SIMPLY INVARIANT SUBSPACES IN $L^p(\Gamma,\mu)$

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1.- Introduction and notation.

Let Γ be a rectifiable Jordan curve in the complex plane and denote by Ω the region inside of Γ. By a result of Caratheodory, each conformal mapping φ, from D (the open unit disc) onto Ω, has a unique extension to $|z|\leq 1$; particulary, its restriction to unit circle T is a parametrization of Γ (We may suppose the origin belongs to Ω and $\varphi(0)=0$). Denote by ψ the inverse function of φ,.

Let μ be a finite nonnegative measure on Γ, which is absolutely continuous with respects to arc length ds ($d\mu=w\,ds$). For $0<p<\infty$, the $L^p(\Gamma,\mu)$ space is the class of complex μ-measurable functions defined on Γ, such that $\int |f|^p d\mu<\infty$, and the $H^p(\Gamma,\mu)$ is the closed span in $L^p(\Gamma,\mu)$ of analytic polynomials $P(z)=\sum_0^n a_k . z^k$.

The measure ν on T given by $d\nu = w\circ\varphi\ |\varphi'|\,d\theta$, where $d\theta$ is the normalized Lebesgue measure, is the image measure of μ by φ. It is obvious $f \in L^p(\Gamma,\mu)$ if and only if $f\circ\varphi \in L^p(T,\nu)$ and, by using a theorem of Mergelyan ([6], 386), $f \in H^p(\Gamma,\mu)$ if and only if $f\circ\varphi \in H^p(T,\nu)$.

We consider on $L^p(\Gamma,\mu)$ the "shift" operator S defined by $S(f)(z)= z\,f(z)$. A closed subespace M of $L^p(\Gamma,\mu)$ is simply invariant (s.i.) under the operator S if $S(M) \subsetneq M$. Our goal in this paper is to study the s.i. subspaces of $L^p(\Gamma,\mu)$ under the operator S, $1<p<\infty$. The problem of determinig the s.i. subspaces was solved by Beurling [1] for the Lebesgue measure and $p=2$. In [3] this result was extended to $H^p(T,\mu)$ when μ is a arbitrary measure wich verifies Szegö's condition, obtaining a paticular description of $H^p(T,\mu)$.

2.- Simply Invariant Subspaces in $L^p(\Gamma,\mu)$.

Let M be a s.i. subspace of $L^p(\Gamma,\mu)$, $1<p<\infty$; select F in $M\setminus zM$, then there exist a function $q\in M$ such that, F-q, is the vector of best approximation of F in zM. The function q is characterized by the condition:

$$\int_\Gamma |q|^{p-2}\bar{q}f\,d\mu=0$$

for every $f\in zM$ ([5])

LEMMA 1.- q is characterized by the following three conditions:

i) $F-q\in zM$

ii) $|q|^p\,d\mu=\|q\|_p^p\,|\psi'|\,ds$

iii) $f\in q\,H^p(\Gamma,|\psi'|ds)$, for all $f\in M$

Proof.-

i) Trivial.

ii) Since ψ is analytic on Ω, continuous on $\bar{\Omega}$ and $\psi(0)=0$, we may approximate ψ on Γ by polynomials which vanish in 0 by using Mergelyian's result. Then, we have $\psi^n q\in zM$ for $n\geq 1$ and

$$0=\int_\Gamma \psi^n q\,|q|^{p-2}\bar{q}\,d\mu=\int_\Gamma \psi^n|q|^p d\mu=\int_T e^{in\theta}|q\circ\varphi|^p d\nu$$

Hence $|q\circ\varphi|^p d\nu=\|q\circ\varphi\|^p_{L^p(T)}d\theta$, since every measure on T is characterized by its Fourier-Stieltjes coefficients.

iii) If f is an element belonging to M, we have, $\psi^n f\in zM$ if $n\geq 1$, and

$$0=\int_\Gamma \psi^n f\,|q|^{p-2}\bar{q}\,d\mu=\int_\Gamma \psi^n f q^{-1}\|q\|_p^p\,|\psi'|\,ds=\int_T e^{in\theta}(fq^{-1}\circ\varphi)\,d\theta$$

Thus $fq^{-1}\in H(\Gamma,|\psi'|ds)$

Conversely, if $f\in M$ by iii) $f=qg$ with $g\in H^p(\Gamma,|\psi'|ds)$ Then, by ii)

$$\int_\Gamma z\,f\,|q|^{p-2}\bar{q}\,d\mu=\int_\Gamma zg|q|^p\,d\mu=\|q\|_p^p\int_\Gamma zg\,|\psi'|\,ds=\|q\|_p^p\int_T \varphi(g\circ\varphi)\,d\theta=0$$

since $\varphi.(g\circ\varphi)\in H(T)$ and $\varphi(0)=0$

LEMMA 2.- A subspace M of $L^p(\Gamma,\mu)$ $1<p<\infty$, is s.i. if and only if there exists $q\in L^p(\Gamma,\mu)$ such that $M=q\,H^p(\Gamma,|\psi'|ds)$. In this case, q is determined by the subspace up to constant factors of modulus one, and it verifies, $|q|^p w=|\psi'|$ a.e. on Γ.

