Orthogonal polynomials associated with an inverse quadratic spectral transform

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ABSTRACT

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials with respect to a quasi-definite linear functional u and $\{Q_n\}_{n\geq 0}$ a sequence of polynomials defined by

$$Q_n(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x), \quad n \geq 1,$$

with $t_n \neq 0$ for $n \geq 2$.

We obtain a new characterization of the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$ with respect to a linear functional v, in terms of the coefficients of a quadratic polynomial h such that $h(x)v = u$.

We also study some cases in which the parameters s_n and t_n can be computed more easily, and give several examples.

Finally, the interpretation of such a perturbation in terms of the Jacobi matrices associated with $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ is presented.

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1. Introduction

Let u be a linear functional defined in the linear space $\mathbb{P}$ of polynomials with complex coefficients. The action of u over an element $p \in \mathbb{P}$ will be denoted by (u, p).

The sequence of complex numbers $\{u_n\}_{n\geq 0}$ such that $u_n = (u, x^n)$, is said to be the moment sequence associated with the linear functional u. The corresponding z-transform $\sum_{n=0}^{\infty} u_n z^{-(n+1)}$ defines a function $S(z)$ which is analytic in a neighbourhood of infinity. This function is known in the literature as a Stieltjes function associated with the linear functional u (see [1]).

The linear functional u defines a bilinear functional on $\mathbb{P} \times \mathbb{P}$ in the following way: $(p, q)_u = (u, pq)$, for every $p, q \in \mathbb{P}$. If $H = [(x^k, x^l)_u]_{k,l=0}^{\infty}$ denotes the Gram matrix of the above bilinear functional with respect to the canonical basis of $\mathbb{P}$, then H is a Hankel matrix, i.e. the entries in every antidiagonal are equal.

The linear functional u is said to be quasi-definite if the leading principal submatrices H_n, $n \geq 0$, of H are non singular. In such a situation, there exists a sequence of monic polynomials $\{P_n\}_{n\geq 0}$ with $\deg P_n = n$ such that $(u, P_n P_m) = k_n \delta_{n,m}$ with $k_n \neq 0$ for all $n \geq 0$. The sequence $\{P_n\}_{n\geq 0}$ is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to the linear functional u.

Taking into account that the multiplication operator by x is a symmetric operator with respect to the bilinear functional $(., .)_u$, the SMOP $\{P_n\}_{n\geq 0}$ satisfies for $n \geq 0$, a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

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with $P_0(x) = 1$ and $P_{-1}(x) = 0$ where $\gamma_n \neq 0$ for every $n \geq 1$. Conversely, if a sequence of monic polynomials $\{P_n\}_{n\geq0}$ satisfies a recurrence relation like the previous one, then there exists a unique quasi-definite linear functional u such that $\{P_n\}_{n\geq0}$ is the corresponding SMOP. This result is known in the literature as Favard's theorem (see [2]).

For the SMOP $\{P_n\}_{n\geq0}$ there exists a sequence $\{\nu_n\}_{n\geq0}$ of linear functionals such that $(\nu_m, P_n) = \delta_{nm}$. This sequence $\{\nu_n\}_{n\geq0}$ is said to be the dual sequence of $\{P_n\}_{n\geq0}$. Notice that $\nu_n = \frac{P_n u}{(u, P_n^2)}$, where for every $p \in \mathbb{P}$, pu denotes the linear functional such that $(pu, q) = (u, pq)$, $q \in \mathbb{P}$. The linear functional $\nu_0 = \frac{u}{(u, P_0^2)}$ is said to be the normalization of u in the sense that $(\nu_0, 1) = 1$. Moreover, the linear functional $(x - a)^{-1} u$ is defined on $\mathbb{P}$ by

$$((x-a)^{-1}u, P) = \left(u, \frac{P(x) - P(a)}{x - a}\right), \quad P \in \mathbb{P}.$$

If $\mathcal{P} = (P_0, P_1, \ldots)^T$ denotes the column vector associated with an SMOP, then the three-term recurrence relation reads $x\mathcal{P} = J_{\mathcal{P}}\mathcal{P}$ where $J_{\mathcal{P}}$ is a tridiagonal matrix such that the diagonal entries are $\{\beta_n\}_{n\geq0}$, the upper diagonal entries are 1, and the subdiagonal entries are $\{\gamma_n\}_{n\geq1}$. It is called a monic Jacobi matrix (see [2]).

Thus, for a quasi-definite linear functional u we have three characteristic elements.

 (i) The Stieltjes function.
 (ii) The SMOP $\{P_n\}_{n\geq0}$.
(iii) The monic Jacobi matrix $J_{\mathcal{P}}$.

The connections between these three basic elements in the theory of orthogonal polynomials with Padé rational approximants, quadrature formula, and spectral theory of symmetric operators have been very fruitful.

In a more general framework, in [3,4] the author deals with quasi-orthogonal polynomial sequences $\{Q_n\}_{n\geq0}$ of order r with respect to a positive Borel measure μ. They are defined by $Q_n(x) = P_n(x) + b_{n,n-1}P_{n-1}(x) + \cdots + b_{n,n-r}P_{n-r}(x)$ with $b_{n,n-r} \neq 0$, where $\{P_n\}_{n\geq0}$ is the sequence of orthonormal polynomials with respect to μ. When the polynomials Q_n are characteristic polynomials of some symmetric tridiagonal matrices with positive subdiagonal entries, many properties of their zeros can be deduced from such a formulation. Furthermore, a positive quadrature formula with n nodes which is exact for polynomials of degree at most $2n - r - 1$, $0 \leq r \leq n$, is based on the zeros of the quasi-orthogonal polynomial Q_n and, as a consequence, for a fixed n, every positive quadrature formula is a Gaussian quadrature formula for some nonnegative measure.

Many authors have been interested in the study of SMOP's associated with some perturbations of linear functionals. In particular, the so-called direct problems deal with some canonical perturbations of linear functionals. For instance, the analysis of the parameters of the three-term recurrence relation when a positive definite linear functional is multiplied by a polynomial (Christoffel transformation) has been done from a numerical point of view in [5], from the point of view of the corresponding monic Jacobi matrices in [6], and taking into account the associated Stieltjes functions, in [7,8]. The relation between the corresponding SMOP in the case of quasi-definite linear functionals has been deduced in [9]. The addition of Dirac masses and their derivatives to a quasi-definite linear functional is called Uvarov transformation (see [10,11,8]). They play an important role in the spectral analysis of higher order linear differential operators whose eigenfunctions are SMOP (see [12]). They are related to bispectral problems and self-similar reductions of differential operators [7]. The connection between the corresponding monic Jacobi matrices has been studied in [6].

On the other hand, given a quasi-definite linear functional u and polynomials p, q the analysis of the quasi-definiteness of a linear functional v such that $pv = qu$ has been done in the framework of the so-called generalized Christoffel formula when u is a positive definite linear functional and q/p is a positive rational function in the support of u (see [5]). In such a case the connection between the monic Jacobi matrices has been done in [13]. For the quasi-definite case, the relation between the corresponding SMOP has been given in [9].

In the general situation, if $\deg p = 1$ and q is a constant, necessary and sufficient conditions for the quasi-definite character of v have been done in [1]. There, the expression of the perturbed SMOP in terms of the initial one as well as the relation between the parameters of the three-term recurrence relation is stated. This perturbation is called the Geronimus transformation. The relation between the corresponding monic Jacobi matrices is analyzed in [6,11]. Finally, for the Stieltjes functions associated with u and v the connection is given in [8].

When $\deg p = \deg q = 1$, the relation between the SMOP associated with u and v is stated in [14].

Eventually, when $p(x) = x^2$ and q is a constant, in [15] the authors focus the attention in the invariance of the semiclassical character for such a transformation. For a more general polynomial transformation and the invariance of the semiclassical character, see [16].

All these cases correspond to the so-called direct problems, i.e., how a perturbation in a linear functional concerns to Stieltjes functions, monic Jacobi matrices, and orthogonal polynomials, respectively. In our contribution, the approach will be completely different and it will be based on the fact that algebraic relations between sequences of monic orthogonal polynomials yield relations between the corresponding linear functionals as well as the corresponding Jacobi matrices. They can be considered as inverse problems.

The study of inverse problems, i.e. to find the relation between two linear functionals assuming their corresponding SMOP's are related in terms of an algebraic expression, that it is said to be of type $k - j$, $Q_n(x) + a_{n,n-1}Q_{n-1}(x) + \cdots + a_{n,n-k+1}Q_{n-k+1}(x) = P_n(x) + b_{n,n-1}P_{n-1}(x) + \cdots + b_{n,n-j+1}P_{n-j+1}(x)$, has been firstly considered by Geronimus in [17] in the

case 1–2. Later on, the relation 1–3 has been analyzed in [18] as well as in [19] from the point of view of the corresponding parameters of the three-term recurrence relation. In this last contribution the authors also deduce the second order linear differential equation satisfied by the perturbed sequence of orthogonal polynomials when the initial sequence is a classical one. We also point out that in [20,21] the authors focus the attention in the study of these perturbed families when the initial orthogonal polynomial sequences are the Chebyshev polynomials of first and second kind, respectively. In particular, the integral representation of the corresponding linear functional is obtained. Relations of type 2–2 have been completely studied in [22]. In a very recent paper [23], the authors solved a problem studied in a particular case by Grinshpun (see [24]), concerning the orthogonality of sequences of monic polynomials $\{Q_n\}_{n\geq 0}$ with $Q_n(x) = P_n(x) + sP_{n-1}(x) + tP_{n-2}(x)$ where $\{P_n\}_{n\geq 0}$ is a given SMOP. Finally, assuming *a priori* the orthogonality of the SMOP $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, from a relation of type $k - j$ the connection between the corresponding linear functionals has been deduced in [25].

