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Asymptotics for a generalization of Hermite polynomials

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Abstract. We consider a generalization of the classical Hermite polynomials by the addition of terms involving derivatives in the inner product. This type of generalization has been studied in the literature from the point of view of the algebraic properties. Thus, our aim is to study the asymptotics of this sequence of nonstandard orthogonal polynomials. In fact, we obtain Mehler–Heine type formulas for these polynomials and, as a consequence, we prove that there exists an acceleration of the convergence of the smallest positive zeros of these generalized Hermite polynomials towards the origin.

Keywords: asymptotics, Hermite polynomials, Mehler–Heine type formulas, zeros, Bessel functions

1. Introduction

The study of the asymptotic behavior of orthogonal polynomials is one of the central problems in the analytic theory of orthogonal polynomials. In this paper we deal with the asymptotic properties of some particular families of polynomials orthogonal with respect to discrete Sobolev inner products of the form:

$$(P, Q) = \int_I P(x)Q(x)\,\mathrm{d}\mu + \mathbb{P}(c)^t A\mathbb{Q}(c),$$

where μ is a positive Borel measure supported on an interval $I \subseteq \mathbb{R}$, $c \in \mathbb{R}$, $A \in \mathbb{R}^{(s,s)}$ is a positive semidefinite matrix and for a polynomial with real coefficients P, $\mathbb{P}(x)$ denotes the column vector $(P(x), P'(x), \ldots, P^{(s-1)}(x))$, being $\mathbb{P}(x)^t$ its transpose. We denote by (Q_n) the sequence of its orthogonal polynomials and by (P_n) the corresponding one with respect to μ. For general measures μ, with bounded support and in the Nevai class $M(0,1)$, some asymptotic results can be seen in [10] and [8], among others.

A natural approach to study the asymptotic properties of Q_n is to compare these polynomials with the standard polynomials P_n, whenever the asymptotic properties of these ones are known. In [4] it was studied in detail the asymptotics of Q_n when $s = 2$ and the measure μ belongs to the Nevai class $M(0,1)$, so the support of μ is bounded. However, when the support of the measure μ is unbounded, we have not a so much general approach (see [9]). In this framework, the Laguerre case was considered

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in [6], where $\mathrm{d}\mu = x^\alpha \mathrm{e}^{-x}\,\mathrm{d}x$, with $\alpha > -1$, and $A \in \mathbb{R}^{(2,2)}$ is a diagonal matrix. There, the authors study the asymptotic properties of the corresponding orthogonal polynomials and of their zeros (see also the survey [9] where a clearer notation is used). Therefore, the motivation of this paper is, on the one hand, to fulfill a gap in the literature taking into account the other classical measure of unbounded support $\mathrm{d}\mu = \mathrm{e}^{-x^2}\mathrm{d}x$, and, on the other hand, to try to obtain a pattern about the scaled asymptotics of Mehler–Heine type for this unbounded case.

Let us consider the inner product

$$(P,Q) = \int_R P(x)Q(x)\mathrm{e}^{-x^2}\mathrm{d}x + \mathbb{P}(0)^t A \mathbb{Q}(0), \quad A \in \mathbb{R}^{(2,2)}, \tag{1}$$

where the matrix A is positive semidefinite.

The corresponding orthogonal polynomials are somewhat different from the modified Hermite polynomials considered in [5] since in that work the entry (2, 2) in the matrix A is zero, that is, the term $P'(0)Q'(0)$ does not appear in the inner product and asymptotic properties were not studied, either. A particular case of the inner product considered here was treated in [11], where the entry (2, 2) in the matrix A is the only one different from zero. In that paper, the zeros of the corresponding orthogonal polynomials were studied from the numerical point of view and a lower bound for the smallest positive zero was given.

It is important to observe that along this paper we will focus our attention on the Mehler–Heine type asymptotics. It is natural that the reader wonders: *Why is this type of asymptotics studied?* In an heuristic way it is possible to guess that the asymptotic behavior of the new orthogonal polynomials and of the standard orthogonal polynomials is the same in all the complex plane except for a neighborhood of the origin. Precisely, the Mehler–Heine type asymptotics describe in a detailed form the asymptotic behavior around the origin. In this paper, we will prove that these two families of orthogonal polynomials have a different behavior around the origin. In this way, Remark 1 of this paper supports the idea that the type of asymptotics that deserves to be studied is the Mehler–Heine type asymptotics since it provides the differences between both sequences of orthogonal polynomials when the degree of the polynomials tends to infinity.

Next, we describe the structure of the paper. In Section 2, we introduce some properties of the classical Hermite polynomials and give expressions of the kernel polynomials and their derivatives which will be used along this paper. In Section 3, we first obtain Mehler–Heine type formulas for the polynomials orthogonal with respect to (1) and we observe that the entries in the matrix A are connected with the order of the Bessel functions which appear in these formulas. We also notice that the nondiagonal case does not add further additional information to the one obtained in the diagonal case concerning this type of asymptotics (see Theorem 1). Furthermore, for the diagonal case, we deduce how the presence of the masses gives an asymptotic behavior of the first positive zero different from the one of the Hermite polynomials. We also observe that the rank of the matrix does not play an important role to state the Mehler–Heine type formulas.

These results are a motivation to find a pattern for this type of asymptotics in a more general framework. Then, looking for this pattern, by using a symmetrization process given in [2], in Section 4 we obtain the asymptotic properties of the polynomials orthogonal with respect to an inner product like (1) when $A \in \mathbb{R}^{(4,4)}$ is a positive semidefinite diagonal matrix. More precisely, we prove that the presence of all the masses produces an increase in four units in the order of the Bessel function appearing in the corresponding Mehler–Heine type formulas (see Theorem 2). In this case, we get a convergence acceleration to 0 of the two smallest positive zeros. Moreover, we prove that the positive definiteness of the matrix A is a necessary condition to increase the convergence acceleration of the zeros (see Remark 5). Finally, a conjecture is raised when $A \in \mathbb{R}^{(2r,2r)}$, $r \geqslant 1$, is a positive definite diagonal matrix. This conjecture can be reformulated in terms of the one stated in [6] for the Laguerre–Sobolev type polynomials.

