, \tag{11.18}$$

where $s_{i,n}$ are complex numbers such that $s_{i,n} = 0$, for $i > n$ and $s_{M,n} \neq 0$, for all $n \geq M$.

Then $(Q_n)_n$ is a MOPS with recurrence coefficients $(\tilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$ where

$$\tilde{\beta}_n = \beta_n^* + s_{1,n+1} - s_{1,n}, \quad n \geq 0, \tag{11.19}$$

$$\tilde{\gamma}_n = \gamma_n^* + s_{1,n}(\beta_n^* - \tilde{\beta}_{n-1}) + s_{2,n+1} - s_{2,n}, \quad n \geq 1, \tag{11.20}$$

if and only if the following formulas hold:

$$B_{i,n} = 0, \quad n \geq i, \quad 2 \leq i \leq M-1, \quad M \geq 3, \tag{11.21}$$

$$B_{M,n} = 0, \quad n \geq M, \quad M \geq 2, \tag{11.22}$$

$$B_{M+1,n} = 0, \quad n \geq M+1, \quad M \geq 1, \tag{11.23}$$

where

$$B_{i,n} := s_{i+1,n} - s_{i+1,n+1} + s_{i,n}(\tilde{\beta}_{n-i} - \beta_n^*) + s_{i-1,n}\tilde{\gamma}_{n+1-i} - s_{i-1,n-1}\gamma_n^*, \tag{11.24}$$

$$B_{M,n} := s_{M,n}(\tilde{\beta}_{n-M} - \beta_n^*) + s_{M-1,n}\tilde{\gamma}_{n+1-M} - s_{M-1,n-1}\gamma_n^*, \tag{11.25}$$

$$B_{M+1,n} := s_{M,n}\tilde{\gamma}_{n-M} - s_{M,n-1}\gamma_n^*. \tag{11.26}$$

Proof Inserting the three-term recurrence relation satisfied by the polynomials R_n and applying (11.18) to $xR_n(x)$, $R_n(x)$ and $R_{n-1}(x)$, successively, we have for $n \geq 2$

$$\begin{aligned} Q_{n+1}(x) = x\left[Q_n + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x)\right] - \beta_n^* \left[Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x)\right] \\ - \gamma_n^* \left[Q_{n-1}(x) + \sum_{i=1}^{M} s_{i,n-1} Q_{n-1-i}(x)\right] - \sum_{i=1}^{M} s_{i,n+1} Q_{n+1-i}(x). \end{aligned} \tag{11.27}$$

Then, we can write

$$
\begin{aligned}
&Q_{n+1}(x) = \\
&xQ_n - (\beta_n^* + s_{1,n+1} - s_{1,n})Q_n(x) - (\gamma_n^* + s_{1,n}\beta_n^* + s_{2,n+1} - s_{2,n})Q_{n-1}(x) \\
&+ \sum_{i=1}^{M} s_{i,n}\,[xQ_{n-i}(x) - Q_{n+1-i}(x)] + \sum_{i=3}^{M} s_{i,n}Q_{n+1-i}(x) \qquad (11.28) \\
&- \beta_n^* \sum_{i=2}^{M} s_{i,n}Q_{n-i}(x) - \sum_{i=3}^{M} s_{i,n+1}Q_{n+1-i}(x) - \gamma_n^* \sum_{i=1}^{M} s_{i,n-1}Q_{n-1-i}(x).
\end{aligned}
$$

Now we suppose that $(Q_n)_n$ is a MOPS with recurrence coefficients $(\tilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$, that is the polynomials satisfy

$$
Q_{n+1}(x) = (x - \tilde{\beta}_n)Q_n(x) - \tilde{\gamma}_n Q_{n-1}(x), \quad n \geq 0, \qquad (11.29)
$$

with $\tilde{\gamma}_n \neq 0$ for all $n \geq 1$. Hence, applying this recurrence relation to every factor $[xQ_{n-i}(x) - Q_{n+1-i}(x)]$ in formula (11.28) and using (11.19) and (11.20), it can be derived for $n \geq 2$

$$
\sum_{i=2}^{M-1} B_{i,n}Q_{n-i}(x) + B_{M,n}Q_{n-M}(x) + B_{M+1,n}Q_{n-1-M}(x) = 0.
$$

Then, the conditions (11.21), (11.22) and (11.23) hold, for $n \geq 2$.

In order to proof the reverse, we observe that formula (11.28) and conditions (11.21), (11.22) and (11.23) yield

$$
\begin{aligned}
&Q_{n+1}(x) - (x - \tilde{\beta}_n)Q_n(x) + \tilde{\gamma}_n Q_{n-1}(x) \qquad (11.30) \\
&= -\sum_{i=1}^{M} s_{i,n}\left[Q_{n+1-i}(x) - (x - \tilde{\beta}_{n-i})Q_{n-i}(x) + \tilde{\gamma}_{n-i}Q_{n-1-i}(x)\right],
\end{aligned}
$$

for $n \geq 2$.