Such a relation appears in the framework of coherent pairs of linear functionals related to Sobolev inner products. This problem has been pointed out in [26] as well as, in a more general framework, in [27].

In [22,26], the orthogonality of the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ is given in terms of some constant sequences. Moreover, these constants are those appearing in the polynomial which relates the corresponding linear functional associated with $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$. More precisely, in [26] for the case 1–2 the authors proved the following results:

Let $\{P_n\}_{n\geq 0}$ be an SMOP with respect to a quasi-definite linear functional u with recurrence coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$. We define a sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ by $Q_n = P_n + \mu_n P_{n-1}$, $n \geq 1$, where μ_n are complex numbers and $\mu_1 \neq 0$. The sequence $\{Q_n\}_{n\geq 0}$ is an SMOP with respect to a quasi-definite linear functional v if and only $P_n^*(x_1; x_1 - \beta_0 + \mu_1, 0) \neq 0$ for all $n \geq 1$ and $\mu_n = -\frac{P_n^*(x_1; x_1 - \beta_0 + \mu_1, 0)}{P_{n-1}^*(x_1; x_1 - \beta_0 + \mu_1, 0)}$, where $\{P_n^*(x; x_1 - \beta_0 + \mu_1, 0)\}_{n\geq 0}$ denotes the co-recursive sequence at level zero corresponding to the sequence $\{P_n\}_{n\geq 0}$ (see [2,28,29]), and $x_1 = \beta_1 - \mu_2 - \frac{\gamma_1}{\mu_1}$.

This condition is equivalent to the fact that for $n \geq 1$ the parameters μ_n do not vanish and satisfy

$$\beta_n - \mu_{n+1} - \frac{\gamma_n}{\mu_n} = x_1, \quad n \geq 1. \tag{1.1}$$

Moreover, $(x - x_1)v = (\beta_0 - x_1 - \mu_1)u$. (see [26, Theorem 2]). This result will be used many times in this paper.

In the same way in [22], the authors studied the relations 2–2 $P_n + s_n P_{n-1} = Q_n + t_n Q_{n-1}$. They obtained the characterization of the orthogonality of the sequence of polynomials $\{Q_n\}_{n\geq 0}$ in terms of the existence of two constant sequences. Besides, these constants $a, \tilde{a}$ are the values appearing in the relation between the two linear functionals, i.e. $(x - \tilde{a})u = k(x - a)v, (a, \tilde{a}, k \in \mathbb{C})$.

A natural question is: Do these constant sequences depend on the coefficients of the polynomial relating the two linear functionals for more general cases?

The aim of our contribution is twofold. First, we find necessary and sufficient conditions that the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$, with $Q_n(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x)$, must satisfy in order to be orthogonal with respect to a linear functional v. Indeed, if u is the linear functional such that $\{P_n\}_{n\geq 0}$ is the corresponding SMOP, then there exists a monic quadratic polynomial $h(x) = x^2 + ax + b$ such that $h(x)v = ku$. We prove that in this case, there exist two constant sequences and the values of these constants a and b are precisely the coefficients and not the zeros of the quadratic polynomial $h(x) = x^2 + ax + b$ such that $h(x)v = ku$. In other words, we deduce two nonlinear constraints for the sequence of the coefficients of the three term recurrence relation and the sequences $\{s_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 2}$. A connection between the corresponding monic Jacobi matrices is stated.

Second, we obtain necessary and sufficient conditions in order to the above relation 1–3 can be decomposed in two relations 1–2 and then proceed by iteration. This is related with a two-step Geronimus transformation. In this case, we prove that the constant sequences can be decomposed in another two simple constant sequences, associated with the zeros of the polynomial $h(x)$. Also, as a consequence, a matrix interpretation using *LU* and *UL* factorization is done. Finally, when $\{P_n\}_{n\geq 0}$ is a symmetric SMOP, i.e. $P_n(-x) = (-1)^n P_n(x)$ and we assume $s_n = 0$, then we prove that the 1–3 relation yields two 1–2 relations.

The structure of the manuscript is the following. In Section 2 we present the basic background and we give a new characterization of the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$ (Theorem 2.2) in terms of some constant relations for $\{\beta_n\}, \{\gamma_n\}, \{s_n\}$, and $\{t_n\}$. They have not been yet studied in the literature. In Section 3 we deduce necessary and sufficient conditions for the split of a 1–3 relation into two 1–2 relations. The matrix interpretation of this problem in terms of monic Jacobi matrices is done. Some illustrative examples are considered.

2. A new characterization of orthogonality

From now on, $\{P_n\}_{n\geq 0}$ will denote an SMOP with respect to a quasi-definite linear functional u which satisfies the three-term recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 0, \tag{2.1}$$
$$P_0(x) = 1, \qquad P_{-1}(x) = 0,$$

where $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ are sequences of complex numbers with $\gamma_n \neq 0$ for $n \geq 1$.

Given two sequences of complex numbers $\{s_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 2}$ we define for $n \geq 1$ the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ such that

$$Q_n(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x), \quad \text{with } t_n \neq 0, \ n \geq 2. \tag{2.2}$$

First, we characterize the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$.

Proposition 2.1. *Let $\{P_n\}_{n\geq 0}$ an SMOP with recurrence coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$. We define a sequence $\{Q_n\}_{n\geq 0}$ of monic polynomials by formula* (2.2)*, i.e.,*

$$Q_n(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x), \quad n \geq 1,$$

where s_n and t_n are complex numbers with $t_n \neq 0$, for all $n \geq 2$. Then $\{Q_n\}_{n\geq 0}$ is an SMOP with recurrence coefficients $\{\tilde{\beta}_n\}_{n\geq 0}$ and $\{\tilde{\gamma}_n\}_{n\geq 1}$ if and only if $\tilde{\gamma}_1 \tilde{\gamma}_2 \neq 0$ and the following formulas hold:

$$s_{n-1}\tilde{\gamma}_n = s_n \gamma_{n-1} + t_n(\beta_{n-2} - \tilde{\beta}_n), \quad n \geq 2, \tag{2.3}$$

$$t_{n-1}\tilde{\gamma}_n = t_n \gamma_{n-2}, \quad n \geq 3, \tag{2.4}$$

where the coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ are defined by

$$\tilde{\beta}_n = \beta_n + s_n - s_{n+1}, \quad n \geq 0, \tag{2.5}$$

$$\tilde{\gamma}_n = \gamma_n + t_n - t_{n+1} + s_n(\beta_{n-1} - \tilde{\beta}_n), \quad n \geq 1, \tag{2.6}$$

with $s_0 = t_0 = t_1 = 0$.

Proof. Inserting formula (2.1) in (2.2) and applying (2.2) to $xP_n(x)$, we get

$$\begin{aligned} Q_{n+1}(x) = {} & xQ_n(x) + (s_{n+1} - s_n - \beta_n)P_n(x) + (t_{n+1} - \gamma_n - s_n\beta_{n-1} - t_n)P_{n-1}(x) \\ & - (s_n\gamma_{n-1} + t_n\beta_{n-2})P_{n-2}(x) - t_n\gamma_{n-2}P_{n-3}(x), \quad n \geq 1, \end{aligned}$$

provided we substitute there $xP_{n-1}(x)$ and $xP_{n-2}(x)$, using again (2.1). Now, formula (2.2) applied to $P_n(x)$ and the definition of $\tilde{\beta}_n$ (see (2.5)), yield

$$\begin{aligned} Q_{n+1}(x) = {} & (x - \tilde{\beta}_n)Q_n(x) + [t_{n+1} - \gamma_n - t_n - s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1})]P_{n-1}(x) \\ & - [s_n\gamma_{n-1} + t_n(s_{n+1} - s_n - \beta_n + \beta_{n-2})]P_{n-2}(x) - t_n\gamma_{n-2}P_{n-3}(x), \quad n \geq 1. \end{aligned}$$

So $\{Q_n\}_{n\geq 0}$ is an SMOP if and only if there exists a sequence of complex numbers $(\tilde{\gamma}_n)_{n\geq 1}$, with $\tilde{\gamma}_n \neq 0$ for $n \geq 1$, such that

$$\begin{aligned} & [t_{n+1} - \gamma_n - t_n - s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1})]P_{n-1}(x) \\ & \quad - [s_n\gamma_{n-1} + t_n(s_{n+1} - s_n - \beta_n + \beta_{n-2})]P_{n-2}(x) - t_n\gamma_{n-2}P_{n-3}(x) \\ & = -\tilde{\gamma}_n Q_{n-1}(x). \end{aligned}$$

Furthermore, $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ are the coefficients of the three-term recurrence relation for Q_n.

Using again (2.2), straightforward calculations yield (2.3) and (2.4). $\quad\square$

This result can be also seen in [18].

Remark. Observe that the conditions $\tilde{\gamma}_n \neq 0$, $\forall n \geq 3$, follow from (2.4).

