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A Theorem of Density for Translation Invariant Subspaces of $L^p(G)$.

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Sunto. – *Si prova che un sottospazio di $L^p(G)$ invariante rispetto alle traslazioni, dove è $1 \leqslant p < \infty$ e G denota un gruppo abeliano localmente compatto, è denso se è invariante rispetto alla moltiplicazione per una funzione non periodica. La dimostrazione si consegue ottenendo previamente una caratterizzazione della più ampia famiglia di funzioni limitate tali che il sottospazio dato sia invariante rispetto alla moltiplicazione per esse.*

1. – Introduction and notation.

In this paper, a theorem of density is obtained for translation invariant subspaces of $L^p(G)$, if $1 \leqslant p < \infty$ and G is a locally compact abelian group. The conditions imposed on the subspaces are different from those which appear in the classical Wiener's theorem. Here, invariance under multiplication by some essentially bounded functions is supposed.

In that follows G is a locally compact abelian group equipped with a Haar measure m. For $1 \leqslant p \leqslant \infty$, we denote by $L^p(G)$ the classical Banach spaces associated with the pair (G, m), by $C_0(G)$ the space of continuous functions on G which vanish at infinity and by $M(G)$ the space of regular Borel measures on G.

Let S be a subspace of $L^p(G)$. If $H \subseteq L^\infty(G)$ and $\mathcal{M} \subseteq M(G)$, we say S is *H-invariant* or *$\mathcal{M}$-invariant* when $\varphi S \subseteq S$ for every $\varphi \in H$ or $\mu * S \subseteq S$ for every $\mu \in \mathcal{M}$. S is called *translation invariant*, if $f \in S$ implies $f_y \in S$ for every $y \in G$, with $f_y(x) = f(xy^{-1})$, $x \in G$. S is *self-adjoint* (or selfconjugate), if $\bar{f} \in S$ whenever $f \in S$.

If $\tilde{f}(x) = \overline{f(x^{-1})}$, we denote $\tilde{S} = \{\tilde{f}; f \in S\}$.

Let B be a Banach space, A a subset of B and S a subspace of B by $[A]$ we denote the span of A in B, i.e., the linear closure of A and by S^0 the *polar* of S, i.e., $S^0 = \{b' \in B'; \langle s, b' \rangle = 0$ for all $s \in S\}$; if $S \subset L^p(G)$, $1 \leqslant p < \infty$, then $S^0 \subset L^{p'}(G)$, $1/p + 1/p' = 1$ and if $S \subset C_0(G)$ then $S^0 \subset M(G)$.

If Γ is a closed subgroup of G, a function φ on G is *Γ-periodic* when $\varphi(xt) = \varphi(x)$ for every $t \in \Gamma$ and $x \in G$. $\Gamma^{\perp}$ is the *orthogonal* subgroup of Γ, i.e., $\Gamma^{\perp} = \{\alpha \in \hat{G};\ \langle x, \alpha \rangle = 1$ for all $x \in \Gamma\}$, where $\hat{G}$ is the dual group of G.

Finally, if T is a closed ideal in $L^1(G)$, its *zero set* is $Z(T) = \{\alpha \in \hat{G};\ \hat{f}(\alpha) = 0$ for all $f \in T\}$. If $\Sigma \subset \hat{G}$, then $K(\Sigma) =$ (*kernel* of Σ) $= \{f \in L^1(G);\ \hat{f}(\Sigma) = 0\}$.

2. – Basic lemmas.

We shall study the largest families $H \subseteq L^\infty(G)$ and $\mathcal{M} \subseteq M(G)$ such that the closure of a subspace S is H-invariant and $\mathcal{M}$-invariant.

LEMMA 1. – *If S is a closed selfadjoint subspace of $L^p(G)$, $1 < p < \infty$, then* $\{\mu \in M(G);\ \mu * S \subseteq S\} = [\tilde{S} * S^0]^0$. (Observe that $S^0 \subset L^{p'}(G)$ and $S * S^0 \subset C_0(G)$, so that the right side represents a closed subspace of $M(G)$).