Since formula (11.29) is true for $n = 1$ and obviously for $n = 0$, we have that it is also true for $n = 2$. Hence by using the induction method we can deduce that the recurrence relation (11.29) holds for all $n \geq 0$.

To conclude it suffices to observe that $\tilde{\gamma}_n \neq 0$ for $n \geq 1$, from (11.23). □

In the following theorem, we link the results studied in Proposition 11.5 and Proposition 11.6 and thus we get to solve the inverse problem in a general case without many calculations.

Theorem 11.7 *Let* $(P_n)_n$ *be a MOPS with respect to a regular functional* $\mathbf{u}$ *with* $(\beta_n)_n$ *and* $(\gamma_n)_n$ *the corresponding sequences of recurrence coefficients. We define* *recursively a sequence $(Q_n)_n$ of monic polynomials by formula (11.1), i.e.*

$$P_n(x) + \sum_{i=1}^{N} r_{i,n} P_{n-i}(x) = Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x), \quad n \geq 0,$$

where $(r_{i,n})_{i=1}^{N}$ and $(s_{i,n})_{i=1}^{M}$ are complex numbers, and such that the conditions det $\mathbf{A} \neq 0$ and $r_{N,n}\, s_{M,n} \neq 0$, for all $n \geq N+M$ are satisfied.

Then $(Q_n)_n$ is a MOPS with recurrence coefficients $(\tilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$, where

$$\tilde{\beta}_n + s_{1,n} - s_{1,n+1} = \beta_n + r_{1,n} - r_{1,n+1}, \quad n \geq 0,$$

$$\begin{aligned}&\tilde{\gamma}_n + s_{1,n}\left(\tilde{\beta}_{n-1} - \tilde{\beta}_n + s_{1,n+1} - s_{1,n}\right) + s_{2,n} - s_{2,n+1} \\ &= \gamma_n + r_{1,n}\left(\beta_{n-1} - \beta_n + r_{1,n+1} - r_{1,n}\right) + r_{2,n} - r_{2,n+1}, \quad n \geq 1.\end{aligned}$$

if and only if the polynomials Q_n satisfy the three-term recurrence relation with $\tilde{\gamma}_n \neq 0$, for $n = 1, 2, \ldots, N$, and the equations (11.14)–(11.16) and (11.21)–(11.23), for $n \geq N+M+1$ hold.

Proof Inserting formula (11.10) in (11.1) and applying (11.1) to $xP_n(x)$, and again (11.10) to $xP_{n-i}(x)$, for $i = 1, 2, \ldots, N$, we get, for $n \geq 1$,

$$\begin{aligned}Q_{n+1}(x) = &- \sum_{i=1}^{M} s_{i,n+1} Q_{n+1-i}(x) + x\left[Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x)\right] \\ &- \beta_n P_n(x) - \gamma_n P_{n-1}(x) + \sum_{i=1}^{N} r_{i,n+1} P_{n+1-i}(x) \\ &- \sum_{i=1}^{N} r_{i,n}\left[P_{n+1-i}(x) + \beta_{n-i} P_{n-i}(x) + \gamma_{n-i} P_{n-1-i}(x)\right].\end{aligned}$$

Now, in the above expression, we apply (11.1) to $P_n(x)$ and we rearrange the formula. Next, we do the same for $P_{n-1}(x)$. Hence, using the auxiliary coefficients (11.12) and (11.13), we can write the above formula in the following way

$$\begin{aligned}Q_{n+1}(x) = &- \sum_{i=1}^{M} s_{i,n+1} Q_{n+1-i}(x) + x\left[Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x)\right] \\ &- \beta_n^*\left[Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x)\right] - \gamma_n^*\left[Q_{n-1}(x) + \sum_{i=1}^{M} s_{i,n-1} Q_{n-1-i}(x))\right] \\ &+ \sum_{i=2}^{N-1} A_{i,n} P_{n-i}(x) + A_{N,n} P_{n-N}(x) + A_{N+1,n} P_{n-(N+1)}(x).\end{aligned} \tag{11.31}$$

Let $(Q_n)_n$ be a MOPS with recurrence coefficients $(\tilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$. Then, applying (11.29) to every factor $xQ_{n-i}(x)$, for $i = 1, \dots, M$ in formula (11.31), and using (11.19), (11.20) and (11.24)–(11.26), we get

$$0 = \sum_{i=2}^{M-1} B_{i,n}Q_{n-i}(x) + B_{M,n}Q_{n-M}(x) + B_{M+1,n}Q_{n-(M+1)}(x) \tag{11.32}$$
$$+ \sum_{i=2}^{N-1} A_{i,n}P_{n-i}(x) + A_{N,n}P_{n-N}(x) + A_{N+1,n}P_{n-(N+1)}(x).$$

