ESTIMATES FOR JACOBI-SOBOLEV TYPE ORTHOGONAL POLYNOMIALS

by

M. Alfaro*, F. Marcellán**, M.L. Rezola*

AMS Subject Classification 1991, (update 1995): 33C25, 42C05

Abstract. Let the Sobolev-type inner product $\langle f, g\rangle = \int_{\mathbb{R}} fg d\mu_0 + \int_{\mathbb{R}} f'g' d\mu_1$ with $\mu_0 = w + M\delta_c$, $\mu_1 = N\delta_c$ where w is the Jacobi weight, c is either 1 or -1 and $M, N \geq 0$. We obtain estimates and asymptotic properties on $[-1, 1]$ for the polynomials orthonormal with respect to $\langle ., .\rangle$ and their kernels. We also compare these polynomials with Jacobi orthonormal polynomials; as a consequence, a result about the convergence acceleration to c of the zeros is given.

Key words: Sobolev-type inner products, orthogonal polynomials, zeros, kernels, asymptotic properties.

Running head: Jacobi-Sobolev Orthogonal Polynomials

---

* This research was partially supported by DGICYT PB93-0228-C02-02

** This research was partially supported by DGICYT PB93-0228-C02-01

1. Introduction

Recently the study of polynomials orthogonal with respect to a nonstandard inner product

$$\langle f,g\rangle = \int_{\mathbb{R}} fg\,d\mu_0 + \sum_{k=1}^{m}\int_{\mathbb{R}} f^{(k)}g^{(k)}d\mu_k$$

has attracted the interest of many researchers. A motivation for such a study appears in a paper by D.C. Lewis ([12]) related to polynomial least square approximations with some smoothness conditions. In fact, this approach becomes valuable when we wish to approximate a function by its projection onto polynomials and, simultaneously, to approximate its derivative by the derivative of the polynomial approximant. Since the derivative of the function is steep, we can expect that the quality of the projection in the conventional L_2 norm deteriorates. The standard projection is poor near the end points whereas the Sobolev projection displays reasonably good behaviour throughout the interval. For some numerical examples see [11].

On the other hand, the applications of polynomial approximation to the numerical simulation of Partial Differential Equations, and more precisely, to elliptic or parabolic type equations are based in the discretization by spectral type methods. They rely on the properties of the interpolation or the Fourier operators. The natural norms involved are those of the Hilbert-Sobolev spaces and approximation results for such operators in this type of norms are required for the numerical analysis of spectral discretizations. In particular, for the Legendre and the Tchebychev case the reader can see [5].

Results concerning algebraic properties as well as the location of the zeros of orthogonal polynomials with respect to the above inner product when $m = 1$ and μ_1 is an atomic measure supported at a point $c \in \mathbb{R}$, have been done (see for instance [1]). From an analytic point of view, the relative asymptotic behaviour of such polynomials when μ_0 belongs to the class $M(0,1)$ has been accomplished in several papers ([2], [13] and [14]). This behaviour is considered in compact sets of $\mathbb{C} \setminus \operatorname{supp}\ \mu_0$.

However, the behaviour of polynomials in $\operatorname{supp}\ \mu_0$ remains an open question. The aim of this paper is to cover this lack in the literature. In fact, a first approach was given by Marcellán and Osilenker [15] when $m = 1$, $d\mu_0 = \chi_{[-1,1]}dx + M(\delta_1 + \delta_{-1})$ and $d\mu_1 = N(\delta_1 + \delta_{-1})$ using some previous work by Bavinck and Meijer ([3], [4]), (δ_c denotes a Dirac measure supported at the point c). Recently, the same authors have considered the case when μ_0 is the Gegenbauer weight, see [8].

In our paper, we will consider $m = 1$

$$\begin{aligned} d\mu_0(x) &= (1-x)^{\alpha}(1+x)^{\beta}\chi_{[-1,1]}(x)dx + M\delta_1(x) \\ d\mu_1(x) &= N\delta_1(x) \end{aligned}$$

with $\alpha > -1$ and $\beta > -1$. Such a kind of polynomials are strongly connected with eigenfunctions of linear differential operators with polynomial coefficients (see [7]) as well as with the analysis of five-diagonal matrices associated with Schrödinger operators ([9]).

In Section 2 we present the basic tools concerning the polynomials orthogonal with respect to the inner product above with special emphasis in the case of the so-called Jacobi-Sobolev type polynomials and some results about Jacobi polynomials which we will need throughout the paper.

In Section 3 we study the behaviour of the coefficients which appear in their representation in terms of Jacobi polynomials and, as a consequence, an estimate for them at the ends of the interval is given and a result about the convergence acceleration of the zeros is derived. We also deal with pointwise analysis and upper bounds for Jacobi-Sobolev type polynomials as well as an upper bound of their uniform norm using the corresponding estimates for standard Jacobi polynomials is obtained.