2. Basic tools

From here on, we will use the standard notation H_n for the monic classical Hermite polynomials orthogonal with respect to the weight e^{-x^2}. The polynomials H_n are symmetric, that is, $H_n(-x) = (-1)^n H_n(x)$, and they satisfy (see [12]):

$$\|H_n\|^2 = \int_R H_n^2(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}n!}{2^n}, \qquad H_n'(x) = nH_{n-1}(x),$$

$$H_{2n+1}(0) = 0, \qquad H_{2n}(0) = \frac{(-1)^n(2n)!}{2^{2n}n!}.$$

The nth kernel for the Hermite polynomials $K_n(x,y) = \sum_{k=0}^{n} \frac{H_k(x)H_k(y)}{\|H_k\|^2}$ satisfies the Christoffel–Darboux formula

$$K_n(x,y) = \frac{1}{\|H_n\|^2}\frac{H_{n+1}(x)H_n(y) - H_{n+1}(y)H_n(x)}{x-y},$$

from which it follows

$$K_{2n+1}(x,0) = K_{2n}(x,0) = \frac{(-1)^n}{n!\sqrt{\pi}}\frac{H_{2n+1}(x)}{x}$$

and

$$K_{2n+1}(0,0) = K_{2n}(0,0) = \frac{(2n+1)!}{\sqrt{\pi}2^{2n}n!^2}.$$

As usual, we denote the derivated kernels by

$$K_n^{(i,j)}(x,y) = \frac{\partial^{i+j}}{\partial x^i\,\partial y^j}K_n(x,y) = \sum_{k=0}^{n}\frac{H_k^{(i)}(x)H_k^{(j)}(y)}{\|H_k\|^2} \quad (i,j \geqslant 0),$$

with the convention $K_n^{(0,0)}(x,y) = K_n(x,y)$.

Observe that the symmetry of Hermite polynomials yields to $K_n^{(i,j)}(0,0) = 0$ for all n, whenever $i+j$ is an odd integer number.

In the next lemma we show some formulas for the derivated kernels that we need throughout the paper.

Lemma 1. *The derivated kernels of the Hermite polynomials satisfy:*

(a)

$$K_{2n}^{(0,1)}(x,0) = K_{2n-1}^{(0,1)}(x,0) = \frac{(-1)^{n-1}}{\sqrt{\pi}(n-1)!}\frac{2xH_{2n}(x)+H_{2n-1}(x)}{x^2},$$

$$K_{2n-1}^{(0,2)}(x,0) = \frac{2(-1)^{n-1}}{\sqrt{\pi}(n-1)!}\frac{2xH_{2n}(x)+(1-2nx^2)H_{2n-1}(x)}{x^3},$$

$$K_{2n}^{(0,3)}(x,0) = \frac{2(-1)^n}{\sqrt{\pi}n!}\frac{(3-6nx^2)H_{2n+1}(x)-(2n+1)(3-2nx^2)xH_{2n}(x)}{x^4}.$$

(b)

$$K_{2n-1}^{(1,1)}(0,0) = K_{2n}^{(1,1)}(0,0) = \frac{(2n+1)!}{3\sqrt{\pi}2^{2n-2}n!(n-1)!},$$

$$K_{2n-1}^{(0,2)}(0,0) = \frac{-(2n-1)!}{3\sqrt{\pi}2^{2n-4}(n-1)!(n-2)!},$$

$$K_{2n-1}^{(2,2)}(0,0) = \frac{(2n-1)!(3n-1)}{15\sqrt{\pi}2^{2n-6}(n-1)!(n-2)!},$$

$$K_{2n}^{(1,3)}(0,0) = \frac{-(2n+1)!}{5\sqrt{\pi}2^{2n-4}n!(n-2)!},$$

$$K_{2n}^{(3,3)}(0,0) = \frac{(2n+1)!(5n-3)}{35\sqrt{\pi}2^{2n-6}n!(n-2)!}.$$

Proof. (a) From the Christoffel–Darboux formula taking successive derivatives with respect to y, evaluating at $y=0$, and using Leibniz's formula, we find for $j=0,1,\dots,$

$$K_n^{(0,j)}(x,0) = \frac{j!}{\|H_n\|^2}\frac{1}{x^{j+1}}\big[P_j(x,0;H_n)H_{n+1}(x) - P_j(x,0;H_{n+1})H_n(x)\big],$$

where $P_j(x,0;f)$ is the jth Taylor polynomial of f at 0 (see, for instance, [4]). Then, the result follows.

(b) To get these formulas it is enough to take into account that, if for each fixed j, we denote by $\sum_{k=j+1}^{n+j+1}\alpha_k x^k$ the $(n+j+1)$th Taylor polynomial of $x^{j+1}K_n^{(0,j)}(x,0)$ at 0, then $K_n^{(i,j)}(0,0) = i!\alpha_{i+j+1}$. Therefore, from the expressions of the kernels obtained in (a) and Taylor's formula for the Hermite polynomials, the result follows after some suitable computations. □

In the next section we will use the Mehler–Heine type formulas for the monic Hermite polynomials H_n: for $j\in\mathbb{Z}$ fixed, it holds

$$\lim_{n\to\infty}\frac{(-1)^n\sqrt{n}}{n!}H_{2n}\left(\frac{x}{2\sqrt{n+j}}\right) = \left(\frac{x}{2}\right)^{1/2}J_{-1/2}(x), \tag{2}$$

$$\lim_{n\to\infty}\frac{(-1)^n}{n!}H_{2n+1}\left(\frac{x}{2\sqrt{n+j}}\right) = \left(\frac{x}{2}\right)^{1/2}J_{1/2}(x), \tag{3}$$

uniformly on compact sets of the complex plane (see, for instance, [1], formulas 22.15.3 and 22.15.4).