Now applying the functional $\overline{P}_{M-1}\mathbf{u}$ to the above equation and taking into account the orthogonality of the polynomials P_n with respect to $\mathbf{u}$, we obtain

$$\langle \overline{P}_{M-1}\mathbf{u}, \sum_{i=2}^{M-1} B_{i,n}Q_{n-i} + B_{M,n}Q_{n-M} + B_{M+1,n}Q_{n-(M+1)} \rangle = 0, \quad n \geq N + M + 1,$$

$$\langle \overline{P}_{M-2}\mathbf{u}, \sum_{i=2}^{M-1} B_{i,n}Q_{n-i} + B_{M,n}Q_{n-M} + B_{M+1,n}Q_{n-(M+1)} \rangle = 0, \quad n \geq N + M,$$

and successively until

$$\langle \overline{P}_{0}\mathbf{u}, \sum_{i=2}^{M-1} B_{i,n}Q_{n-i} + B_{M,n}Q_{n-M} + B_{M+1,n}Q_{n-(M+1)} \rangle = 0, \quad n \geq N + 2.$$

Thus, for $n \geq N + M + 1$, we have

$$\sum_{i=2}^{M-1} B_{i,n}\langle \overline{P}_j\,\mathbf{u}, Q_{n-i}\rangle + B_{M,n}\langle \overline{P}_j\,\mathbf{u}, Q_{n-M}\rangle + B_{M+1,n}\langle \overline{P}_j\,\mathbf{u}, Q_{n-(M+1)}\rangle = 0, \tag{11.33}$$

for $j = 0, 1, \dots, M - 1$.

Fixed an positive integer n ($n \geq N + M + 1$), we consider the homogeneous system (11.33) of M equations and M unknowns $B_{i,n}$, $i = 2, \dots, M + 1$, whose associated matrix is $\widetilde{\mathbf{B}}_{n-2}^T$ that is the transpose of the matrix $\widetilde{\mathbf{B}}_{n-2}$. Applying the hypotheses det$\mathbf{A} \neq 0$, $s_{M,n} \neq 0$, for all $n \geq N + M$, and (b) of the Lemma 11.1, we get det $\widetilde{\mathbf{B}}_{n-2}^T \neq 0$, for all $n \geq N + M + 1$. Therefore the system has a unique solution that is the trivial solution

$$B_{i,n} = 0, \quad i = 2, \dots, M + 1,$$

so the equations (11.21)–(11.23) hold for $n \geq N + M + 1$. The other equations (11.14)–(11.16) for $n \geq N + M + 1$ are also fulfilled simply noticing that from the equation (11.32) we obtain

$$\sum_{i=2}^{N-1} A_{i,n} P_{n-i}(x) + A_{N,n} P_{n-N}(x) + A_{N+1,n} P_{n-(N+1)}(x) = 0,$$

and the sequence of polynomials $(P_n)_n$ is a basis.

Conversely, notice that the equations (11.14)–(11.16) in formula (11.31) yield

$$\begin{aligned} Q_{n+1}(x) = & -\sum_{i=1}^{M} s_{i,n+1} Q_{n+1-i}(x) + x \left[Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x) \right] \\ & - \beta_n^* \left[Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x)) \right] - \gamma_n^* \left[Q_{n-1}(x) + \sum_{i=1}^{M} s_{i,n-1} Q_{n-1-i}(x) \right], \end{aligned}$$

for $n \geq N + M + 1$. We observe that this is formula (11.27) in the Proposition 11.6 which can be rewritten as (11.28). Then taking into account the equations (11.21)–(11.23) we get (11.30) for $n \geq N + M + 1$. Thus, since by hypotheses the polynomials Q_n satisfy the three-term recurrence relation with $\tilde{\gamma}_n \neq 0$, for $n = 1, 2, \ldots, N$, we obtain that these polynomials satisfy the same three-term recurrence relation for all n. Besides from (11.16) and (11.23) we get that $\tilde{\gamma}_n \neq 0$, for all $n \geq N + 1$, and therefore the sequence $(Q_n)_n$ is a MOPS. □

Remark. Observe that to prove the first part of the Theorem, i.e. with the assumption that the sequence $(Q_n)_n$ is orthogonal, it is enough to require only the initial conditions (11.3), taking into account (a) of Theorem 11.3. However for the converse we need that the more extensive condition $r_{N,n}\, s_{M,n} \neq 0$ for all $n \geq N + M$ hold.

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