Finally, in Section 4 we obtain some bounds and estimates for the kernels associated with the polynomials considered above. In particular, the analogue of a very well known result by Máté-Nevai-Totik concerning Christoffel functions is deduced.

In such a way we can give a complete answer in order to estimate the behaviour on $[-1,1]$ of such polynomials. Notice that some of the results above, when $d\mu_0 = wdx + M\delta_c$ where w is a generalized Jacobi weight and $\mu_k = 0$ $(k = 1, ..., m)$, have been obtained in [10].

2. Representation formulas and basic results

Let μ be a positive Borel measure on $\mathbb{R}$ whose moments are finite and whose support is an infinite set.

We consider the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} fg d\mu + Mf(c)g(c) + Nf'(c)g'(c) \qquad M, N \geq 0 \qquad c \in \mathbb{R} \tag{1}$$

Let p_n and q_n be the polynomials orthonormal with respect to the measure μ and the inner product (1), respectively.

Denote $q_n(x) = \gamma_n x^n + ...$ and $p_n(x) = k_n x^n + ...$. The Fourier expansion of q_n in terms of p_k $(k = 0, ..., n)$ leads to

$$q_n(x) = \frac{\gamma_n}{k_n} p_n(x) - Mq_n(c)K_{n-1}(x, c) - Nq_n'(c)K_{n-1}^{(0,1)}(x, c) \tag{2}$$

We have used the abbreviation

$$K_n^{(r,s)}(x, y) = \sum_{k=0}^{n} p_k^{(r)}(x)p_k^{(s)}(y) = \frac{\partial^{r+s}}{\partial x^r \partial y^s} K_n(x, y)$$

where, as usual, $K_n(x, y) = \sum_{k=0}^{n} p_k(x)p_k(y)$.

If we take derivatives in (2) with respect to x and evaluating at $x = c$, the values of $q_n(c)$ and $q_n'(c)$ can be expressed by

$$q_n(c) = \frac{\gamma_n}{k_n D_n} [p_n(c)\{1 + NK_{n-1}^{(1,1)}(c, c)\} - Np_n'(c)K_{n-1}^{(0,1)}(c, c)]$$
$$q_n'(c) = \frac{\gamma_n}{k_n D_n} [-Mp_n(c)K_{n-1}^{(0,1)}(c, c) + p_n'(c)\{1 + MK_{n-1}(c, c)\}]$$

where

$$D_n = 1+MK_{n-1}(c,c)+NK_{n-1}^{(1,1)}(c,c)+MN[K_{n-1}(c,c)K_{n-1}^{(1,1)}(c,c)-(K_{n-1}^{(0,1)}(c,c))^2] \quad (3)$$

(note that $D_n = D_n(M,N,c) > 0$ for all $M \geq 0$, $N \geq 0$ and $c \in \mathbb{R}$).

Let $p_n(x;\mu_j) = k_n(\mu_j)x^n + ...$, $j = 0,1,2,...$, the orthonormal polynomials with respect to the measure $d\mu_j = (x-c)^{2j}d\mu$ (where $\mu_0 = \mu$) and $K_n(x,y;\mu_j)$ the corresponding kernels. Expanding $(x-c)p_{n-1}(x;\mu_{j+1})$ in terms of $p_k(x;\mu_j)$ we obtain (see [1, Lemma 2.1])

$$(x-c)p_{n-1}(x;\mu_{j+1}) = \frac{k_{n-1}(\mu_{j+1})}{k_n(\mu_j)}[p_n(x;\mu_j) - \frac{p_n(c;\mu_j)}{K_{n-1}(c,c;\mu_j)}K_{n-1}(x,c;\mu_j)]$$

Using the orthonormality of the polynomials $p_{n-1}(x;\mu_{j+1})$ and $p_n(x;\mu_j)$ and the reproducing property of the kernels $K_{n-1}(x,c;\mu_j)$ we have

$$\left(\frac{k_n(\mu_j)}{k_{n-1}(\mu_{j+1})}\right)^2 = 1 + \frac{p_n(c;\mu_j)^2}{K_{n-1}(c,c;\mu_j)}$$

We want to point out that

$$\lim_n \frac{k_n(\mu_j)}{k_{n-1}(\mu_{j+1})} = 1 \qquad \text{whenever} \qquad \mu_j \in M(0,1) \quad c \in [-1,1] \quad (4)$$

(see [17, Theorem 3 on p.26]), that we will use later.