PROOF. – We begin by the identity

$$\int_G (\tilde{\tilde{f}} * g)\, d\mu = \int_G g(\mu * f)\, dm \qquad (f \in L^p,\ g \in L^{p'}) .$$

If $\mu \in M(G)$ is such $\mu * S \subseteq S$, and $h \in \tilde{S} * S^0$, it follows that $\int_G h\, d\mu = 0$, and therefore $\mu \in [\tilde{S} * S^0]^0$.

The argument can be reversed to prove that $\mu \in [\tilde{S} * S^0]^0$ implies $\mu * S \subseteq S$, since ${}^0(S^0) = S$ by Hahn-Banach theorem.

COROLLARY 1. – *If S is as in lemma 1, the following are equivalent:*

a) *S is translation invariant (S is $\mathcal{M}$-invariant, with $\mathcal{M} = \{\delta_x;\ x \in G\}$);*

b) *S is $\mathcal{M}$-invariant for some $\sigma(M(G), C_0(G))$-dense family $\mathcal{M}$ in $M(G)$;*

c) *S is $M(G)$-invariant.*

LEMMA 2. – *If S is a closed, selfadjoint and translation invariant subspace of $L^p(G)$, $1 \leqslant p < \infty$, then,*

$$\{\varphi \in L^\infty(G);\ \varphi S \subseteq S\} = \{\varphi \in L^\infty(G);\ \varphi \text{ is } \Sigma^{\perp}\text{-periodic}\}$$

where Σ is the following subgroup of dual group:

$$\Sigma = \{\alpha \in \hat{G};\ \alpha S \subseteq S\} .$$

PROOF. – Let T be, the closed span of SS^0 in $L^1(G)$. We are going to prove:

a) $T^0 = \{\varphi \in L^\infty(G);\ \varphi S \subseteq S\}$

b) $T = K(\Sigma)$

c) $T^0 = \{\varphi \in L^\infty(G);\ \varphi$ is $\Sigma^\perp$-periodic$\}$.

a) Let φ be an element of $L^\infty(G)$ such that $\varphi S \subseteq S$. It follows that $\int_G \varphi f g\, dm = 0$, for all $f \in S$, $g \in S^0$ and therefore $\varphi \in T^0$. Conversely, if $\varphi \in T^0$, $\int_G \varphi f g\, dm = 0$ for all $f \in S$, $g \in S^0$, and since ${}^0(S^0) = S$ by Hahn-Banach theorem, $\varphi f \in S$, for all $f \in S$.

b) T is a closed ideal of $L^1(G)$ because of T is a closed translation invariant subspace of $L^1(G)$. Moreover $Z(T) = \Sigma$. Indeed, $\alpha \in Z(T)$ iff $\int_G f(x) g(x) \langle x, \bar{\alpha}\rangle\, dm(x) = 0$, for all $f \in S$, $g \in S^0$. As S is a closed selfadjoint subspace of $L^p(G)$, that is equivalent to $f\bar{\alpha} \in S$, for all $f \in S$, and thus $\alpha \in \Sigma$.

Since S is a closed subspace, Σ is a closed subgroup and, therefore, a spectral set (see [4]). Thus, $K(\Sigma) = T$.

c) Let φ be a function of $L^\infty(G)$ such that $\varphi(xt) = \varphi(x)$ for every $t \in \Sigma^\perp$ and $x \in G$. Then, Weil's formula (see [2]) implies

$$\int_G \varphi h\, dm = \int_{G/\Sigma^\perp} \widetilde{h\varphi}\, d\tilde{m} = \int_{G/\Sigma^\perp} \varphi \tilde{h}\, d\tilde{m}$$

where $\tilde{m}$ is a normalised Haar measure on $G/\Sigma^\perp$ and

$$\tilde{h}(\tilde{x}) = \int_{\Sigma^\perp} h(xt)\, dm_{\Sigma^\perp}(t)\ .$$

Now, we use the fact that $\hat{h}/\Sigma = (\tilde{h})^\wedge$ and we infer from $Z(T) = \Sigma$ that $\tilde{h} = 0$ $\tilde{m}$-a.e. and therefore

$$\int_G \varphi h\, dm = 0 \qquad \text{for every } h \in T\,.$$

Conversely, assume that $g \in T^0$. For every $f \in L^1(G)$, $(f - f_t)^\wedge(\alpha) = 0$ for every $\alpha \in \Sigma$, so that we have $f - f_t \in T$ for every $t \in \Sigma^\perp$. Hence

$$0 = \int_G (f - f_t) g\, dm = \int_G f(g - g_{t^{-1}})\, dm$$

whenever $f \in L^1(G)$ and thus, $g(tx) = g(x)$ a.e. x, if $t \in \Sigma^\perp$.