Remind that the Bessel function J_α of the first kind of order α ($\alpha \in \mathbb{R}$) is defined by

$$J_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{z}{2}\right)^{2n+\alpha}$$

(if α is a negative integer, we assume $n \geqslant -\alpha$). Therefore $z^{-\alpha} J_\alpha(z)$ is an entire function which does not vanish at the point 0.

It is well known that the Bessel functions satisfy the following recurrence relation (see, for instance, [12]):

$$J_{\alpha-1}(z) + J_{\alpha+1}(z) = \frac{2\alpha}{z} J_\alpha(z). \tag{4}$$

In the sequel, the notation $\alpha_n \sim \beta_n$ means that $\alpha_n/\beta_n \to 1$ when n goes to infinity.

3. Hermite–Sobolev type polynomials

We denote by Q_n^λ the monic polynomials orthogonal with respect to the inner product

$$(P, Q) = \int_{\mathbb{R}} P(x)Q(x)\mathrm{e}^{-x^2}\,\mathrm{d}x + \mathbb{P}(0)^t A\mathbb{Q}(0), \tag{5}$$

where $A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}$ with $M_0 \geqslant 0, M_1 \geqslant 0, \lambda \in \mathbb{R}$ and $M_0 M_1 - \lambda^2 \geqslant 0$.

It is obvious that the Hermite polynomials are a particular case of the polynomials Q_n^λ, as it is enough to take the matrix A as the zero matrix.

The algebraic properties of these polynomials Q_n^λ have been studied in [4]. In the next proposition, we obtain the estimates for the coefficients which appear in the representation of the Hermite–Sobolev type polynomials in terms of (H_n).

We want to make a comment about notation. Obviously, the polynomials orthogonal with respect to (5) depend on the parameters M_0, M_1 and λ, but there is usually no confusion denoting them by Q_n. However, we want to consider two different situations, the nondiagonal case $\lambda \neq 0$ (so $M_0 > 0, M_1 > 0$) and the diagonal case $\lambda = 0$ (so $M_0 \geqslant 0, M_1 \geqslant 0$). Thus, we do not write the index λ whenever $\lambda = 0$, that is, in the diagonal case.

Proposition 1. *Let Q_n^λ be the monic polynomials orthogonal with respect to the inner product* (5). *Then, for $n \geqslant 1$, the following formulas hold:*

$$Q_{2n}^\lambda(x) = H_{2n}(x) - a_n^\lambda \frac{H_{2n-1}(x)}{x} - b_n^\lambda \frac{2xH_{2n}(x) + H_{2n-1}(x)}{x^2}, \tag{6}$$

$$Q_{2n+1}^\lambda(x) = H_{2n+1}(x) - c_n^\lambda \frac{H_{2n+1}(x)}{x} - d_n^\lambda \frac{2xH_{2n}(x) + H_{2n-1}(x)}{x^2}, \tag{7}$$

where

$$\lim_{n\to\infty} a_n = \begin{cases} 0, & \textit{if } M_0 = 0, M_1 \geqslant 0, \\ -\frac{1}{2}, & \textit{if } M_0 > 0, M_1 \geqslant 0, \end{cases} \qquad \lim_{n\to\infty} a_n^\lambda = \begin{cases} 0, & \textit{if } M_0M_1 - \lambda^2 = 0, \\ -\frac{1}{2}, & \textit{if } M_0M_1 - \lambda^2 > 0, \end{cases}$$

$$\lim_{n\to\infty} \sqrt{n}b_n = \lim_{n\to\infty} \sqrt{n}b_n^\lambda = 0; \qquad \lim_{n\to\infty} \sqrt{n}c_n = \lim_{n\to\infty} \sqrt{n}c_n^\lambda = 0,$$

$$\lim_{n\to\infty} d_n = \begin{cases} 0, & \textit{if } M_0 \geqslant 0, M_1 = 0, \\ -\frac{3}{4}, & \textit{if } M_0 \geqslant 0, M_1 > 0, \end{cases} \qquad \lim_{n\to\infty} d_n^\lambda = -\frac{3}{4} \quad \textit{if } M_0M_1 - \lambda^2 \geqslant 0.$$

Proof. The algebraic relations (6) and (7) can be deduced in the usual way expanding Q_n^λ in terms of the orthogonal system (H_n) and taking into account the expressions of $K_n(x,0)$ and $K_n^{(0,1)}(x,0)$ (see also Proposition 6 in [4]). The coefficients are given by

$$a_n^\lambda = \frac{(-1)^{n-1}}{\sqrt{\pi}(n-1)!}\frac{H_{2n}(0)}{\Delta_{2n}^\lambda}\left[M_0 + \left(M_0M_1 - \lambda^2\right)K_{2n-1}^{(1,1)}(0,0)\right],$$

$$b_n^\lambda = \frac{(-1)^{n-1}}{\sqrt{\pi}(n-1)!}\frac{H_{2n}(0)}{\Delta_{2n}^\lambda}\lambda,$$

$$c_n^\lambda = \frac{(-1)^{n}}{\sqrt{\pi}n!}\frac{(2n+1)H_{2n}(0)}{\Delta_{2n+1}^\lambda}\lambda,$$

$$d_n^\lambda = \frac{(-1)^{n-1}}{\sqrt{\pi}(n-1)!}\frac{(2n+1)H_{2n}(0)}{\Delta_{2n+1}^\lambda}\left[M_1 + \left(M_0M_1 - \lambda^2\right)K_{2n}(0,0)\right],$$

where

$$\Delta_n^\lambda = 1 + M_0K_{n-1}(0,0) + M_1K_{n-1}^{(1,1)}(0,0) + \left(M_0M_1 - \lambda^2\right)K_{n-1}(0,0)K_{n-1}^{(1,1)}(0,0).$$