Since the polynomials $p_n(x;\mu_1)$ satisfy

$$K_n(c,c)p_n(x;\mu_1) = \frac{k_n(\mu_1)}{k_{n+1}}[p'_{n+1}(c)K_n(x,c) - p_{n+1}(c)K_n^{(0,1)}(x,c)]$$

we can write $q_n(c)$ and $q'_n(c)$ as follows

$$\begin{aligned} q_n(c) &= \frac{\gamma_n}{k_nD_n}[p_n(c) - N\frac{k_n}{k_{n-1}(\mu_1)}p'_{n-1}(c;\mu_1)K_{n-1}(c,c)] \\ q'_n(c) &= \frac{\gamma_n}{k_nD_n}[p'_n(c) + M\frac{k_n}{k_{n-1}(\mu_1)}p_{n-1}(c;\mu_1)K_{n-1}(c,c)\}] \end{aligned} \quad (5)$$

If we represent the kernels $K_{n-1}(x,c)$ and $K_{n-1}^{(0,1)}(x,c)$ in terms of the polynomials $p_n(x)$ and $p_n(x;\mu_j)$ with $j = 1,2$ we can obtain (see [1, Proposition 2.2])

Proposition 1. *Let p_n be the orthonormal polynomials for the measure μ and $c \in \mathbb{R}$ such that the condition $p_n(c)p_{n-1}(c;\mu_1) \neq 0$ is satisfied for every $n \in \mathbb{N}$. Then, the polynomials q_n orthonormal with respect to the inner product (1) verify the formula*

$$q_n(x) = A_np_n(x) + B_n(x-c)p_{n-1}(x;\mu_1) + C_n(x-c)^2p_{n-2}(x;\mu_2) \quad (6)$$

with

$$A_n = \frac{\gamma_n}{k_n}(1-\alpha_n) \qquad B_n = \frac{\gamma_n}{k_{n-1}(\mu_1)}(\alpha_n - \beta_n) \qquad C_n = \frac{\gamma_n}{k_{n-2}(\mu_2)}\beta_n \tag{6.1}$$

where

$$1-\alpha_n = D_n^{-1}\left[1 - N\frac{k_n}{k_{n-1}(\mu_1)}\frac{p'_{n-1}(c;\mu_1)}{p_n(c)}K_{n-1}(c,c)\right] \tag{6.2}$$

$$\beta_n = NK_{n-2}(c,c;\mu_1)D_n^{-1}\left[\frac{k_{n-1}(\mu_1)}{k_n}\frac{p'_n(c)}{p_{n-1}(c;\mu_1)} + MK_{n-1}(c,c)\right] \tag{6.3}$$

Remark. Since all the zeros of the polynomials $p_n(x)$ and $p_{n-1}(x;\mu_1)$ are in the interior of the convex hull of supp μ, then the formula (6) is true whenever c is not an interior point of the convex hull of supp μ.

From (2), it is obvious that

$$\frac{\gamma_n}{k_n} = \int_{\mathbb{R}} q_n p_n d\mu = \langle q_n, p_n\rangle - Mp_n(c)q_n(c) - Np'_n(c)q'_n(c)$$

and then by straightforward calculations we find, (see [1] or [2])

$$\frac{\gamma_n}{k_n} = \left(\frac{D_n}{D_{n+1}}\right)^{1/2} \tag{7}$$

In the sequel we consider the inner product (1) when the measure μ is the Jacobi weight and $c=1$, that is

$$\langle f, g\rangle = \int_{[-1,1]} fgw_{\alpha,\beta}dx + Mf(1)g(1) + Nf'(1)g'(1) \tag{8}$$

where $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha,\beta > -1$ and $M, N \geq 0$.

Let $P_n^{(\alpha,\beta)}$ be the Jacobi polynomials with the normalization condition $P_n^{(\alpha,\beta)}(1) = \dfrac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!}$ and $p_n^{(\alpha,\beta)}$ the Jacobi orthonormal polynomials. We denote by $q_n^{(\alpha,\beta)}$ the polynomials orthonormal with respect to the inner product (8).

Some basic properties of Jacobi polynomials, (see [18], Chapter IV), we will need in the following, are given below. Throughout this paper we use the notation $z_n \cong w_n$ when the sequence z_n/w_n converges to 1.

$$P_n^{(\alpha,\beta)}(1) \cong \frac{n^\alpha}{\Gamma(\alpha+1)} \tag{9}$$

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{10}$$

$$\|P_n^{(\alpha,\beta)}\|^2 = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)} \cong 2^{\alpha+\beta}n^{-1} \tag{11}$$

$$a_n = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n n!\Gamma(n+\alpha+\beta+1)} \cong 2^{n+\alpha+\beta}(\pi n)^{-1/2} \tag{12}$$

where $P_n^{(\alpha,\beta)}(x) = a_n x^n + ...$

From (9)-(12), we have for Jacobi orthonormal polynomials:

$$p_n^{(\alpha,\beta)}(1) \cong \frac{n^{\alpha+(1/2)}}{2^{(\alpha+\beta)/2}\Gamma(\alpha+1)} \tag{13}$$

$$(p_n^{(\alpha,\beta)})'(1) \cong \frac{n^{\alpha+(5/2)}}{2^{(\alpha+\beta+2)/2}\Gamma(\alpha+2)} \tag{14}$$