If the sequence $\{Q_n\}_{n\geq 0}$ given by (2.2) is an SMOP with respect to a quasi-definite linear functional v, then it is well known, see [18,19], that the linear functional u is a quadratic polynomial modification of v, that is, $h(x)v = ku$ with $h(x) = x^2 + ax + b$ and k is a nonzero complex number.

In the next theorem, we show that the orthogonality of the sequence $\{Q_n\}_{n\geq 0}$ can be characterized by the fact that there are two sequences depending on the parameters $s_n, t_n, \beta_n, \gamma_n$ which remain constant and these constants are precisely the coefficients a and b of the polynomial h.

Theorem 2.2. *Let $\{P_n\}_{n\geq 0}$ be an SMOP with recurrence coefficients $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$. We define a sequence $\{Q_n\}_{n\geq 0}$ of monic polynomials by the formula* (2.2)*, i.e.,*

$$Q_n(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x), \quad n \geq 1,$$

where s_n and t_n are complex numbers with $t_n \neq 0$, for all $n \geq 2$ and $s_0 = t_0 = t_1 = 0$. Then the following statements are equivalent:

(i) *$\{Q_n\}_{n\geq 0}$ is an SMOP.*

(ii) *There exist two complex numbers a and b such that, for $n \geq 1$,*

$$A_n := \frac{s_n}{t_{n+1}}[\gamma_{n+1} + t_{n+1} - t_{n+2} + s_{n+1}(\beta_n - \beta_{n+1} - s_{n+1} + s_{n+2})] + s_{n+1} - \beta_{n-1} - \beta_n = a, \tag{2.7}$$

$$B_n := \frac{1}{t_{n+1}}[\gamma_{n+1} - t_{n+2} + s_{n+1}(\beta_n - \beta_{n+1} - s_{n+1} + s_{n+2})][\gamma_n + t_n - t_{n+1} + s_n(s_{n+1} - \beta_n)]$$
$$+ t_n - \gamma_{n-1} + (s_{n+1} - \beta_n)(s_n - \beta_{n-1}) = b, \tag{2.8}$$

where, by convention, we take $\gamma_0 = 0$. Besides, the values of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in (2.6) have to be different from zero.
    Furthermore, if u and v are the linear functionals associated with the sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively, normalized by $\langle u, 1 \rangle = \langle v, 1 \rangle = 1$, then

$$(x^2 + ax + b)v = ku \tag{2.9}$$

with $k \in \mathbb{C} \setminus \{0\}$.

Proof. We write the proof of this theorem in terms of $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ and then it is enough to use (2.5) and (2.6).
(i)⟹(ii).
    If $\{Q_n\}_{n\geq 0}$ is an SMOP, by Proposition 2.1, (2.3) and (2.4) hold. Inserting in (2.3) the expression of γ_{n-1} in terms of $\tilde{\gamma}_{n+1}$ given by (2.4) we get for $n \geq 2$,

$$\frac{s_{n-1}}{t_n}\tilde{\gamma}_n - \tilde{\beta}_{n-2} - \tilde{\beta}_{n-1} + s_{n-2} = \frac{s_n}{t_{n+1}}\tilde{\gamma}_{n+1} - \tilde{\beta}_{n-1} - \tilde{\beta}_n + s_{n-1} = a \tag{2.10}$$

and then we get (2.7) from (2.5) and (2.6).
    Now, we will deduce (2.8).
    From the definition of $\tilde{\gamma}_n$, using (2.4) and (2.5) we have

$$\tilde{\gamma}_n = \frac{t_{n+1}}{t_{n+2}}\tilde{\gamma}_{n+2} + t_n - t_{n+1} + s_n(\tilde{\beta}_{n-1} - s_{n-1} + s_n - \tilde{\beta}_n), \quad n \geq 1.$$

Multiplying in the above expression by $\tilde{\gamma}_{n+1}/t_{n+1}$

$$\frac{\tilde{\gamma}_n\tilde{\gamma}_{n+1}}{t_{n+1}} + (s_{n-1} - \tilde{\beta}_{n-1})\frac{s_n\tilde{\gamma}_{n+1}}{t_{n+1}} = \frac{\tilde{\gamma}_{n+1}\tilde{\gamma}_{n+2}}{t_{n+2}} + (s_n - \tilde{\beta}_n)\frac{s_n\tilde{\gamma}_{n+1}}{t_{n+1}} + \left(\frac{t_n}{t_{n+1}} - 1\right)\tilde{\gamma}_{n+1}.$$

Using (2.10), for $n + 1$ instead of n, in the expression $\frac{s_n\tilde{\gamma}_{n+1}}{t_{n+1}}$ which appears in the right hand side of the above formula, for $n \geq 1$ we obtain

$$\frac{\tilde{\gamma}_n\tilde{\gamma}_{n+1}}{t_{n+1}} + (s_{n-1} - \tilde{\beta}_{n-1})\left(\frac{s_n\tilde{\gamma}_{n+1}}{t_{n+1}} - \tilde{\beta}_n\right) = \frac{\tilde{\gamma}_{n+1}\tilde{\gamma}_{n+2}}{t_{n+2}} + (s_n - \tilde{\beta}_n)\left(\frac{s_{n+1}\tilde{\gamma}_{n+2}}{t_{n+2}} - \tilde{\beta}_{n+1}\right)$$
$$+ \left(\frac{t_n}{t_{n+1}} - 1\right)\tilde{\gamma}_{n+1} + s_n(\tilde{\beta}_{n-1} + s_n - s_{n-1} - \tilde{\beta}_n).$$

    Besides, from (2.4) and according to the definition of $\tilde{\gamma}_{n-1}$ we have, for $n \geq 1$,

$$\frac{t_n}{t_{n+1}}\tilde{\gamma}_{n+1} = \tilde{\gamma}_{n-1} - t_{n-1} + t_n - s_{n-1}(\tilde{\beta}_{n-2} + s_{n-1} - s_{n-2} - \tilde{\beta}_{n-1})$$

and, therefore, for $n \geq 1$

$$\frac{\tilde{\gamma}_n\tilde{\gamma}_{n+1}}{t_{n+1}} + (s_{n-1} - \tilde{\beta}_{n-1})\left(\frac{s_n\tilde{\gamma}_{n+1}}{t_{n+1}} - \tilde{\beta}_n\right) + t_{n-1} - \tilde{\gamma}_{n-1} - \tilde{\gamma}_n + s_{n-1}(\tilde{\beta}_{n-2} + s_{n-1} - s_{n-2} - \tilde{\beta}_{n-1})$$
$$= \frac{\tilde{\gamma}_{n+1}\tilde{\gamma}_{n+2}}{t_{n+2}} + (s_n - \tilde{\beta}_n)\left(\frac{s_{n+1}\tilde{\gamma}_{n+2}}{t_{n+2}} - \tilde{\beta}_{n+1}\right) + t_n - \tilde{\gamma}_n - \tilde{\gamma}_{n+1} + s_n(\tilde{\beta}_{n-1} + s_n - s_{n-1} - \tilde{\beta}_n) = b. \tag{2.11}$$

    Thus, by straightforward computations, using (2.5) and (2.6) we obtain (2.8).
(ii)⟹(i).
    Notice first that (2.7) and (2.8) are equivalent to (2.10) and (2.11). We will prove that (2.10) and (2.11) yield (2.4). Taking into account the new expression of $\frac{s_{n+1}\tilde{\gamma}_{n+2}}{t_{n+2}}$, obtained from (2.10) written for $n + 1$ instead of n, we can reformulate (2.11)

$$\frac{\tilde{\gamma}_n\tilde{\gamma}_{n+1}}{t_{n+1}} + (s_{n-1} - \tilde{\beta}_{n-1})\left(\frac{s_n\tilde{\gamma}_{n+1}}{t_{n+1}} - \tilde{\beta}_n\right) + t_{n-1} - \tilde{\gamma}_{n-1} + s_{n-1}(\tilde{\beta}_{n-2} + s_{n-1} - s_{n-2} - \tilde{\beta}_{n-1})$$
$$= \frac{\tilde{\gamma}_{n+1}\tilde{\gamma}_{n+2}}{t_{n+2}} + (s_n - \tilde{\beta}_n)\left[\frac{s_n\tilde{\gamma}_{n+1}}{t_{n+1}} - \tilde{\beta}_{n-1} + s_{n-1} - s_n\right] + t_n - \tilde{\gamma}_{n+1} + s_n(\tilde{\beta}_{n-1} + s_n - s_{n-1} - \tilde{\beta}_n).$$

Using the definition of $\tilde{\gamma}_{n-1}$ in the left hand side of the above expression and simplifying we get

$$\frac{\tilde{\gamma}_{n+1}}{t_{n+1}}[\tilde{\gamma}_n + s_n(\tilde{\beta}_n - s_n + s_{n-1} - \tilde{\beta}_{n-1}) + t_{n+1}] = \gamma_{n-1} + \frac{\tilde{\gamma}_{n+1}\tilde{\gamma}_{n+2}}{t_{n+2}}.$$

Using again the definition of $\tilde{\gamma}_n$ we have

$$\frac{\tilde{\gamma}_{n+1}}{t_{n+1}}(\gamma_n + t_n) = \gamma_{n-1} + \frac{\tilde{\gamma}_{n+1}\tilde{\gamma}_{n+2}}{t_{n+2}},$$

that is, for $n \geq 1$

$$\frac{\tilde{\gamma}_{n+1}}{t_{n+1}}\left(\gamma_n - \frac{t_{n+1}}{t_{n+2}}\tilde{\gamma}_{n+2}\right) = \gamma_{n-1} - \frac{t_n}{t_{n+1}}\tilde{\gamma}_{n+1}. \tag{2.12}$$

Since $\gamma_0 - \frac{t_1}{t_2}\tilde{\gamma}_2 = 0$, we deduce $\gamma_n = \frac{t_{n+1}}{t_{n+2}}\tilde{\gamma}_{n+2}$, for $n \geq 1$, and therefore (2.4) holds. Finally, we point out that from (2.4) and (2.10) we can derive (2.3).