From the estimates of $H_{2n}(0), K_n(0,0), K_n^{(1,1)}(0,0)$ and Δ_n^λ, according to the different cases, and by using Stirling's formula adequately, it can be deduced after suitable computations:

$$a_n = 0 \quad \text{if } M_0 = 0, M_1 \geqslant 0; \qquad \lim_{n\to\infty} a_n = -1/2 \quad \text{if } M_0 > 0, M_1 \geqslant 0,$$

$$\lim_{n\to\infty} na_n^\lambda = \frac{-3M_0}{8M_1} \quad \text{if } M_0M_1 - \lambda^2 = 0,$$

$$\lim_{n\to\infty} a_n^\lambda = -1/2 \quad \text{if } M_0M_1 - \lambda^2 > 0,$$

$$b_n = 0 \quad \text{if } M_0 \geqslant 0, M_1 \geqslant 0,$$

$$\lim_{n\to\infty} nb_n^\lambda = \frac{-3\lambda}{8M_1} \quad \text{if } M_0M_1 - \lambda^2 = 0,$$

$$\lim_{n\to\infty} n^{3/2}b_n^\lambda = \frac{-3\pi\lambda}{16(M_0M_1 - \lambda^2)} \quad \text{if } M_0M_1 - \lambda^2 > 0,$$

$$
\begin{aligned}
&c_n = 0 \quad \text{if } M_0 \geqslant 0, M_1 \geqslant 0,\\
&\lim_{n\to\infty} n c_n^\lambda = \frac{3\lambda}{4M_1} \quad \text{if } M_0M_1 - \lambda^2 = 0,\\
&\lim_{n\to\infty} n^{3/2} c_n^\lambda = \frac{3\pi\lambda}{8(M_0M_1 - \lambda^2)} \quad \text{if } M_0M_1 - \lambda^2 > 0,\\
&d_n = 0 \quad \text{if } M_0 \geqslant 0, M_1 = 0; \qquad \lim_{n\to\infty} d_n = -3/4 \quad \text{if } M_0 \geqslant 0, M_1 > 0,\\
&\lim_{n\to\infty} d_n^\lambda = -3/4 \quad \text{if } M_0M_1 - \lambda^2 \geqslant 0,
\end{aligned}
$$

and the result follows. □

Remark 1. Taking into account the relative asymptotics for the monic Hermite polynomials (which can be obtained from Perron's formula, see [12], Section 8.22), i.e.,

$$
\lim_{n\to\infty} \sqrt{n}\frac{H_{2n}(x)}{H_{2n+1}(x)} = -\operatorname{sgn}\bigl(\operatorname{Im}(x)\bigr)\mathrm{i},
$$

the scaled asymptotics for monic Hermite polynomials (see [13], p. 126), i.e., for $j \in \mathbb{Z}$ fixed and being $\varphi(x) = x + \sqrt{x^2-1}$ the conformal mapping of $\mathbb{C} \setminus [-1,1]$ onto the exterior of the closed unit disk, we have

$$
\lim_{n\to\infty} \sqrt{n}\frac{H_{n-1}(\sqrt{n+j}x)}{H_n(\sqrt{n+j}x)} = \frac{\sqrt{2}}{\varphi(x/\sqrt{2})},
$$

and applying Proposition 1, we can deduce after several computations that

$$
\begin{aligned}
&Q_n^\lambda(x) = H_n(x)\bigl(1 + \mathrm{o}(1)\bigr),\\
&Q_n^\lambda(\sqrt{n}x) = H_n(\sqrt{n}x)\bigl(1 + \mathrm{o}(1)\bigr),
\end{aligned}
$$

hold uniformly on compact sets of $\mathbb{C} \setminus \mathbb{R}$ and $\mathbb{C} \setminus [-\sqrt{2}, \sqrt{2}]$, respectively. Therefore, the polynomials Q_n^λ have the same outer strong asymptotics and Plancherel–Rotach type behavior as the Hermite polynomials.

We focus our attention on Mehler–Heine type formulas for the polynomials Q_n^λ. These formulas are interesting twofold: on the one hand, they provide the scaled asymptotics of Q_n^λ on compact sets of the complex plane and, on the other hand, they supply us with asymptotic information about the location of the zeros of Q_n^λ in terms of the zeros of other simple and known special functions.

With the previous results we are ready to prove the scaled asymptotics for the polynomials Q_n^λ orthogonal with respect to (5) where A is not the zero matrix.

Theorem 1. *The following Mehler–Heine type formulas hold:*

(a) *For generalized Hermite polynomials of even degree*

$$\lim_{n\to\infty}\frac{(-1)^n\sqrt{n}}{n!}Q_{2n}\left(\frac{x}{2\sqrt{n}}\right)=\begin{cases}\left(\frac{x}{2}\right)^{1/2}J_{-1/2}(x), & \text{if rank } A=1, M_0=0,\\ -\left(\frac{x}{2}\right)^{1/2}J_{3/2}(x), & \text{if rank } A=1, M_0>0,\\ -\left(\frac{x}{2}\right)^{1/2}J_{3/2}(x), & \text{if rank } A=2,\end{cases}$$

$$\lim_{n\to\infty}\frac{(-1)^n\sqrt{n}}{n!}Q^{\lambda}_{2n}\left(\frac{x}{2\sqrt{n}}\right)=\begin{cases}\left(\frac{x}{2}\right)^{1/2}J_{-1/2}(x), & \text{if rank } A=1,\\ -\left(\frac{x}{2}\right)^{1/2}J_{3/2}(x), & \text{if rank } A=2.\end{cases}$$