From these formulas we can deduce

Lemma 1. *The following estimates hold:*

$$K_n(1,1) \cong \frac{n^{2\alpha+2}}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha+2)} \tag{15}$$

$$K_n^{(0,1)}(1,1) \cong \frac{n^{2\alpha+4}}{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\alpha+3)} \tag{16}$$

$$K_n^{(1,1)}(1,1) \cong \frac{\alpha+2}{2^{\alpha+\beta+3}\Gamma(\alpha+2)\Gamma(\alpha+4)}n^{2\alpha+6} \tag{17}$$

Proof: Because of the reproducing property of the kernels, $K_n(x,1)$ is a polynomial of degree n, orthogonal with respect to the weight $w_{\alpha+1,\beta}$, that is, for each n there exists a constant c_n such that $K_n(x,1) = c_n p_n^{(\alpha+1,\beta)}(x)$. Comparing the leading coefficients we get

$$K_n(x,1) = \frac{\|P_n^{(\alpha+1,\beta)}\|}{\|P_n^{(\alpha,\beta)}\|}\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}p_n^{(\alpha,\beta)}(1)p_n^{(\alpha+1,\beta)}(x) \tag{18}$$

Now, (15) follows from (11) and (13).

If we derive (18) and evaluating at $x=1$, by using (11), (13) and (14), we deduce (16).

To obtain the estimate for $K_n^{(1,1)}(1,1)$ we can consider the formula

$$K_n(1,1)K_n^{(1,1)}(1,1) - (K_n^{(0,1)}(1,1))^2 = K_{n-1}(1,1;w_{\alpha+2,\beta})K_n(1,1) \tag{19}$$

(see [1, Formula (2.9′)]). Now (17) follows from (19), taking into account (15) and (16). ∎

This lemma and (19) allow us to deduce easily the asymptotic behaviour of D_n, (see formula (3)).

From now on C will denote a positive constant independent of n, but possibly different in each ocurrence.

Lemma 2. *There exists a positive constant C such that:*
a) if $MN > 0$, then

$$D_n \cong MN[K_{n-1}(1,1)K_{n-1}^{(1,1)}(1,1) - (K_{n-1}^{(0,1)}(1,1))^2] \cong Cn^{4\alpha+8}$$

b) if $M = 0$ and $N > 0$ then

$$D_n \cong NK_{n-1}^{(1,1)}(1,1) \cong Cn^{2\alpha+6}$$

Taking in mind (7), a consequence of the lemma above is the following

Corollary 1. *Let k_n and γ_n be the leading coefficients of the polynomials $p_n^{(\alpha,\beta)}$ and $q_n^{(\alpha,\beta)}$ respectively. Then $\lim_n \frac{\gamma_n}{k_n} = 1$.*

3. Jacobi-Sobolev polynomials $q_n^{(\alpha,\beta)}$: Estimates on $[-1,1]$ and zeros

In this section, the representation of the polynomials $q_n^{(\alpha,\beta)}$ orthonormal with respect to (8) in terms of Jacobi polynomials plays an important role. According to Proposition 1, the corresponding formula is

$$q_n^{(\alpha,\beta)}(x) = A_n p_n^{(\alpha,\beta)}(x) + B_n(x-1)p_{n-1}^{(\alpha+2,\beta)}(x) + C_n(x-1)^2 p_{n-2}^{(\alpha+4,\beta)}(x) \tag{20}$$

We begin by analyzing the size of the coefficients.

Theorem 1. *There exists a positive constant C such that:*
a) if $MN > 0$ then, $A_n \cong -Cn^{-2\alpha-2}$ $\quad B_n \cong Cn^{-2\alpha-2}$ $\quad C_n \cong 1$
b) if $M = 0$ and $N > 0$ then, $A_n \cong \frac{-1}{\alpha+2}$ $\quad B_n \cong 1$ $\quad C_n \cong \frac{1}{\alpha+2}$.

Proof: Firstly, note that because of (4) and Corollary 1, $\frac{\gamma_n}{k_n}$, $\frac{\gamma_n}{k_{n-1}(w_{\alpha+2,\beta})}$ and $\frac{\gamma_n}{k_{n-2}(w_{\alpha+4,\beta})}$ converge to 1. So, from (6.1), the asymptotic behaviour of A_n, B_n and C_n only depends on α_n and β_n.

a) Assume $MN > 0$. Using (13)-(15), we can see that, in formula (6.2), the term in brackets tends to $-\infty$ like $-n^{2\alpha+6}$. Since, by Lemma 2, $D_n \cong Cn^{4\alpha+8}$ it follows that $\alpha_n \to 1$ and $A_n \cong -Cn^{-2\alpha-2}$.

Applying formulas (13)-(15) and Lemma 2 in (6.3), we obtain that $\beta_n \to 1$; hence $\alpha_n - \beta_n \to 0$. Handling as above, it is not difficult to deduce that $B_n \cong Cn^{-2\alpha-2}$.