To conclude the proof it remains to deduce the relation between the functionals u and v in terms of the constants a and b.

If we expand the linear functional u in the dual basis $\{\frac{Q_j v}{\langle v, Q_j^2\rangle}\}_{j\geq 0}$ of the polynomials $\{Q_j\}_{j\geq 0}$ (see [11]) and taking into account (2.2), then

$$u = \sum_{j=0}^{2}\frac{\langle u, Q_j\rangle}{\langle v, Q_j^2\rangle}Q_j v = \left(\frac{t_2}{\tilde{\gamma}_1\tilde{\gamma}_2}Q_2 + \frac{s_1}{\tilde{\gamma}_1}Q_1 + 1\right)v.$$

Introducing the explicit expression of the polynomials Q_2 and Q_1 given by the three-term recurrence relation we obtain

$$u = \frac{t_2}{\tilde{\gamma}_1\tilde{\gamma}_2}\left[x^2 + \left(\frac{s_1\tilde{\gamma}_2}{t_2} - \tilde{\beta}_0 - \tilde{\beta}_1\right)x + \left(\frac{\tilde{\gamma}_1\tilde{\gamma}_2}{t_2} - \tilde{\beta}_0\frac{s_1\tilde{\gamma}_2}{t_2} - \tilde{\gamma}_1 + \tilde{\beta}_0\tilde{\beta}_1\right)\right]v. \quad \square$$

Remark. The proof of the theorem has been written in terms of $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ not only for simplicity. If we work in a similar way as above but trying to look for that all the expressions appearing therein depend only on β_n and γ_n, then we can obtain similar formulas for a and b but a problem appears when we want to come back. Indeed, instead of (2.12) we obtain

$$\frac{\gamma_{n-1}}{t_n}\left[\tilde{\gamma}_n - t_n\frac{\tilde{\gamma}_{n-2}}{t_{n-1}}\right] = \tilde{\gamma}_{n+1} + t_{n+1}\frac{\gamma_{n-1}}{t_n}, \quad n \geq 3,$$

and so we can not achieve (2.4). The main reason is that the trivial extension of (2.4) to $n = 2$ ($t_1 = \gamma_0 = 0$) allows us to obtain the value of γ_0 in terms of $\tilde{\gamma}_2$ as $\gamma_0 = t_1\frac{\tilde{\gamma}_2}{t_2}$ but not conversely, i.e. $\tilde{\gamma}_2$ in terms of γ_0.

A matrix interpretation.

In the sequel, we present a matrix interpretation of these results in terms of the monic Jacobi matrices associated with the SMOP's $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, respectively.

Let $\mathcal{P} = (P_0, P_1, \ldots)^T$ and $\mathcal{Q} = (Q_0, Q_1, \ldots)^T$ be the column vectors associated with these orthogonal families, and $J_{\mathcal{P}}$ and $J_{\mathcal{Q}}$ the corresponding monic Jacobi matrices. Then, the three-term recurrence relations for such SMOP's read $x\mathcal{P} = J_{\mathcal{P}}\mathcal{P}$ and $x\mathcal{Q} = J_{\mathcal{Q}}\mathcal{Q}$.

Next, we will describe a method to find the matrix $J_{\mathcal{Q}}$ using the matrix $J_{\mathcal{P}}$ and the polynomial $h(x) = x^2 + ax + b$.

Taking into account (2.2) we may write $\mathcal{Q} = \mathcal{M}\mathcal{P}$ where $\mathcal{M} = (m_{k,j})$ is a banded lower triangular matrix such that $m_{k,k} = 1$ and $m_{k,j} = 0$ for $k - j > 2$, then $x\mathcal{M}\mathcal{P} = J_{\mathcal{Q}}\mathcal{M}\mathcal{P}$ and, as a consequence, $J_{\mathcal{P}}\mathcal{P} = \mathcal{M}^{-1}J_{\mathcal{Q}}\mathcal{M}\mathcal{P}$. Thus, we get

$$\mathcal{M}J_{\mathcal{P}} = J_{\mathcal{Q}}\mathcal{M}.$$

On the other hand, from the classical Christoffel formula (see [5]) we can express $h(x)\mathcal{P}$ using the matrix representation

$$h(x)\mathcal{P} = \mathcal{N}\mathcal{Q}$$

where $\mathcal{N}$ is a banded upper triangular matrix such that $n_{k,k+2} = 1$ and $n_{k,j} = 0$ for $j - k > 2$. Thus $h(x)\mathcal{P} = \mathcal{N}\mathcal{M}\mathcal{P}$, and then

$$J_{\mathcal{P}}^2 + aJ_{\mathcal{P}} + bI = \mathcal{N}\mathcal{M}. \tag{2.13}$$

But, from $h(x)\mathcal{Q} = \mathcal{M}\mathcal{N}\mathcal{Q}$, we get

$$J_{\mathcal{Q}}^2 + aJ_{\mathcal{Q}} + bI = \mathcal{M}\mathcal{N}. \tag{2.14}$$

As a conclusion, we can summarize our process as follows.

Step 1. Given $J_{\mathcal{P}}$, we find the polynomial matrix $J_{\mathcal{P}}^2 + aJ_{\mathcal{P}} + bI$.

Step 2. From $\mathcal{M}$ and (2.13) we find $\mathcal{N}$.

Step 3. From (2.14) we obtain the polynomial matrix $J_{\mathcal{Q}}^2 + aJ_{\mathcal{Q}} + bI$.

Step 4. Taking into account that $J_{\mathcal{Q}}$ is a tridiagonal matrix, from step 3 we can deduce $J_{\mathcal{Q}}$, since $(J_{\mathcal{Q}} + \frac{a}{2} I)^2 = \mathcal{M}\mathcal{N} - (b - \frac{a^2}{4}) I$.

A similar analysis for the truncated matrices yields the principal submatrices of $J_{\mathcal{Q}}$ can be deduced from a rank-one perturbation of the corresponding leading principal submatrices of $J_{\mathcal{P}}$.

Let us denote $(\mathcal{P})_n = (P_0, P_1, \ldots, P_n)^T$ and $(\mathcal{Q})_n = (Q_0, Q_1, \ldots, Q_n)^T$. Then the corresponding three-term recurrence relations read

$$x(\mathcal{P})_n = (J_{\mathcal{P}})_{n+1}(\mathcal{P})_n + P_{n+1}e_{n+1}, \tag{2.15}$$

$$x(\mathcal{Q})_n = (J_{\mathcal{Q}})_{n+1}(\mathcal{Q})_n + Q_{n+1}e_{n+1}, \tag{2.16}$$

where the symbol $(A)_n$ stands for the truncation of any infinite matrix A at level n, that is, $(J_{\mathcal{P}})_{n+1}$ denotes the leading principal submatrrix of $J_{\mathcal{P}}$ of size $(n + 1) \times (n + 1)$, and $e_{n+1} = (0, \ldots, 0, 1)^T \in \mathbb{R}^{n+1}$.

Taking into account

$$(\mathcal{Q})_n = (\mathcal{M})_{n+1}(\mathcal{P})_n, \tag{2.17}$$

and replacing (2.17) in (2.16) we get

$$\begin{aligned} x(\mathcal{M})_{n+1}(\mathcal{P})_n &= (J_{\mathcal{Q}})_{n+1}(\mathcal{M})_{n+1}(\mathcal{P})_n + (P_{n+1} + s_{n+1}P_n + t_{n+1}P_{n-1})e_{n+1} \\ &= [(J_{\mathcal{Q}})_{n+1}(\mathcal{M})_{n+1} + e_{n+1}(s_{n+1}e_{n+1}^T + t_{n+1}d_{n+1}^T)](\mathcal{P})_n + P_{n+1}e_{n+1}, \end{aligned}$$

where $d_{n+1} = (0, \ldots, 0, 1, 0)^T \in \mathbb{R}^{n+1}$.

Thus, from (2.15)

$$(\mathcal{M})_{n+1}(J_{\mathcal{P}})_{n+1} = (J_{\mathcal{Q}})_{n+1}(\mathcal{M})_{n+1} + e_{n+1}(s_{n+1}e_{n+1}^T + t_{n+1}d_{n+1}^T).$$

So, in order to obtain $(J_{\mathcal{Q}})_{n+1}$ we proceed in three steps.

Step 1. Obtain $(\mathcal{M})_{n+1}(J_{\mathcal{P}})_{n+1}$.

Step 2. Subtract from the previous one the rank-one matrix $e_{n+1}(s_{n+1}e_{n+1}^T + t_{n+1}d_{n+1}^T)$. This leads to a matrix G_{n+1}.