(b) *For generalized Hermite polynomials of odd degree*

$$\lim_{n\to\infty}\frac{(-1)^n}{n!}Q_{2n+1}\left(\frac{x}{2\sqrt{n}}\right)=\begin{cases}\left(\frac{x}{2}\right)^{1/2}J_{1/2}(x), & \text{if rank } A=1, M_1=0,\\ -\left(\frac{x}{2}\right)^{1/2}J_{5/2}(x), & \text{if rank } A=1, M_1>0,\\ -\left(\frac{x}{2}\right)^{1/2}J_{5/2}(x), & \text{if rank } A=2,\end{cases}$$

$$\lim_{n\to\infty}\frac{(-1)^n}{n!}Q^{\lambda}_{2n+1}\left(\frac{x}{2\sqrt{n}}\right)=\begin{cases}-\left(\frac{x}{2}\right)^{1/2}J_{5/2}(x), & \text{if rank } A=1,\\ -\left(\frac{x}{2}\right)^{1/2}J_{5/2}(x), & \text{if rank } A=2.\end{cases}$$

All the limits hold uniformly on compact sets of the complex plane.

Proof. (a) We have only written the proof for the nondiagonal case ($\lambda \neq 0$), since the diagonal case ($\lambda = 0$) can be deduced in a similar way. From formula (6) in Proposition 1, we have

$$\begin{aligned}\frac{(-1)^n\sqrt{n}}{n!}Q^{\lambda}_{2n}\left(\frac{x}{2\sqrt{n}}\right)&=\frac{(-1)^n\sqrt{n}}{n!}H_{2n}\left(\frac{x}{2\sqrt{n}}\right)\\&\quad+2a^{\lambda}_n\frac{1}{x}\frac{(-1)^{n-1}}{(n-1)!}H_{2n-1}\left(\frac{x}{2\sqrt{n}}\right)\\&\quad+4b^{\lambda}_n\sqrt{n}\left[\frac{1}{x^2}\frac{(-1)^{n-1}}{(n-1)!}H_{2n-1}\left(\frac{x}{2\sqrt{n}}\right)-\frac{1}{x}\frac{(-1)^n\sqrt{n}}{n!}H_{2n}\left(\frac{x}{2\sqrt{n}}\right)\right].\end{aligned}$$

Now, we must distinguish two different cases according to the asymptotic behavior of the coefficients a^{λ}_n and b^{λ}_n. Handling formulas (2)–(4) adequately, we get:

- If rank $A = 1$ (that is $M_0M_1 - \lambda^2 = 0, M_0 > 0, M_1 > 0$), since $a_n^\lambda \to 0$ and $\sqrt{n}b_n^\lambda \to 0$, then

$$\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}}{n!} Q_{2n}^\lambda\left(\frac{x}{2\sqrt{n}}\right) = \left(\frac{x}{2}\right)^{1/2} J_{-1/2}(x).$$

- If rank $A = 2$ (that is $M_0M_1 - \lambda^2 > 0, M_0 > 0, M_1 > 0$), since $2a_n^\lambda \to -1$ and $\sqrt{n}b_n^\lambda \to 0$, then

$$\begin{aligned}&\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}}{n!} Q_{2n}^\lambda\left(\frac{x}{2\sqrt{n}}\right)\\ &\quad= \left(\frac{x}{2}\right)^{1/2}\left[J_{-1/2}(x) - \frac{1}{x}J_{1/2}(x)\right] = -\left(\frac{x}{2}\right)^{1/2} J_{3/2}(x).\end{aligned}$$

All the limits hold uniformly on compact sets of $\mathbb{C}$.

(b) Upon in the case $M_1 = 0$ and $\lambda = 0$ ($Q_{2n+1} = H_{2n+1}$), the asymptotic behavior of $\sqrt{n}c_n^\lambda$ and d_n^λ is independent from the occurrence of the parameter λ in the inner product, where the coefficients c_n^λ and d_n^λ are those ones appearing in formula (7). So, we do not write the index λ. Since $\sqrt{n}c_n \to 0$ and $4d_n \to -3$, using (2), (3) and property (4), we get:

$$\begin{aligned}\lim_{n\to\infty} \frac{(-1)^n}{n!} Q_{2n+1}\left(\frac{x}{2\sqrt{n}}\right) &= \lim_{n\to\infty}\left\{\frac{(-1)^n}{n!} H_{2n+1}\left(\frac{x}{2\sqrt{n}}\right)\right.\\ &\quad - 2\sqrt{n}c_n \frac{1}{x}\frac{(-1)^n}{n!} H_{2n+1}\left(\frac{x}{2\sqrt{n}}\right)\\ &\quad \left. - 4d_n\left[\frac{1}{x}\frac{(-1)^n\sqrt{n}}{n!} H_{2n}\left(\frac{x}{2\sqrt{n}}\right) - \frac{1}{x^2}\frac{(-1)^{n-1}}{(n-1)!} H_{2n-1}\left(\frac{x}{2\sqrt{n}}\right)\right]\right\}\\ &= \left(\frac{x}{2}\right)^{1/2}\left[J_{1/2}(x) + \frac{3}{x}\left(J_{-1/2}(x) - \frac{1}{x}J_{1/2}(x)\right)\right]\\ &= \left(\frac{x}{2}\right)^{1/2}\left[J_{1/2}(x) - \frac{3}{x}J_{3/2}(x)\right] = -\left(\frac{x}{2}\right)^{1/2} J_{5/2}(x),\end{aligned}$$

holds uniformly on compact sets of $\mathbb{C}$. □

Remark 2. Observe that the nondiagonal case does not add further additional information to the one obtained in the diagonal case concerning the asymptotic behavior of the scaled polynomials.