The result for C_n is immediate.

b) Assume $M = 0$ and $N > 0$. Lemma 2 and formulas (13)-(15) lead to

$$D_n^{-1} \frac{NK_{n-1}(1,1)(p_{n-1}^{(\alpha+2,\beta)})'(1)}{p_n^{(\alpha,\beta)}(1)} \to \frac{1}{\alpha+2}$$

which, since $D_n \cong Cn^{2\alpha+6}$, implies that $1 - \alpha_n \to -1/(\alpha+2)$. As to β_n, arguing in a similar way we get that $\beta_n \to 1/(\alpha+2)$ and the assertion follows. ∎

Now, we can give the asymptotic behaviour of the polynomials $q_n^{(\alpha,\beta)}$ and $(q_n^{(\alpha,\beta)})'$ at the ends of the interval $[-1,1]$ for $M \geq 0$ and $N > 0$.

Theorem 2. *There exists a positive constant C such that the following estimates*

$$q_n^{(\alpha,\beta)}(-1) \cong p_n^{(\alpha,\beta)}(-1) \cong C(-1)^n n^{\beta+(1/2)}$$

$$(q_n^{(\alpha,\beta)})'(-1) \cong (p_n^{(\alpha,\beta)})'(-1) \cong C(-1)^n n^{\beta+(5/2)}$$

$$q_n^{(\alpha,\beta)}(1) \cong \begin{cases} -Cn^{-\alpha-(3/2)} & \text{if } MN > 0 \\ -Cn^{\alpha+(1/2)} & \text{if } M = 0,\ N > 0 \end{cases}$$

$$(q_n^{(\alpha,\beta)})'(1) \cong Cn^{-\alpha-(7/2)}$$

hold.

Proof: Evaluating formula (20) at $x = -1$ and taking into account that $p_n^{(\alpha,\beta)}(-x) = (-1)^n p_n^{(\beta,\alpha)}(x)$, for all $x \in [-1,1]$, we have

$$q_n^{(\alpha,\beta)}(-1) = (-1)^n[A_n p_n^{(\alpha,\beta)}(1) + 2B_n p_{n-1}^{(\beta,\alpha+2)}(1) + 4C_n p_{n-2}^{(\beta,\alpha+4)}(1)]$$

Theorem 1 and (13) yield $\lim_n \dfrac{q_n^{(\alpha,\beta)}(-1)}{p_n^{(\alpha,\beta)}(-1)} = 1$, whenever $M \geq 0$ and $N > 0$.

Deriving in the expression above of $q_n^{(\alpha,\beta)}(x)$ and proceeding as before, from (13), (14), and Theorem 1, we obtain that $\lim_n \dfrac{(q_n^{(\alpha,\beta)})'(-1)}{(p_n^{(\alpha,\beta)})'(-1)} = 1$, whenever $M \geq 0$ and $N > 0$.

To give the asymptotic behaviour at the point 1, we can use similar arguments. However, we want to point out that to estimate $(q_n^{(\alpha,\beta)})'(1)$ it is easier to apply formula (5) written for Jacobi polynomials and $c = 1$, that is

$$(q_n^{(\alpha,\beta)})'(1) = \frac{\gamma_n}{k_n D_n}[(p_n^{(\alpha,\beta)})'(1) + M\frac{k_n}{k_{n-1}(w_{\alpha+2,\beta})} p_{n-1}^{(\alpha+2,\beta)}(1) K_{n-1}(1,1)]$$

Now it suffices to apply (4), (13)-(15), Lemma 2 and Corollary 1. ∎

Note that the polynomials orthogonal with respect to the measure $\mu + M\delta_1$ are orthogonal with respect to the inner product (1) with $c = 1$, $M > 0$ and $N = 0$. Next we summarize for this situation the main results of this section:

Lemma 3. *Whenever $M > 0$ and $N = 0$, there exists a positive constant C such that,*

$$D_n \cong MK_{n-1}(1,1) \cong Cn^{2\alpha+2}$$

$$A_n \cong Cn^{-2\alpha-2} \qquad B_n \cong 1 \qquad C_n = 0$$

$$q_n^{(\alpha,\beta)}(1) \cong Cn^{-\alpha-(3/2)} \qquad (q_n^{(\alpha,\beta)})'(1) \cong Cn^{\alpha+(5/2)}$$

$$q_n^{(\alpha,\beta)}(-1) \cong p_n^{(\alpha,\beta)}(-1) \cong C(-1)^n n^{\beta+(1/2)}$$
$$(q_n^{(\alpha,\beta)})'(-1) \cong (p_n^{(\alpha,\beta)})'(-1) \cong C(-1)^n n^{\beta+(5/2)}$$

The results above allow us to deduce some information about the zeros of $q_n^{(\alpha,\beta)}$.