Step 3. $(J_{\mathcal{Q}})_{n+1} = G_{n+1}[(\mathcal{M})_{n+1}]^{-1}$.

In the case of quasi-orthogonal polynomials associated with positive Borel measures, a similar matrix approach has been considered in [3, Theorem 3.1] and [4, Theorem 3.2] for real symmetric tridiagonal (Jacobi) matrices. Indeed the *n*-th quasi-orthogonal polynomial can be represented as the characteristic polynomial of a symmetric tridiagonal matrix whose entries are explicitly given in terms of the entries of the initial Jacobi matrix associated with the positive Borel measure. Nevertheless, not every quasi-orthogonal polynomial can be represented as the characteristic polynomial of a symmetric tridiagonal matrix. Indeed, in [4, Theorem 3.3] a sufficient condition for such a representation is given assuming the quasi-orthogonal polynomial of degree *n* is represented as a linear combination of three consecutive monic orthogonal polynomials.

3. Reducible cases

Let $\{P_n\}_{n\geq 0}$ be an SMOP with respect to a quasi-definite linear functional u, and $\{Q_n\}_{n\geq 0}$ a sequence of monic polynomials strictly quasi-orthogonal of order two, satisfying the relation (2.2).

As a consequence of the Proposition 2.1 we have that $\{Q_n\}_{n\geq 0}$ is an SMOP if and only if the parameters s_n and t_n satisfy, for $n \geq 3$,

$$t_{n+1} = t_n \left(1 - \frac{\gamma_{n-2}}{t_{n-1}}\right) + \gamma_n + s_n(\beta_{n-1} - \beta_n + s_{n+1} - s_n),$$

$$s_{n+1} = -\frac{s_n}{t_n}\gamma_{n-1} + \beta_n - \beta_{n-1} + s_3 + \frac{s_2}{t_2}\gamma_1 - \beta_2 - \beta_1,$$

with some initial conditions.

In this section we will study two particular situations where the computations of these parameters can be simplified.

3.1. Iterative case

Notice that if $\{Q_n\}_{n\geq0}$ satisfies (2.2), then the polynomials Q_n cannot be represented as a linear combination of the two consecutive polynomials P_n and P_{n-1}. A natural question arises: Can the SMOP $\{Q_n\}_{n\geq0}$ be generated from $\{P_n\}_{n\geq0}$ in two steps with the help of an intermediate SMOP? In other words: Do two sequences of nonzero complex numbers $\{\lambda_n\}_{n\geq1}$ and $\{\mu_n\}_{n\geq1}$ exist such that $\{R_n\}_{n\geq0}$ defined by $R_n = P_n + \mu_n P_{n-1}$ is an SMOP and $Q_n = R_n + \lambda_n R_{n-1}$? The interest of this question is to simplify the computation of the parameters s_n and t_n, in this case via the parameters λ_n and μ_n.

If the answer is positive, then we will say that the representation (2.2) is iterative. For these cases we have $s_n = \lambda_n + \mu_n$ and $t_n = \lambda_n \mu_{n-1}$.

Now, we give a characterization of the iterative representations when both $\{P_n\}_{n\geq0}$ and $\{Q_n\}_{n\geq0}$ are SMOP's.

Proposition 3.1. *Let $\{P_n\}_{n\geq0}$ and $\{Q_n\}_{n\geq0}$ be two SMOP's with respect to the quasi-definite linear functionals u and v, respectively, which are related by (2.2). In this case, the linear functionals satisfy $(x^2 + ax + b)v = (x - x_1)(x - x_2)v = ku$. We denote by $\{\beta_n\}_{n\geq0}, \{\gamma_n\}_{n\geq1}$ and $\{\tilde{\beta}_n\}_{n\geq0}, \{\tilde{\gamma}_n\}_{n\geq1}$ the coefficients of the three-term recurrence relations of $\{P_n\}_{n\geq0}$ and $\{Q_n\}_{n\geq0}$, respectively.*

Then, the representation (2.2) is iterative if and only if there exist a complex number α and a sequence of complex numbers $\{\mu_n\}_{n\geq1}$, with $\mu_1 \neq s_1$ and $\mu_n \neq 0, n \geq 1$, such that

$$C_n := \beta_n - \mu_{n+1} - \frac{\gamma_n}{\mu_n} = \alpha, \quad n \geq 1, \tag{3.1}$$

and

$$\mu_{n+1} = s_{n+1} - \frac{t_{n+1}}{\mu_n}, \quad n \geq 1. \tag{3.2}$$

Under such conditions, α is one of the zeros x_1 or x_2 of $h(x) := x^2 + ax + b$.

Proof. If the representation is iterative then there exists a sequence of complex numbers $\{\mu_n\}_{n\geq1}$, with $\mu_n \neq 0$, such that $R_n = P_n + \mu_n P_{n-1}$ is an SMOP. Then, from Theorem 2 in [26] (and its proof), we have

$$(x - \alpha)w = -\lambda u, \qquad C_n := \beta_n - \mu_{n+1} - \frac{\gamma_n}{\mu_n} = \alpha, \quad n \geq 1,$$

where α and λ are certain complex numbers, $\lambda \neq 0$, and w is the quasi-definite functional with respect to which $\{R_n\}_{n\geq0}$ is orthogonal. Substituting R_n in (2.2), we get

$$Q_1 = R_1 + s_1 - \mu_1, \qquad Q_n = R_n + (s_n - \mu_n)R_{n-1} + [t_n - \mu_{n-1}(s_n - \mu_n)]P_{n-2}, \quad n \geq 2,$$

and since (2.2) is iterative, then we also have $Q_n = R_n + \lambda_n R_{n-1}$, with $\lambda_n \neq 0$ for all $n \geq 1$. Therefore $\lambda_n = s_n - \mu_n$ for all $n \geq 1$ and $t_n - \mu_{n-1}(s_n - \mu_n) = 0$ for all $n \geq 2$. Hence (3.2) follows and, furthermore $\mu_1 \neq s_1$ holds. Moreover, since $\{R_n\}_{n\geq0}$ is an SMOP with respect to w and $Q_n = R_n + \lambda_n R_{n-1}$, being $\{Q_n\}_{n\geq0}$ an SMOP with respect to v, then using Theorem 2 again in [26] we find $(x - \beta)v = -\mu w$, where β and μ are complex numbers, $\mu \neq 0$. Thus, since by hypothesis we also have $(x - x_1)(x - x_2)v = ku$, we find the relation

$$(x - x_1)(x - x_2)\,v = \frac{k}{\mu\lambda}(x - \alpha)(x - \beta)v.$$

Since v is quasi-definite, this gives $k = \mu\lambda$ and $\{x_1, x_2\} = \{\alpha, \beta\}$, hence α is indeed equal to x_1 or to x_2.

Conversely, given $\{\mu_n\}_{n\geq1}$ in the above conditions, from [26, Theorem 2], (3.1) implies that the sequence $\{R_n\}_{n\geq0}$ defined by $R_n = P_n + \mu_n P_{n-1}, n \geq 1$, is an SMOP with respect to a quasi-definite linear functional w such that $(x - \alpha)w = k_1 u$ where $k_1 \in \mathbb{C} \setminus \{0\}$.

Taking $\lambda_n = s_n - \mu_n, n \geq 1$, we have $\lambda_n \neq 0, n \geq 1$, and $t_n = \lambda_n \mu_{n-1}, n \geq 2$. So, we can write

$$Q_n = P_n + (\lambda_n + \mu_n)P_{n-1} + \lambda_n \mu_{n-1} P_{n-2} = R_n + \lambda_n R_{n-1}, \quad n \geq 1.$$

This means that (2.2) is iterative. $\quad\square$

Remark. There is another constant sequence involved in this topic. Indeed, as a consequence of the above proof and according to [26, Theorem 1], we get

$$D_n = \tilde{\beta}_n - \lambda_n - \frac{\tilde{\gamma}_{n+1}}{\lambda_{n+1}} = x_2, \text{ (or } x_1), \quad n \geq 0,$$

with $\lambda_0 = 0$. Moreover, we point out that, in general, the characterization of the orthogonality of a family of polynomials $\{Q_n\}_{n\geq0}$ defined by (2.2) can be given (see Theorem 2.2) in terms of two sequences $\{A_n\}$ and $\{B_n\}$ which remain constant and these constant values are precisely the coefficients of the quadratic polynomial $h(x)$ which appears in the relation $h(x)v = ku$. But if we have an iterative representation those sequences $\{A_n\}$ and $\{B_n\}$ (which in fact are very complicated) can be decomposed in two other simple sequences $\{C_n\}$ and $\{D_n\}$ associated with the zeros of the polynomial $h(x)$. More precisely, when formula (2.2) is iterative then $C_n + D_n = -A_n$ and $C_n D_n = B_n$.

Observe that, from (3.2), μ_n can be expressed as a finite continued fraction

$$\mu_{n+1} = s_{n+1} - \cfrac{t_{n+1}}{s_n - \cfrac{t_n}{\ddots \atop s_2 - \frac{t_2}{\mu_1}}}.$$

Next, for the sake of an example we are going to construct a family of monic polynomials orthogonal with respect to a quasi-definite linear functional v satisfying $x^2 v = u$, where u is the Laguerre functional, represented (in the sense of distribution theory) by the weight function $x^\alpha e^{-x} \chi_{(0,+\infty)}(x)$, being $\alpha > 0$.