For this reason, from now on, we will only consider the inner product (5) with $\lambda = 0$, i.e., the diagonal case.

Remark 3. We want to emphasize the fact that what it is really important in order to have a different result from the Hermite case is the presence of the masses either M_0 for the polynomials Q_{2n} or M_1 for Q_{2n+1} and not the rank of A. Observe how the presence of these masses implies, in addition to a change of sign, an increase in the order of the corresponding Bessel functions appearing in Theorem 1.

On the other hand, in the next corollary we will show a remarkable difference between the zeros of (H_n) and (Q_n) with respect to the convergence acceleration to 0.

Before analyzing this, we recall (see [12]) that the zeros of the Hermite polynomials are real, simple and symmetric. We denote by $(x_{n,k})_{k=1}^{[n/2]}$ the positive ones in increasing order. It is worth pointing out that they satisfy the interlacing property $0 < x_{n+1,1} < x_{n,1} < x_{n+1,2} < \cdots$, and that $x_{n,k} \to_n 0$ for every fixed k.

Let $(j_{\alpha,k})_{k\geqslant 1}$ be the positive zeros of the Bessel function J_α in increasing order. Then, formulas (2) and (3) and Hurwitz's theorem lead us to

$$2\sqrt{n}x_{2n,k} \underset{n}{\rightarrow} j_{-1/2,k} \quad (k \geqslant 1),$$

$$2\sqrt{n}x_{2n+1,k} \underset{n}{\rightarrow} j_{1/2,k} \quad (k \geqslant 1),$$

and therefore

$$x_{n,k} \sim \frac{C_k}{\sqrt{n}} \quad (k \geqslant 1),$$

where, for every k, C_k is a positive constant.

Concerning the zeros of Q_n we know that all of them are real, simple and symmetric and they interlace with those of H_n (see [3]). We denote by $(\xi_{n,k})_{k=1}^{[n/2]}$ the positive zeros of Q_n in increasing order. In this case, it also happens that $\xi_{n,k} \to_n 0$ for each fixed k.

From Theorem 1 and Hurwitz's theorem and taking into account the multiplicity of 0 as a zero of the limit functions in Theorem 1 we achieve the following corollary.

Corollary 1. *Let $(\xi_{n,k})_{k=1}^{[n/2]}$ be the positive zeros of Q_n in increasing order. Then*

(a) *If $M_0 = 0$,*

$$2\sqrt{n}\xi_{2n,k} \underset{n}{\rightarrow} j_{-1/2,k} \quad (k \geqslant 1).$$

If $M_0 > 0$,

$$\sqrt{n}\xi_{2n,1} \underset{n}{\rightarrow} 0,$$

$$2\sqrt{n}\xi_{2n,k} \underset{n}{\rightarrow} j_{3/2,k-1} \quad (k \geqslant 2).$$

(b) *If $M_1 = 0$,*

$$2\sqrt{n}\xi_{2n+1,k} \underset{n}{\rightarrow} j_{1/2,k} \quad (k \geqslant 1).$$

If $M_1 > 0$,

$$\sqrt{n}\xi_{2n+1,1} \underset{n}{\rightarrow} 0,$$

$$2\sqrt{n}\xi_{2n+1,k} \underset{n}{\rightarrow} j_{5/2,k-1} \quad (k \geqslant 2).$$

Observe that in all the cases we have $\xi_{n,k} \sim \frac{C_k}{\sqrt{n}}$ $(k \geqslant 2)$. However, there exist only two situations for which the asymptotic behavior of the first positive zero is different from the one of the Hermite polynomials. They correspond to $M_0 > 0$ for even degree polynomials and $M_1 > 0$ for odd degree polynomials, then $\sqrt{n}\xi_{n,1} \underset{n}{\to} 0$. Thus, the presence of the masses M_0 and M_1 in the inner product (5) produces a convergence acceleration to 0 of two zeros of the polynomials (Q_n), namely, the first positive zero and its symmetric one.

4. Mehler–Heine type formulas: The diagonal case

The comments and results in the previous section are a motivation to study what happens with these properties when the matrix A is a positive semidefinite and diagonal.

We begin considering a diagonal matrix $A \in \mathbb{R}^{(4,4)}$. Thus, we introduce the inner product

$$(P, Q) = \int_{\mathbb{R}} P(x)Q(x)\mathrm{e}^{-x^2}\,\mathrm{d}x + \sum_{i=0}^{3} M_i P^{(i)}(0)Q^{(i)}(0), \tag{8}$$

with $M_i \geqslant 0$, $i = 0, 1, 2, 3$. We denote by S_n the monic orthogonal polynomials with respect to (8).

Notice that in this case the polynomials S_n are symmetric, i.e., $S_n(-x) = (-1)^n S_n(x)$. This does not occur for the polynomials Q_n^λ considered in the previous section when $\lambda \neq 0$. Therefore, because of this symmetry, we can transform the inner product (8) into a Laguerre–Sobolev type inner product and so we can establish a simple relation between the polynomials S_n and the polynomials studied in [6] and [9]. This technique is known as a symmetrization process. In fact, in [7] this process is considered for standard inner products associated with positive measures. The simplest case of this situation is the relation between monic Laguerre polynomials and Hermite polynomials, that is (see [7] or [12]),

$$H_{2n}(x) = L_n^{(-1/2)}(x^2), \qquad H_{2n+1}(x) = xL_n^{(1/2)}(x^2), \quad n \geqslant 0.$$

Later in [2] the authors generalize the symmetrization process in the framework of Sobolev type orthogonal polynomials.