Proof.- Let M be a s.i. subspace of $L^p(\Gamma,\mu)$ and q as in lemma 1. After a normalization we may suppose $|q|^p w=|\psi|$ a.e. on Γ. From lemma 1-iii), $M \subseteq q\, H^p(\Gamma,|\psi|ds)$. Now we shall prove that $fq \in M$ if $f \in H^p(\Gamma,|\psi'|ds)$. For each $\varepsilon>0$, we can find a polynomial $P(z)$ such that $\|f-P\|<\varepsilon$ in $L^p(\Gamma, |\psi'|ds)$-norm. Then $\|fq-Pq\|<\varepsilon$ in $L^p(\Gamma,d\mu)$ and $fq \in M$, since $q \in M$ and M is a closed subspace.

Conversely, it is clear that $M=q\, H^p(\Gamma,|\psi'|ds)$ is an invariant subspace. Since $zM=M$ implies $\varphi H^p(T)=H^p(T)$, M must be a s.i. subspace because, in other case, φ would be an outer function and this contradicts the fact that $\varphi(0)=0$.

Finally, assume that there are q_1, q_2, that satisfy the conditions of lemma 2 and $q_1 H^p(\Gamma,|\psi'|ds)=q_2 H^p(\Gamma,|\psi'|ds)$. It follows that $|q_1|=|q_2|$, $q_1=q_2 g$ with $g \in H^p(\Gamma,|\psi'|ds)$ and $q_2=q_1 h$ with $h \in H^p(\Gamma,|\psi'|ds)$. Then $gh=1$ and since $|g|=|h|=1$, we obtain $g=\bar{h}$. Hence $\mathrm{Re}\,(h) \in H^p(\Gamma,|\psi'|ds)$ and this shows that h is constant.

We can apply preceding lemma when $w=1$ and $M=H^p(\Gamma,ds)$. Indeed, if M is not s.i. subspace of $L^p(\Gamma,ds)$, $zH^p(\Gamma,ds)= H^p(\Gamma,ds)$. Then, $\varphi H^p(T,|\varphi'|d\theta)=H^p(T,|\varphi'|d\theta)$ and $1/\varphi \in H^p(T)$, which is not true $(\varphi(0)=0)$.

We obtain

THEOREM 3.- $M \subseteq L^p(\Gamma,\mu)$, $1<p<\infty$, is a s.i. subspace if and only if $M= Q\, H^p(\Gamma,ds)$, where $Q \in L^p(\Gamma,\mu)$, $|Q|^p w=1$ and Q is unique up to constant factors of modulus one.

Proof.- $M=q\, H^p(\Gamma,|\psi'|ds)$ and $H^p(\Gamma,ds)=q_1 H^p(\Gamma,|\psi'|ds)$, then the results holds for $Q= q/q_1$.

3.- The Space $H^p(\Gamma,\mu)$.

Now we shall study when the space $H^p(\Gamma,\mu)$ is a s.i. subspace of $L^p(\Gamma,\mu)$ and in this case we shall obtain an explicit description of Q. In order to obtain this, first we must consider when

$H^p(\Gamma,\mu)$ is a proper subspace of $L^p(\Gamma,\mu)$. In [7] p. 341, it appears that $H^p(\Gamma,\mu) \subsetneq L^p(\Gamma,\mu)$ iff $|\psi'| \log w \in L^1(\Gamma, ds)$ (i.e. $\log w \circ \varphi \in L^1(T)$). We shall give a sufficient condition on weight w, restricting the class of curves.

Definition.- Let Γ be a rectifiable Jordan curve. An arc along Γ with endpoints z_1 and z_2 will be denoted (z_1, z_2). Γ is said to be a chord-arc curve if there is a constant C such that for all points z_1, z_2 of Γ, $s(z_1, z_2) \leq C \, |z_1 - z_2|$, where $s(z_1, z_2)$ is the arc length of z_1, z_2.