Example. Laguerre polynomials.

Let $\{L_n^\alpha\}_{n\geq 0}$ be the sequence of monic Laguerre polynomials orthogonal with respect to the positive definite linear functional u defined by the weight function $x^\alpha e^{-x} \chi_{(0,+\infty)}(x)$ with $\alpha > 0$. We can take the auxiliary polynomials $R_n = L_n^{\alpha-1}$ satisfying $R_n(x) = L_n^\alpha(x) + n L_{n-1}^\alpha(x)$ (see [2]) and the new polynomials Q_n such that $Q_n(x) = R_n(x) + \lambda_n R_{n-1}(x)$. Then the polynomials Q_n satisfy the relation (2.2) with $s_n = n + \lambda_n$ and $t_n = (n-1)\lambda_n$, $n \geq 1$. It is well known that the recurrence coefficients of $L_n^{\alpha-1}$, $\alpha > 0$, are $\beta_n = 2n + \alpha$, $n \geq 0$ and $\gamma_n = n(n + \alpha - 1)$, $n \geq 1$.

Using formula (1.1) for this case, having $x_1 = 0$ and $\lambda_2 = 2 + \alpha - \alpha/\lambda_1$, by induction we can derive that, for $\alpha > 0$ and $\alpha \neq 1$, the values of the parameters λ_n in terms of λ_1 are

$$\lambda_n = n \frac{\Gamma(\alpha)(\alpha - \lambda_1) + (\lambda_1 - 1)\frac{\Gamma(n+\alpha)}{\Gamma(n+1)}}{\Gamma(\alpha)(\alpha - \lambda_1) + (\lambda_1 - 1)\frac{\Gamma(n-1+\alpha)}{\Gamma(n)}}, \tag{3.3}$$

and then v is quasi-definite if and only if

$$\Gamma(n)\Gamma(\alpha)(\alpha - \lambda_1) + (\lambda_1 - 1)\Gamma(n-1+\alpha) \neq 0, \quad n \geq 1.$$

Notice that if $\alpha \in \mathbb{N} \setminus \{1\}$ then λ_n is a rational function of n, namely,

$$\lambda_n = n \frac{\Gamma(\alpha)(\alpha - \lambda_1) + (\lambda_1 - 1)(\alpha + n - 1)\cdots(n+1)}{\Gamma(\alpha)(\alpha - \lambda_1) + (\lambda_1 - 1)(\alpha + n - 2)\cdots n}.$$

If $\alpha = 1$, then, by induction, we can also obtain, for $n \geq 1$

$$\lambda_n = n \frac{(\lambda_1 - 1)(1 + 1/2 + \cdots + 1/n) + 1}{(\lambda_1 - 1)(1 + 1/2 + \cdots + 1/(n-1)) + 1}, \tag{3.4}$$

and v is quasi-definite if and only if

$$(\lambda_1 - 1)(1 + 1/2 + \cdots + 1/n) + 1 \neq 0, \quad n \geq 1.$$

The quasi-definite linear functional v is given by

$$v = (\alpha - \lambda_1)x^{-1}w + v_0\delta_0,$$

where w is the positive-definite linear functional for monic Laguerre polynomials $\{L_n^{\alpha-1}\}_{n\geq 0}$. From $xv = (\alpha - \lambda_1)w$ it follows $-Q_1(0)v_0 = (\alpha - \lambda_1)\Gamma(\alpha)$. Since $Q_1(0) = R_1(0) + \lambda_1 = L_1^\alpha(0) + \lambda_1 + 1 = \lambda_1 - \alpha$ we achieve $v_0 = \Gamma(\alpha)$.

In particular for $\alpha > 1$, we can write

$$v = (\alpha - \lambda_1)\ell + (\lambda_1 - 1)\Gamma(\alpha - 1)\delta_0,$$

where ℓ is the positive-definite linear functional for monic Laguerre polynomials $\{L_n^{\alpha-2}\}_{n\geq 0}$.

A matrix interpretation.

Next, as in the general case, we also present a matrix interpretation for these iterative cases.

If w denotes the corresponding linear functional for $\{R_n\}_{n\geq 0}$, then $(x - x_1)w = k_1 u$ where x_1 is a zero of $h(x)$ as well as $(x - x_2)v = k_2 w$ where x_2 is the other zero of $h(x)$.

It is well known (see [5]) that

$$(x - x_1)P_n = R_{n+1} + r_n R_n, \quad n \geq 0, \tag{3.5}$$

as well as

$$(x - x_2)R_n = Q_{n+1} + r_n' Q_n, \quad n \geq 0,$$

with $r_n, r_n' \neq 0$.

As before, $\mathcal{P} = (P_0, P_1, \ldots)^T$, $\mathcal{R} = (R_0, R_1, \ldots)^T$, and $\mathcal{Q} = (Q_0, Q_1, \ldots)^T$, and then the corresponding three-term recurrence relations read as

$$x\mathcal{P} = J_\mathcal{P}\mathcal{P}, \qquad x\mathcal{R} = J_\mathcal{R}\mathcal{R}, \qquad x\mathcal{Q} = J_\mathcal{Q}\mathcal{Q}. \tag{3.6}$$

On the other hand, from $R_n = P_n + \mu_n P_{n-1}$ and (3.5) we have the matrix representations

$$\mathcal{R} = L_1 \mathcal{P}, \qquad (x - x_1)\mathcal{P} = U_1 \mathcal{R} \tag{3.7}$$

where L_1 is a lower bidiagonal matrix with 1 as diagonal entries and U_1 is an upper bidiagonal matrix with 1 as entries in the upper diagonal given explicitly by

$$L_1 = \begin{pmatrix} 1 & & & & \\ \mu_1 & 1 & & & \\ & \mu_2 & 1 & & \\ & & \mu_3 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad U_1 = \begin{pmatrix} r_0 & 1 & & & \\ & r_1 & 1 & & \\ & & r_2 & 1 & \\ & & & r_3 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

Notice that from (3.6) and (3.7) we get

$$J_{\mathcal{P}} - x_1 I = U_1 L_1 \tag{3.8}$$

as well as $L_1^{-1}(J_{\mathcal{R}} - x_1 I)\mathcal{R} = U_1 \mathcal{R}$. In other words

$$J_{\mathcal{R}} - x_1 I = L_1 U_1. \tag{3.9}$$

Proceeding in a similar way

$$\mathcal{Q} = L_2 \mathcal{R}, \qquad (x - x_2)\mathcal{R} = U_2 \mathcal{Q}$$

where L_2 is a lower bidiagonal matrix with 1 as entries in the diagonal and U_2 is an upper bidiagonal matrix with 1 as entries in the upper diagonal given explicitly by

$$L_2 = \begin{pmatrix} 1 & & & & \\ \lambda_1 & 1 & & & \\ & \lambda_2 & 1 & & \\ & & \lambda_3 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} r'_0 & 1 & & & \\ & r'_1 & 1 & & \\ & & r'_2 & 1 & \\ & & & r'_3 & \ddots \\ & & & & \ddots \end{pmatrix}.$$

Thus, we get

$$J_{\mathcal{R}} - x_2 I = U_2 L_2 \tag{3.10}$$

and

$$J_{\mathcal{Q}} - x_2 I = L_2 U_2. \tag{3.11}$$

As a consequence we can summarize the process as follows.

Step 1. Given $J_{\mathcal{P}}$, from L_1 and (3.8) we get U_1.

Step 2. From (3.9) we get $J_{\mathcal{R}}$.

Step 3. Given $J_{\mathcal{R}}$, from L_2 and (3.10) we get U_2.

Step 4. From (3.11) we get $J_{\mathcal{Q}}$.

Notice that this is essentially the iteration of the so-called Geronimus transformation in the framework of linear spectral transformations of Stieltjes functions associated with quasi-definite linear functionals (see [30]).

Now, as in the general case, we analyze the case of the truncated matrices.

The three-term recurrence relation for the sequence $\{R_n\}_{n \geq 0}$ reads

$$x(\mathcal{R})_n = (J_{\mathcal{R}})_{n+1}(\mathcal{R})_n + R_{n+1} e_{n+1}. \tag{3.12}$$

Since, by (3.7),

$$(\mathcal{R})_n = (L_1)_{n+1}(\mathcal{P})_n, \tag{3.13}$$

where $(L_1)_{n+1}$ is a lower bidiagonal matrix, namely, the $(n+1) \times (n+1)$ leading principal submatrix of L_1, replacing (3.13) in (3.12)

$$x(\mathcal{P})_n = [(L_1)_{n+1}]^{-1}[(J_{\mathcal{R}})_{n+1}(L_1)_{n+1} + \mu_{n+1} e_{n+1} e_{n+1}^T](\mathcal{P})_n + P_{n+1} e_{n+1}.$$

Thus, from (2.15) we get $(L_1)_{n+1}(J_{\mathcal{P}})_{n+1} = (J_{\mathcal{R}})_{n+1}(L_1)_{n+1} + \mu_{n+1} e_{n+1} e_{n+1}^T$,

i.e.

$$(J_{\mathcal{R}})_{n+1} = (L_1)_{n+1}[(J_{\mathcal{P}})_{n+1} - \mu_{n+1} e_{n+1} e_{n+1}^T][(L_1)_{n+1}]^{-1}.$$

As a consequence, $(J_\mathscr{R})_{n+1}$ is similar to the matrix

$$(J_\mathscr{P})_{n+1} - \mu_{n+1}e_{n+1}e_{n+1}^T$$

that is, to the matrix obtained when we subtract μ_{n+1} in the entry $(n+1) \times (n+1)$ of the Jacobi matrix $(J_\mathscr{P})_{n+1}$ and the other entries remain invariant.