Thus, applying Theorem 2 in [2] in a straightforward way we obtain that

$$S_{2n}(x) = L_n^{(-1/2, M_0, 4M_2)}(x^2) \quad \text{and} \quad S_{2n+1}(x) = xL_n^{(1/2, M_1, 36M_3)}(x^2),$$

where $(L_n^{(-1/2,M_0,4M_2)})$ and $(L_n^{(1/2,M_1,36M_3)})$ are the sequences of monic orthogonal polynomials with respect to

$$(P, Q)_1 = \int_0^\infty P(x)Q(x)x^{-1/2}\mathrm{e}^{-x}\,\mathrm{d}x + M_0 P(0)Q(0) + 4M_2 P'(0)Q'(0),$$

$$(P, Q)_{1^*} = \int_0^\infty P(x)Q(x)x^{1/2}\mathrm{e}^{-x}\,\mathrm{d}x + M_1 P(0)Q(0) + 36M_3 P'(0)Q'(0),$$

respectively. The Mehler–Heine type formulas for the orthogonal polynomials with respect to the above inner products were obtained in [6] and later reformulated more clearly in [9]. Observe that the inner products considered in those articles are $\frac{(P,Q)_1}{\sqrt{\pi}}$ and $\frac{2(P,Q)_{1^*}}{\sqrt{\pi}}$, respectively. Taking into account that the

Mehler–Heine type formulas do not depend on the explicit value of the masses M_i, but only on whether the masses are positive or not, see Proposition 2.10 in [9], we deduce the following result directly:

Theorem 2. *The polynomials S_n satisfy the following Mehler–Heine type formulas.*

(a) *For polynomials of even degree:*

$$\lim_{n\to\infty}\frac{(-1)^n\sqrt{n}}{n!}S_{2n}\left(\frac{x}{2\sqrt{n}}\right)$$
$$=\begin{cases}\left(\frac{x}{2}\right)^{1/2}J_{-1/2}(x), & \text{if } M_0=0, M_2=0,\\ -\left(\frac{x}{2}\right)^{1/2}J_{3/2}(x), & \text{if } M_0>0, M_2=0,\\ \left(\frac{x}{2}\right)^{1/2}\left[\frac{2}{3}J_{7/2}(x)-J_{3/2}(x)-\frac{2}{3}J_{-1/2}(x)\right], & \text{if } M_0=0, M_2>0,\\ \left(\frac{x}{2}\right)^{1/2}J_{7/2}(x), & \text{if } M_0>0, M_2>0.\end{cases}$$

(b) *For polynomials of odd degree:*

$$\lim_{n\to\infty}\frac{(-1)^n}{n!}S_{2n+1}\left(\frac{x}{2\sqrt{n}}\right)$$
$$=\begin{cases}\left(\frac{x}{2}\right)^{1/2}J_{1/2}(x), & \text{if } M_1=0, M_3=0,\\ -\left(\frac{x}{2}\right)^{1/2}J_{5/2}(x), & \text{if } M_1>0, M_3=0,\\ \left(\frac{x}{2}\right)^{1/2}\left[\frac{2}{5}J_{9/2}(x)-J_{5/2}(x)-\frac{2}{5}J_{1/2}(x)\right], & \text{if } M_1=0, M_3>0,\\ \left(\frac{x}{2}\right)^{1/2}J_{9/2}(x), & \text{if } M_1>0, M_3>0.\end{cases}$$

All the limits hold uniformly on compact sets of the complex plane.

Notice that in the even case as well as in the odd case, the presence of the two relevant masses produces an increase in four units in the order of the Bessel functions appearing in the corresponding Mehler–Heine type formulas. As we have proved, we get an increase in two units in the order of Bessel functions, when only the first masses appear in (8), i.e., $M_0 > 0$ and $M_2 = 0$ or $M_1 > 0$ and $M_3 = 0$ in the respective cases.

Remark 4. We can also observe that the rank of the matrix A is not relevant to establish the Mehler–Heine type formulas as we have said in the previous section.

In the next corollary we only display the results which are different from those ones obtained before.

Corollary 2. *Let $(\xi_{n,k})_{k=1}^{[n/2]}$ be the positive zeros of S_n in increasing order. Then*

(a) *If $M_0 = 0$ and $M_2 > 0$,*

$$\xi_{2n,k} \sim \frac{C_k}{\sqrt{n}} \quad (k \geqslant 1).$$

If $M_0 > 0$ and $M_2 > 0$,

$$\sqrt{n}\xi_{2n,k} \underset{n}{\rightarrow} 0 \quad (k = 1, 2),$$

$$\xi_{2n,k} \sim \frac{C_k}{\sqrt{n}} \quad (k \geqslant 3).$$

(b) *If $M_1 = 0$ and $M_3 > 0$,*

$$\xi_{2n+1,k} \sim \frac{C_k}{\sqrt{n}} \quad (k \geqslant 1).$$

If $M_1 > 0$ and $M_3 > 0$,

$$\sqrt{n}\xi_{2n+1,k} \underset{n}{\rightarrow} 0 \quad (k = 1, 2),$$

$$\xi_{2n+1,k} \sim \frac{C_k}{\sqrt{n}} \quad (k \geqslant 3).$$

In all the cases, for every k, C_k is a positive constant.

Remark 5. We want to point out that a singular fact occurs when there is a gap in the set of the masses, namely $M_0 = 0, M_2 > 0$ or $M_1 = 0, M_3 > 0$, in the respective cases. This difference appears as much in the expression of the limit function (a particular linear combination of Bessel functions, see Theorem 2) as in the convergence acceleration to 0 of the zeros. Observe that in order to get a convergence acceleration to 0 of four zeros of (S_n) (the two smallest positive zeros and their symmetric ones) it is necessary that all the masses M_i appearing in the inner product (8) are positive.