It is known that Jacobi orthonormal polynomials satisfy the Mehler-Heine formula (see [18, Theorem 8.1.1]):

$$\lim_n n^{-\alpha-1/2} p_n^{(\alpha,\beta)}(1-\frac{z^2}{2n^2}) = \lim_n n^{-\alpha-1/2} p_n^{(\alpha,\beta)}(\cos\frac{z}{n}) = 2^{-\frac{\alpha+\beta}{2}} \left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z)$$

uniformly for z on compact sets of $\mathbb{C}$, where J_α is the Bessel function of order α.

Using Hurwitz's theorem, we get that, if $-1 < x_{nn} < ... < x_{1n} < 1$ are the zeros of $p_n^{(\alpha,\beta)}$ and we write $x_{kn} = \cos\theta_{kn} (0 < \theta_{kn} < \pi, 1 \le k \le n)$, then $\lim_n n\theta_{kn} = j_{k\alpha}$ where $j_{k\alpha}$ is the kth positive zero of J_α. From this and taking into account that $0 < j_{k\alpha} < j_{k+1,\alpha}$ (see [18] and [19, chapter XV]), it can be derived that $1 - x_{kn} \cong \dfrac{C}{n^2}$.

Concerning the zeros of $q_n^{(\alpha,\beta)}$, denoted by $(\xi_{kn})_{k=1}^n$, we know that all of them are real and simple and at least $n-1$ are in the interval $(-1,1)$.

Theorem 2 and Lemma 3 imply that, for n large enough, either $q_n^{(\alpha,\beta)}(1) < 0$ or $q_n^{(\alpha,\beta)}(1) > 0$ according to either $N > 0$ or $N = 0$, respectively. So we have that $-1 < \xi_{nn} < \ldots < \xi_{2n} < 1 < \xi_{1n}$ whenever $N > 0$ while $-1 < \xi_{nn} < \ldots < \xi_{1n} < 1$ whenever $N = 0$. Moreover, $\lim_n \xi_{1n} = 1$, (see [1, Proposition 3.3]).

Theorem 3. *Let $(\xi_{kn})_{k=1}^n$ be the zeros of $q_n^{(\alpha,\beta)}$ in decreasing order. Then*

a) *If $MN > 0$*

$$n^2(1-\xi_{kn}) \to 0, \quad k = 1, 2$$

$$1 - \xi_{kn} \cong \frac{C}{n^2} \qquad (k \ge 3)$$

b) If $M = 0, N > 0$:

$$1 - \xi_{kn} \cong \frac{C}{n^2} \qquad (k \ge 1)$$

c) If $M > 0, N = 0$:

$$n^2(1-\xi_{1n}) \to 0$$
$$1 - \xi_{kn} \cong \frac{C}{n^2} \qquad (k \ge 2)$$

Proof: Formula (20) evaluated at $1 - \dfrac{z^2}{2n^2}$ and Mehler-Heine formula lead to

$$\lim_n n^{-\alpha-1/2} q_n^{(\alpha,\beta)}(1-\frac{z^2}{2n^2}) = 2^{-\frac{\alpha+\beta}{2}}(\lim_n A_n)\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z) -$$
$$2^{-\frac{\alpha+\beta+4}{2}}(\lim_n B_n) z^2 \left(\frac{z}{2}\right)^{-(\alpha+2)} J_{\alpha+2}(z) + 2^{-\frac{\alpha+\beta+8}{2}}(\lim_n C_n) z^4 \left(\frac{z}{2}\right)^{-(\alpha+4)} J_{\alpha+4}(z)$$

uniformly for z on compact sets of $\mathbb{C}$

a) Suppose $MN > 0$. Using the estimates for the coefficients A_n, B_n, and C_n (see Theorem 1), we get

$$\lim_n n^{-\alpha-1/2} q_n^{(\alpha,\beta)}(1-\frac{z^2}{2n^2}) = 2^{-\frac{\alpha+\beta+8}{2}} z^4 \left(\frac{z}{2}\right)^{-(\alpha+4)} J_{\alpha+4}(z)$$

uniformly for z on compact sets of $\mathbb{C}$. As a consequence of Hurwitz's theorem, we obtain

$$n^2(1-\xi_{1n}) \to 0$$
$$n^2(1-\xi_{2n}) \to 0$$
$$n^2(1-\xi_{kn}) \to \frac{1}{2}(j_{k-2,\alpha+4})^2 \qquad (k \geq 3)$$

b) For $M = 0, N > 0$, we have

$$\lim_n n^{-\alpha-1/2} q_n^{(\alpha,\beta)}(1-\frac{z^2}{2n^2}) = -\frac{2^{-\frac{\alpha+\beta}{2}}}{\alpha+2}\left(\frac{z}{2}\right)^{-\alpha}[J_\alpha(z) + (\alpha+2)J_{\alpha+2}(z) - J_{\alpha+4}(z)]$$

uniformly for z on compact sets of $\mathbb{C}$.