On the other hand, we have, by (3.5) and (3.7),

$$(x - x_1)(\mathscr{P})_n = (U_1)_{n+1}(\mathscr{R})_n + R_{n+1}e_{n+1}, \tag{3.14}$$

where $(U_1)_{n+1}$ is an upper bidiagonal matrix.

According to (2.15) and (3.13), (3.14) reads

$$((J_\mathscr{P})_{n+1} - x_1 I_{n+1})(\mathscr{P})_n + P_{n+1}e_{n+1} = (U_1)_{n+1}(L_1)_{n+1}(\mathscr{P})_n + (P_{n+1} + \mu_{n+1}P_n)e_{n+1}.$$

Equivalently,

$$(J_\mathscr{P})_{n+1} - \mu_{n+1}e_{n+1}e_{n+1}^T - x_1 I_{n+1} = (U_1)_{n+1}(L_1)_{n+1}.$$

But from (3.14)

$$[(L_1)_{n+1}]^{-1}((J_\mathscr{R})_{n+1} - x_1 I_{n+1})(L_1)_{n+1} = (U_1)_{n+1}(L_1)_{n+1}.$$

Thus,

$$(J_\mathscr{R})_{n+1} - x_1 I_{n+1} = (L_1)_{n+1}(U_1)_{n+1}.$$

In a similar way,

$$(J_\mathscr{R})_{n+1} - \lambda_{n+1}e_{n+1}e_{n+1}^T - x_2 I_{n+1} = (U_2)_{n+1}(L_2)_{n+1},$$

and then

$$(J_\mathscr{Q})_{n+1} - x_2 I_{n+1} = (L_2)_{n+1}(U_2)_{n+1}.$$

3.2. The symmetric case

Assume that the sequence $\{P_n\}_{n\geq 0}$ is orthogonal with respect to a symmetric linear functional u. Then $\beta_n = 0$, $n \geq 0$, and there exist two polynomial sequences $\{V_n\}_{n\geq 0}$ and $\{V_n^*\}_{n\geq 0}$ such that for all n, $P_{2n}(x) = V_n(x^2)$ and $P_{2n+1}(x) = xV_n^*(x^2)$.

It is known (see [2]) that $\{V_n\}_{n\geq 0}$ and $\{V_n^*\}_{n\geq 0}$ are SMOP's with respect to the linear functionals σ_u and $x\sigma_u$ where $(\sigma_u, x^n) = (u, x^{2n})$, $n \geq 0$.

If the sequence $\{Q_n\}_{n\geq 0}$ is defined by (2.2), in general, the polynomials Q_n are not symmetric. More precisely, Q_n are symmetric polynomials if and only if $s_n = 0$ for all $n \geq 1$.

Again, if $\{Q_n\}_{n\geq 0}$ is an SMOP with respect to a symmetric linear functional v, then by a symmetrization process there exist $\{R_n\}_{n\geq 0}$ and $\{R_n^*\}_{n\geq 0}$ SMOP's with respect to σ_v and $x\sigma_v$, respectively, satisfying $Q_{2n}(x) = R_n(x^2)$ and $Q_{2n+1}(x) = xR_n^*(x^2)$. In this case, from (2.2), we have for ≥ 1

$$R_n(x) = V_n(x) + t_{2n}V_{n-1}(x) \tag{3.15}$$

$$R_n^*(x) = V_n^*(x) + t_{2n+1}V_{n-1}^*(x) \tag{3.16}$$

and the coefficients t_{2n} and t_{2n+1} can be computed using the Theorem 2 in [26]. Besides, as a consequence of the Theorem 2.2, we get $a = 0$ and the relation between the linear functionals u and v is: $(x^2 + b)v = ku$.

Now, we construct some families of monic symmetric orthogonal polynomials which have reducible representations but not iterative representations.

Examples.

Hermite generalized polynomials.

Let $\{H_n^\mu\}_{n\geq 0}$ be the sequence of monic Hermite generalized polynomials orthogonal with respect to the positive definite linear functional u defined by the weight function $|x|^{2\mu}e^{-x^2}$ with $\mu > -1/2$ (see [2, p. 157]). Consider the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ orthogonal with respect to a symmetric quasi-definite functional v satisfying $x^2v = u$. Notice that, now the representation (2.2) is clearly not iterative because xv is not a quasi-definite linear functional.

By a symmetrization process, σ_u is the functional defined by the weight function $x^{\mu-1/2}e^{-x}\chi_{(0,+\infty)}(x)$ and we can write $H_{2n}^\mu(x) = L_n^{\mu-1/2}(x^2)$, $H_{2n+1}^\mu(x) = xL_n^{\mu+1/2}(x^2)$.

From (3.3) and (3.4) we get the explicit expressions of the parameters t_{2n} in terms of t_2, for $\mu > -1/2$, $\mu \neq 1/2$, and $\mu = 1/2$, respectively.

Otherwise, the relation between the two linear functionals u and v yields $x\sigma_v = \sigma_u$. Then the polynomials R_n^* are orthogonal with respect to $x^{\mu-1/2}e^{-x}$ and thus $t_{2n+1} = n$ for all $n \geq 1$. Moreover the quasi-definite linear functional v is given in terms of the generalized Hermite linear functional u by

$$v = x^{-2}u + v_0\delta_0$$

for $\mu > -1/2$. From $x^2v = u$, it follows $-Q_2(0)v_0 = \Gamma(\mu + 1/2)$ and since $Q_2(0) = R_1(0) = -\mu - 1/2 + t_2$ we have $v_0 = \frac{\Gamma(\mu+1/2)}{\mu+1/2-t_2}$.

In particular, for $\mu > 1/2$

$$v = \ell + \frac{t_2 - 1}{\mu + 1/2 - t_2} \Gamma(\mu - 1/2)\delta_0,$$

where ℓ is the positive-definite linear functional corresponding to the generalized Hermite SMOP $\{H_n^{\mu-1}\}_{n \geq 0}$.

Chebyshev polynomials of the second kind.

Let $\{U_n\}_{n \geq 0}$ be the sequence of monic Chebyshev polynomials of the second kind, orthogonal with respect to the positive definite linear functional u defined by the weight function $(1 - x^2)^{1/2}\chi_{(-1,1)}(x)$, with the recurrence coefficients $\beta_n = 0$, $n \geq 0$ and $\gamma_n = \frac{1}{4}$, $n \geq 1$. Consider the SMOP $\{Q_n\}_{n \geq 0}$ orthogonal with respect to a symmetric quasi-definite linear functional v satisfying $(x^2 - 1)v = u$.

By the symmetrization process described above, we have $\{V_n\}_{n \geq 0}$, $\{R_n\}_{n \geq 0}$ are SMOP's orthogonal with respect to the linear functionals σ_u and σ_v, respectively, where σ_u is defined by the weight function $(1 - x)^{1/2}x^{-1/2}\chi_{(0,1)}(x)$. Besides, using (9.9) in [2, p. 46] the recurrence coefficients for the polynomials V_n are, for $n \geq 1$, $\beta_n^{\sigma_u} = \gamma_{2n} + \gamma_{2n+1} = 1/2$, $\gamma_n^{\sigma_u} = \gamma_{2n}\gamma_{2n-1} = 1/16$, and $\beta_0^{\sigma_u} = \gamma_1 = 1/4$.

Moreover, the relation between the two linear functionals $(x^2 - 1)v = u$, yields $(x - 1)\sigma_v = \sigma_u$. Now, applying in (3.15) Theorem 2 in [26], the parameters t_{2n} satisfy $\beta_n^{\sigma_u} - t_{2n+2} - \frac{\gamma_n^{\sigma_u}}{t_{2n}} = 1$ for all $n \geq 1$, i.e. $t_{2n+2} = -(\frac{1}{2} + \frac{1}{16t_{2n}})$ and $k = -(\frac{3}{4} + t_2)\frac{v_0}{u_0}$. Thus, by induction, we can derive the values of parameters t_{2n} in terms of t_2. Indeed

$$t_{2n} = -\frac{1}{4}\frac{n(4t_2 + 1) - 1}{(n - 1)(4t_2 + 1) - 1}. \tag{3.17}$$

On the other hand, since $(x - 1)x\sigma_v = x\sigma_u$, we can apply in (3.16) Theorem 2 in [26]. Thus, the parameters t_{2n+1} satisfy $\beta_n^{x\sigma_u} - t_{2n+3} - \frac{\gamma_n^{x\sigma_u}}{t_{2n+1}} = 1$ for all $n \geq 1$, and since, using (9.10) in [2, p. 46], $\beta_n^{x\sigma_u} = \gamma_{2n+1} + \gamma_{2n+2} = 1/4$ and $\gamma_n^{x\sigma_u} = \gamma_{2n}\gamma_{2n+1} = 1/16$, we get $t_{2n+3} = -(\frac{1}{2} + \frac{1}{16t_{2n+1}})$. Again, by induction, we can derive the values of parameters t_{2n+1} in terms of t_3, and taking into account that t_2 and t_3 are related by $t_3 = \frac{12t_2+1}{4(1-4t_2)}$ we have

$$t_{2n+1} = -\frac{1}{4}\frac{2n(4t_2 + 1) + (4t_2 - 1)}{2(n - 1)(4t_2 + 1) + (4t_2 - 1)}. \tag{3.18}$$