REMARK 1. – If we consider the real-valued space $L^p(G)$, the previous lemma remains to hold provided that we define as Σ the subgroup:

$$\Sigma = \{\alpha \in \hat{G};\ (\mathrm{Re}\,\alpha)S \subseteq S \text{ and } (\mathrm{Im}\,\alpha)S \subseteq S\}\,.$$

3. – Main results.

Our first result is essentially known, though we cannot find a explicit reference for it. Since it is used in proving the second theorem, we prefer to state it for sake of completeness:

THEOREM 1. – *If S is a translation invariant and $\hat{G}$-invariant subspace of $L^p(G)$, $1 \leqslant p < \infty$, then S is dense in $L^p(G)$.*

PROOF. – Let g be a function of S^0. For every $f \in S$ and $\alpha \in \hat{G}$ is $\alpha f \in S$, then

$$(gf)\hat{}\,(\alpha) = \int_G \bar{\alpha} f g\, dm = 0\,.$$

Therefore, $gf = 0$ m-a.e. for all $f \in S$. Since, there is no set of positive measure on which all the functions of S vanish (because S is translation invariant), $g = 0$, i.e., $S^0 = \{0\}$ and thus $\bar{S} = L^p(G)$.

REMARK 2. – This is a particular case of a more general result which can be stated for an arbitrary measure space: « If the subspace $S \subset L^p$ has full support (i.e., there is no set of positive measure on which all the functions of S vanish) and S is H-invariant for some $H \subseteq L^\infty$ such that the σ-algebra generated by H is equivalent to the σ-algebra of all measurables sets, then $\bar{S} = L^p$ ». Details and further generalizations are contained in [3].

THEOREM 2. – *Let S be a translation invariant, H-invariant, selfadjoint subspace of $L^p(G)$, $1 \leqslant p < \infty$, $H \neq \emptyset$, $\Sigma_1 = \{t \in G;\ \varphi$ is t-periodic $\forall \varphi \in H\}$, Σ the same from lemma 2 and e the unit element in G. If $\Sigma_1 \cap \Sigma^\perp = \{e\}$, then S is dense in $L^p(G)$.*

PROOF. – It follows from lemma 2, that

$$H \subseteq \{\varphi \in L^\infty(G);\ \varphi \text{ is } \Sigma_0^\perp\text{-periodic}\}$$

with $\Sigma_0 = \{\alpha \in \hat{G};\ \alpha\bar{S} \subseteq \bar{S}\}$ and then $(\Sigma_0)^\perp \subseteq \Sigma_1$. Moreover, $(\Sigma_0)^\perp \subseteq \Sigma^\perp$ and then $\Sigma_0^\perp = \{e\}$. Therefore $\Sigma_0 = \hat{G}$, and the result follows from theorem 1.

REMARK 3. – The condition S is selfadjoint is not superfluous, since $H^p(T) \subsetneq L^p(T)$ is not selfadjoint but verifies the other hypothesis of theorem 2.

COROLLARY 2. – *If S is a selfadjoint, translation invariant subspace of $L^p(G)$, $1 \leqslant p < \infty$, and there exists $\varphi \in L^\infty(G)$ no periodic such that $\varphi S \subseteq S$, then S is dense.*

REMARK 4. – All the preceding results are true in $L^\infty(G)$, if we considered the $*$-weak topology in $L^\infty(G)$, and the closure of S is taken in this topology.

REMARK 5. – It is possible, to extend the results obtained in this paper to a more general situation. For example, if S is a subspace of $L^\varrho(G)$ (a Köthe function space, see [5]) where ϱ is a saturated, absolutely continuous, translation invariant and symetrical norm. (A lot of the important classical Banach function spaces are contained in this class for suitable ϱ's).

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