Bessel functions satisfy the recurrence relation (see [18, (1.71.5)])

$$2\alpha z^{-1} J_\alpha(z) = J_{\alpha-1}(z) + J_{\alpha+1}(z)$$

so, after straightforward calculations, it follows

$$\lim_n n^{-\alpha-1/2} q_n^{(\alpha,\beta)}(1-\frac{z^2}{2n^2}) =$$
$$2^{-\frac{\alpha+\beta+4}{2}}\left[r_4(z)\left(\frac{z}{2}\right)^{-(\alpha+4)} J_{\alpha+4}(z) + s_2(z)\left(\frac{z}{2}\right)^{-(\alpha+3)} J_{\alpha+3}(z)\right]$$

uniformly for z on compact sets of $\mathbb{C}$, where $r_4(z) = \frac{z^2}{4}[z^2 + 4(\alpha+1)]$ and $s_2(z) = -(\alpha+1)[z^2+4(\alpha+3)]$.

Since $r_4(z) > 0$ and $s_2(z) < 0$ for all $z \in (0, +\infty)$ and the positive zeros of $J_{\alpha+3}$ interlace with those of $J_{\alpha+4}$, then between two consecutive positive zeros of $J_{\alpha+3}$ there is precisely one zero of $r_4(z)\left(\frac{z}{2}\right)^{-(\alpha+4)} J_{\alpha+4}(z) + s_2(z)\left(\frac{z}{2}\right)^{-(\alpha+3)} J_{\alpha+3}(z)$. Besides, this last function does not vanish at 0. Then, again by Hurwitz's theorem, it follows that

$$1 - \xi_{kn} \cong \frac{C}{n^2} \qquad (k \geq 1)$$

c) Proceeding as above, for $M > 0, N = 0$, we obtain that

$$\lim_n n^{-\alpha-1/2} q_n^{(\alpha,\beta)}(1 - \frac{z^2}{2n^2}) = -2^{-\frac{\alpha+\beta+4}{2}} z^2 \left(\frac{z}{2}\right)^{-(\alpha+2)} J_{\alpha+2}(z)$$

uniformly for z on compact sets of $\mathbb{C}$ and hence

$$n^2(1 - \xi_{1n}) \to 0$$

$$n^2(1 - \xi_{kn}) \to \frac{1}{2}(j_{k-1,\alpha+2})^2 \qquad (k \geq 2)$$

holds. ∎

Remark. Note that, the theorem above says that the convergence to 1 of the biggest or the two biggest zeros of Jacobi polynomials may be accelerated adding to the inner product either a Dirac's delta at the point 1 or a Dirac's delta plus a term involving the first derivative at the point 1, respectively.

Next we are going to estimate the polynomials $q_n^{(\alpha,\beta)}$ in $[-1, 1]$. The asymptotic behaviour of $q_n^{(\alpha,\beta)}(x)$ for x on compact sets of $\mathbb{C} \setminus [-1, 1]$ is well known since Lemma 16 on p.132 in [17] and Theorem 4 in [14] lead to $\lim_n \frac{q_n^{(\alpha,\beta)}(x)}{p_n^{(\alpha,\beta)}(x)} = 1$ uniformly for x on compact sets of $\mathbb{C} \setminus [-1, 1]$, whenever $M \geq 0$ and $N \geq 0$. (Concerning the asymptotic behaviour of $p_n^{(\alpha,\beta)}(x)$ out of $[-1, 1]$, see [17, Theorem 8.21.7]).

First, we recall a property satisfied by Jacobi polynomials.

Lemma 16 on page 83 in [17] shows that there is a constant C independent of x and n such that

$$|p_n^{(\alpha,\beta)}(x)| \leq C(1 - x + n^{-2})^{-(\alpha/2)-(1/4)}(1 + x + n^{-2})^{-(\beta/2)-(1/4)} \tag{21}$$

for all $x \in [-1, 1]$ and $n \geq 1$, with $\alpha, \beta > -1$. In the sequel C will denote a positive constant independent of n and x, but possibly different in each ocurrence.

We will find that similar bounds are valid for the polynomials $q_n^{(\alpha,\beta)}$ with $M, N \geq 0$.

Theorem 4. *There exists a constant C such that for each $x \in [-1, 1]$, $n \geq 1$ and $\alpha, \beta > -1$*

$$|q_n^{(\alpha,\beta)}(x)| \leq C(1 - x + n^{-2})^{-(\alpha/2)-(1/4)}(1 + x + n^{-2})^{-(\beta/2)-(1/4)} \tag{22}$$

Proof: It suffices to prove the result for n large enough.

Since the coefficients A_n, B_n and C_n are bounded (see Theorem 1 and Lemma 3) and the boundedness (21) for $p_n^{(\alpha,\beta)}(x)$ is also true for $(1 - x)p_{n-1}^{(\alpha+2,\beta)}(x)$ and $(1 - x)^2 p_{n-2}^{(\alpha+4,\beta)}(x)$ for all $x \in [-1, 1]$ and $n \geq 2$, the statement follows. ∎

As a consequence, whenever $\alpha, \beta \geq -1/2$, we get a bound independent of n

$$|q_n^{(\alpha,\beta)}(x)| \leq C(1 - x)^{-(\alpha/2)-(1/4)}(1 + x)^{-(\beta/2)-(1/4)}$$

for all $x \in (-1,1)$.