In [6] a nice conjecture was stated for the orthogonal polynomials with respect to a Laguerre–Sobolev type inner product involving r masses at the origin. This conjecture was reformulated with a clearer notation in the survey paper [9]. Therefore, according to our previous results it is natural to pose a similar one for the orthogonal polynomials, Q_n, with respect to the inner product

$$(P, Q) = \int_{\mathbb{R}} P(x)Q(x)\mathrm{e}^{-x^2}\,\mathrm{d}x + \sum_{i=0}^{2r-1} M_i P^{(i)}(0)Q^{(i)}(0), \quad r \geqslant 1, M_i \geqslant 0.$$

Then, it should be true the following conjecture.

Conjecture. If $M_i > 0, i = 0, 1, \ldots, 2r-1$, with $r \geqslant 1$, then

$$\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}}{n!} Q_{2n}\left(\frac{x}{2\sqrt{n}}\right) = (-1)^r \left(\frac{x}{2}\right)^{1/2} J_{-1/2+2r}(x),$$

$$\lim_{n\to\infty} \frac{(-1)^n}{n!} Q_{2n+1}\left(\frac{x}{2\sqrt{n}}\right) = (-1)^r \left(\frac{x}{2}\right)^{1/2} J_{1/2+2r}(x),$$

uniformly on compact sets of $\mathbb{C}$.

Now, by using again the symmetrization process given in [2], Theorem 2, we can rewrite our polynomials Q_n as

$$Q_{2n}(x) = L_n^{(-1/2,N_0,\ldots,N_{2r-2})}(x^2) \quad \text{and} \quad Q_{2n+1}(x) = xL_n^{(1/2,N_1,\ldots,N_{2r-1})}(x^2),$$

where $(L_n^{(-1/2,N_0,\ldots,N_{2r-2})})$ and $(L_n^{(1/2,N_1,\ldots,N_{2r-1})})$ are the sequences of monic orthogonal polynomials with respect to

$$(P,Q)_{r-1} = \int_0^\infty P(x)Q(x)x^{-1/2}\mathrm{e}^{-x}\,\mathrm{d}x + \sum_{i=0}^{r-1} N_{2i}P^{(i)}(0)Q^{(i)}(0),$$

$$(P,Q)_{(r-1)^*} = \int_0^\infty P(x)Q(x)x^{1/2}\mathrm{e}^{-x}\,\mathrm{d}x + \sum_{i=0}^{r-1} N_{2i+1}P^{(i)}(0)Q^{(i)}(0),$$

where

$$N_0 = M_0, \qquad N_{2i} = (i+1)_i^2 M_{2i} \quad \text{and} \quad N_{2i+1} = (i+1)_{i+1}^2 M_{2i+1}$$

and $(a)_i$ denotes the Pochhammer symbol, that is,

$$(a)_i = a(a+1)\cdots(a+i-1), \qquad (a)_0 = 1.$$

We have proved this conjecture for $r = 1$ and $r = 2$ in Theorems 1 and 2, respectively. However, the techniques used in this paper and in [6] do not seem the most adequate ones to prove the general case. We want to highlight that in solving the conjecture for the Laguerre case in [6] we have solved the one for the Hermite case.

Finally, it is worth observing that as a consequence the following result could be deduced for the positive zeros of Q_n:

$$\sqrt{n}\xi_{n,k} \underset{n}{\to} 0 \quad (k = 1, 2, \ldots, r),$$

$$\xi_{n,k} \sim \frac{C_k}{\sqrt{n}} \quad (k \geqslant r+1).$$

So, the presence of all the constants $M_i > 0, i = 0, 1, \ldots, 2r-1$, in the above generalized inner product would induce a convergence acceleration to 0 of $2r$ zeros of the polynomials (Q_n), namely, the r smallest positive zeros and their symmetric ones.

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References

[1] M. Abramowitz and I.A. Stegun (eds), *Handbook of Mathematical Functions*, 10th edn, Dover, New York, 1972.
[2] M. Alfaro, F. Marcellán, H.G. Meijer and M.L. Rezola, Symmetric orthogonal polynomials for Sobolev-type inner products, *J. Math. Anal. Appl.* **184** (1994), 360–381.
[3] M. Alfaro, F. Marcellán, M.L. Rezola and A. Ronveaux, On orthogonal polynomials of Sobolev-type: Algebraic properties and zeros, *SIAM J. Math. Anal.* **23** (1992), 737–757.
[4] M. Alfaro, F. Marcellán, M.L. Rezola and A. Ronveaux, Sobolev-type orthogonal polynomials: The nondiagonal case, *J. Approx. Theory* **83** (1995), 266–287.
[5] R. Álvarez-Nodarse, J. Arvesú and F. Marcellán, Modifications of quasi–definite linear functionals via addition of delta and derivatives of delta Dirac functions, *Indag. Math. (N.S.)* **15** (2004), 1–20.
[6] R. Álvarez-Nodarse and J.J. Moreno-Balcázar, Asymptotic properties of generalized Laguerre orthogonal polynomials, *Indag. Math. (N.S.)* **15** (2004), 151–165.
[7] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon & Breach, New York, 1978.
[8] G. López, F. Marcellán and W. Van Assche, Relative asymptotics for polynomials orthogonal with respect to a discrete Sobolev inner product, *Constr. Approx.* **11** (1995), 107–137.
[9] F. Marcellán and J.J. Moreno-Balcázar, Asymptotics and zeros of Sobolev orthogonal polynomials on unbounded supports, *Acta Appl. Math.* **94** (2006), 163–192.
[10] F. Marcellán and W. Van Assche, Relative asymptotics for orthogonal polynomials with a Sobolev inner product, *J. Approx. Theory* **72** (1992), 192–209.
[11] M.C. Recchioni, Quadratically convergent method for simultaneously approaching the roots of polynomial solutions of a class of differential equations: Application to orthogonal polynomials, *Numer. Algorithms* **28** (2001), 285–308.
[12] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1975.
[13] W. Van Assche, *Asymptotics for Orthogonal Polynomials*, Lecture Notes in Mathematics, Vol. 1265, Springer, Berlin, 1987.