If Γ is a chord-arc curve then Ω is a Smirnov domain (φ' is outer) and these are the most general domains where the system of orthogonal polynomials spans $H^2(\Gamma, ds)$.

Definition.- If $1<p<\infty$ and w is a nonnegative function on Γ, integrable with respect to ds, we say that $w \in A_p(\Gamma)$ if there exists a constant $C_p>0$ such that for all intervals $J \subseteq \Gamma$

$$\left(\frac{1}{S(J)} \int_J w \, ds \right) \left(\frac{1}{S(J)} \int_J w^{-1/p-1} ds \right)^{p-1} \leq C_p$$

Where $S(J)$ is the arc length of J.

This definition constituies the natural extension of the classes A_p of Muckenhoupt (see [2]). The class A_1 is the limit of the classes A_p when $p \to 1$ and A_∞ is the union of all classes A_p.

In the segment that follows we suppose that Γ is a chord-arc curve and then $|\varphi'| \in A_q$ for some $q \in [1,\infty)$ (see [8]).

By using Hölder's inequality we can obtain that $\log w \in L^q(\Gamma, ds)$ implies $H^p(\Gamma,\mu) \subsetneq L^p(\Gamma,\mu)$ (see [4]). In this case, it is not difficult to see that $H^p(\Gamma,\mu)$ is a s.i. subspace. Hence, when theorem 3 is used, $H^p(\Gamma,\mu) = Q \, H^p(\Gamma, ds)$ where $Q=q/q_1$.

The function $1/q\circ\varphi$ is outer in $H^p(T)$. Indeed, if $h\in H^p(\Gamma,|\psi'|ds)$, we have $qh\in H^p(\Gamma,\mu)$ and then there is a sequence of polynomials $\{P_n\}_1^\infty$, such that $\int_\Gamma |qh-P_n|^p d\mu \to 0 \quad (n\to\infty)$.
Consequently, $\int_T |h\circ\varphi - 1/q\circ\varphi\ P_n\circ\varphi|^p d\theta \to 0 \quad (n\to\infty)$.
By application of Mergelyan's result we can get another sequence of polynomials Q_n such that

$$\int_T |h\circ\varphi - 1/q\circ\varphi\ Q_n|^p d\theta \to 0 \qquad (n\to\infty)$$

and then $1/q\circ\varphi$ is outer.

But if g is an outer function in $H^p(T)$ then $g=\exp(u+i\tilde{u})$, where u is real, $\tilde{u}$ is the conjugate function of u and both u, $\exp(pu)$ are integrable. Since $|q\circ\varphi|^p w\circ\varphi = |\psi'\circ\varphi|$ and $1/q\circ\varphi$ is outer, we have

$$q\circ\varphi = (w\circ\varphi/|\psi'\circ\varphi|)^{-1/p} \exp|-i/p\ (\log w\circ\varphi/|\psi'\circ\varphi|)^\sim|$$

By the some reasons

$$q_1\circ\varphi = (1/|\psi'\circ\varphi|)^{-1/p}\exp|-i/p\ (\log 1/|\psi'\circ\varphi|)^\sim|$$

and then it follows that

$$(q/q_1)\circ\varphi = (w\circ\varphi)^{-1/p}\exp|-i/p\ \log(w\circ\varphi)^\sim|$$

and

$$Q=w^{-1/p}\exp|-i/p\ (\log w)^\sim|$$

where ~ represents the conjugate fuction operator on Γ defined by $\tilde{f}=(f\circ\varphi)^\sim\circ\psi$ wich coincides with the natural definition for polynomials, i.e., if $f=\mathrm{Re}\,(P(z)|_\Gamma)$ then $\tilde{f}=\mathrm{Im}\,(P(z)|_\Gamma)$.

Denoting $K_p=Q$, we obtain the following representation for $H^p(\Gamma,\mu)$ space.

THEOREM 4.- If $1<p<\infty$, $H^p(\Gamma,\mu)=K_p\ H^p(\Gamma,ds)$ where

$$K_p=w^{-1/p}\exp|-i/p\ (\log w)^\sim|$$

COROLLARY 5.- If Γ is a chord-arc curve, $(\log w)^q\in L^1(\Gamma,ds)$ and $1<p<\infty$, the simply invariant subspaces of $L^p(\Gamma,\mu)$ are the form

$M=u\ H^p(\Gamma,\mu)$, where $|u|=1$ a.e. and u is unique up to constant factors of modulus one.

For $\Gamma=T$ and w=1 we have Beurling's theorem.

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