Moreover, v is quasi-definite if and only if the following conditions hold:

$$t_2 \neq -3/4, \qquad n(4t_2 + 1) - 1 \neq 0, \quad n \geq 1, \quad \text{and} \quad 2n(4t_2 + 1) + (4t_2 - 1) \neq 0, \quad n \geq 0.$$

The expression of v in terms of u_τ (the Chebyshev functional of the first kind) is $v = -u_\tau + \frac{(v_0+\pi)}{2}(\delta_1 + \delta_{-1})$. From $(x^2 - 1)v = u$ it is not difficult to see that $-v_0(Q_2(0) + 1) = u_0 = \pi/2$ and since $Q_2(0) = R_1(0) = U_2(0) + t_2 = t_2 - 1/4$, we have $v_0 = \frac{-\pi/2}{t_2+3/4}$. Then

$$v = -u_\tau + \frac{\pi}{2}\frac{t_2 + 1/4}{t_2 + 3/4}(\delta_1 + \delta_{-1}).$$

See also [21].

Chebyshev polynomials of the first kind.

Let $\{T_n\}_{n \geq 0}$ be the sequence of monic Chebyshev polynomials of the first kind, orthogonal with respect to the positive definite linear functional u defined by the weight function $(1-x^2)^{-1/2}\chi_{(-1,1)}(x)$. The recurrence coefficients are $\beta_n = 0$, $n \geq 0$, $\gamma_n = \frac{1}{4}$, $n \geq 2$, and $\gamma_1 = 1/2$. Consider an SMOP $\{Q_n\}_{n \geq 0}$ orthogonal with respect to a symmetric quasi-definite linear functional v satisfying $(x^2 - 1)v = u$.

Thus, $\{V_n\}_{n \geq 0}$, $\{R_n\}_{n \geq 0}$ are SMOP's with respect to the linear functionals σ_u and σ_v, respectively, where σ_u is defined by the weight function $(1 - x)^{-1/2}x^{-1/2}\chi_{(0,1)}(x)$. Besides, the recurrence coefficients for the polynomials V_n are, $\beta_n^{\sigma_u} = 1/2$, $\gamma_n^{\sigma_u} = 1/16$, $n \geq 2$, and $\gamma_1^{\sigma_u} = 1/8$.

Since $(x - 1)\sigma_v = \sigma_u$, applying in (3.15) Theorem 2 in [26], the parameters t_{2n} are

$$t_{2n} = -\frac{1}{4}\frac{n(2t_2 + 1) - 1}{(n - 1)(2t_2 + 1) - 1}, \quad n \geq 2. \tag{3.19}$$

Using the same arguments, from (3.16) we obtain

$$t_{2n+1} = -\frac{1}{4}\frac{2n(2t_2 + 1) + (2t_2 - 1)}{2(n - 1)(2t_2 + 1) + (2t_2 - 1)}, \quad n \geq 1. \tag{3.20}$$

Furthermore, v is quasi-definite if and only if the following conditions hold:

$$t_2 \neq -1/2, \qquad n(2t_2 + 1) - 1 \neq 0, \quad n \geq 1, \quad \text{and} \quad 2n(2t_2 + 1) + (2t_2 - 1) \neq 0, \quad n \geq 0.$$

The expression of v in terms of u is

$$v = -(1-x^2)^{-1}u + \frac{v_0}{2}(\delta_1 + \delta_{-1}).$$

Again, as in the previous case, v_0 satisfies $-v_0(Q_2(0) + 1) = u_0 = \pi$ and since $Q_2(0) = R_1(0) = T_2(0) + t_2 = t_2 - 1/2$ we achieve $v_0 = \frac{-2\pi}{2t_2+1}$.

See also [20].

Acknowledgements

References

[1] P. Maroni, Prolégomènes à l'étude des polynômes orthogonaux semi-classiques, Ann. Math. Pura Appl. 4 (1987) 165–184.
[2] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[3] Y. Xu, A characterization of positive quadrature formulae, Math. Comp. 62 (1994) 703–718.
[4] Y. Xu, Quasi-orthogonal polynomials, quadrature, and interpolation, J. Math. Anal. Appl. 182 (1994) 779–799.
[5] W. Gautschi, An algorithmic implementation of the Generalized Christoffel Theorem, in: G. Hammerlin (Ed.), Numerical Integration, in: ISNM, vol. 57, Birkhauser Verlag, Basel, 1982, pp. 89–106.
[6] M.I. Bueno, F. Marcellán, Darboux transformations and perturbations of linear functionals, Linear Algebra Appl. 384 (2004) 215–242.
[7] P. Maroni, Sur la suite de polynômes orthogonaux associée à la forme $u = \delta_c + \lambda(x-c)^{-1}L$, Period. Math. Hung. 21 (1990) 223–248.
[8] V. Spiridonov, L. Vinet, A. Zhedanov, Spectral transformations, self-similar reductions and orthogonal polynomials, J. Phys. A: Math. Gen. 30 (1997) 7621–7637.
[9] P. Maroni, Tchebychev forms and their perturbed as second degree forms, Ann. Numer. Math. 2 (1995) 123–143.
[10] F.A. Grunbaum, L. Haine, E. Horozov, Some functions that generalize the Krall–Laguerre polynomials, J. Comput. Appl. Math. 106 (1998) 271–297.
[11] P. Maroni, Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques, in: C. Brezinski, et al. (Eds.), Orthogonal Polynomials and Their Applications, in: IMACS Ann. Comput. Appl. Math., vol. 9, 1991, pp. 95–130.
[12] F.A. Grunbaum, L. Haine, Orthogonal polynomials satisfying differential equations: the role of the Darboux transformation, in: D. Levi, et al. (Eds.), Symmetries and Integrability of Difference Equations, in: CRM Proceedings and Lecture Notes, vol. 9, Amer. Math. Soc., RI, 1996, pp. 143–154.
[13] S. Elhay, J. Kautsky, Jacobi matrices for measures modified by a rational factor, Numer. Algorithms 6 (1994) 205–227.
[14] M. Alfaro, F. Marcellán, A. Peña, M.L. Rezola, On rational transformations of linear functionals: direct problem, J. Math. Anal. Appl. 298 (2004) 171–183.
[15] D. Beghdadi, P. Maroni, On the inverse problem of the product of a semiclassical form by a polynomial, J. Comput. Appl. Math. 88 (1998) 377–399.
[16] P. Maroni, Semi-classical character and finite type relations between polynomial sequences, Appl. Numer. Math. 31 (1999) 295–330.
[17] Ya.L. Geronimus, On polynomials orthogonal with regard to a given sequence of numbers, Comm. Inst. Sci. Math. Mec. Univ. Kharkoff 17 (1940) 3–18.
[18] A. Branquinho, F. Marcellán, Generating new classes of orthogonal polynomials, Int. J. Math. Math. Sci. 19 (1996) 643–656.
[19] C. Hounga, M.N. Houkonnou, A. Ronveaux, New families of orthogonal polynomials, J. Comput. Appl. Math. 193 (2006) 474–483.
[20] J. Wimp, H. Kiesel, Some sequences arising in the creation of new orthogonal polynomials, in: Annie Cuyt (Ed.), Nonlinear Numerical Methods and Rational Approximation II, Kluwer Acad. Publ., Dordrecht, 1994, pp. 17–24.
[21] J. Wimp, H. Kiesel, Non-linear recurrence relations and some derived orthogonal polynomials, Ann. Numer. Math. 2 (1995) 169–180.
[22] M. Alfaro, F. Marcellán, A. Peña, M.L. Rezola, On linearly related orthogonal polynomials and their functionals, J. Math. Anal. Appl. 287 (2003) 307–319.
[23] M. Alfaro, F. Marcellán, A. Peña, M.L. Rezola, When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials? J. Comput. Appl. Math. 233 (2010) 1446–1452.
[24] Z. Grinshpun, Special linear combinations of orthogonal polynomials, J. Math. Anal. Appl. 299 (2004) 1–18.
[25] J. Petronilho, On the linear functionals associated to linearly related sequences of orthogonal polynomials, J. Math. Anal. Appl. 315 (2006) 379–393.
[26] F. Marcellán, J. Petronilho, Orthogonal polynomials and coherent pairs: the classical case, Indag. Math. (NS) 6 (1995) 287–307.
[27] M. de Jesus, J. Petronilho, On linearly related sequences of derivatives of orthogonal polynomials, J. Math. Anal. Appl. 347 (2008) 482–492.
[28] T.S. Chihara, On co-recursive orthogonal polynomials, Proc. Amer. Math. Soc. 8 (1957) 899–905.
[29] F. Marcellán, J.S. Dehesa, A. Ronveaux, On orthogonal polynomials with perturbed recurrence relations, J. Comput. Appl. Math. 30 (1990) 203–212.
[30] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. 85 (1997) 67–83.