In particular, if $\alpha = \beta = 0$, we have $|q_n^{(\alpha,\beta)}(x)| \leq C(1-x^2)^{-(1/4)}$ for all $x \in (-1,1)$. A similar result has been obtained in [15] for the polynomials orthonormal with respect to the inner product $\langle f,g \rangle = \int_{[-1,1]} fg d\mu_0 + \int_{[-1,1]} f'g' d\mu_1$ with $d\mu_0 = \frac{1}{2}dx + M(\delta_1 + \delta_{-1})$ and $d\mu_1 = N(\delta_1 + \delta_{-1})$.

Now, from Theorem 4, we can deduce an upper bound of the maximum of $q_n^{(\alpha,\beta)}(x)$ on $[-1,1]$.

Corollary 2. *There exists a constant C such that for each $n \geq 1$ we have*

$$\max_{-1\leq x\leq 1} |q_n^{(\alpha,\beta)}(x)| \leq \begin{cases} Cn^{q+(1/2)} & \text{if } q \geq -1/2 \\ C & \text{if } q \leq -1/2 \end{cases}$$

where $q = \max\{\alpha, \beta\}$.

Proof: The inequalities $1 \leq 1 + x + n^{-2} \leq 3$ and $n^{-2} \leq 1 - x + n^{-2} \leq 2$ hold for $x \in [0,1]$. Therefore, from (22), it follows that

$$|q_n^{(\alpha,\beta)}(x)| \leq \begin{cases} Cn^{\alpha+(1/2)} & \text{if } \alpha \geq -1/2 \\ C & \text{if } \alpha \leq -1/2 \end{cases}$$

for all $x \in [0,1]$.

A similar argument leads to

$$|q_n^{(\alpha,\beta)}(x)| \leq \begin{cases} Cn^{\beta+(1/2)} & \text{if } \beta \geq -1/2 \\ C & \text{if } \beta \leq -1/2 \end{cases}$$

for all $x \in [-1,0]$. The assertion follows easily. ∎

Concerning the asymptotic behaviour of the $q_n^{(\alpha,\beta)}$ on $[-1,1]$, by the previous Section we know estimates for these polynomials at the end points of the support of the Jacobi weight. What about the asymptotic behaviour of the $q_n^{(\alpha,\beta)}$ on $(-1,1)$?

The Jacobi orthonormal polynomials verify

$$p_n^{(\alpha,\beta)}(x) = r_n^{\alpha,\beta}(1-x)^{-(\alpha/2)-(1/4)}(1+x)^{-(\beta/2)-(1/4)}\cos(k\theta+\gamma) + O(n^{-1}) \tag{23}$$

$k = n + \frac{\alpha+\beta+1}{2}$, $\gamma = -(\alpha+1)\pi/2$ and $r_n^{\alpha,\beta} = \dfrac{2^{(\alpha+\beta+1)/2}(\pi n)^{-1/2}}{\|P_n^{(\alpha,\beta)}\|} \to \left(\dfrac{2}{\pi}\right)^{1/2}$
uniformly for x on compact sets of $(-1,1)$, (see [18, Theorem 8.21.8]).

Now, we will show that the polynomials $q_n^{(\alpha,\beta)}$ have a similar asymptotic behaviour to the one of $p_n^{(\alpha,\beta)}$ on the interval $(-1,1)$.

Theorem 5. *Let $q_n^{(\alpha,\beta)}$ be the polynomials orthonormal with respect to (8) and A_n, B_n and C_n the corresponding coefficients which appear in formula (20). Then*

$$q_n^{(\alpha,\beta)}(x) = s_n^{\alpha,\beta}(1-x)^{-(\alpha/2)-(1/4)}(1+x)^{-(\beta/2)-(1/4)}\cos(k\theta+\gamma) + O(n^{-1})$$

$$s_n^{\alpha,\beta} = A_n r_n^{\alpha,\beta} + B_n r_{n-1}^{\alpha+2,\beta} + C_n r_{n-2}^{\alpha+4,\beta} \to \left(\frac{2}{\pi}\right)^{1/2}$$

uniformly for x on compact sets of $(-1,1)$. Therefore, $\lim_n [q_n^{(\alpha,\beta)}(x) - p_n^{(\alpha,\beta)}(x)] = 0$ uniformly for x on compact sets of $(-1,1)$.

Proof: By (20) and (23), we have

$$\begin{aligned} q_n^{(\alpha,\beta)}(x) = {} & (1-x)^{-(\alpha/2)-(1/4)}(1+x)^{-(\beta/2)-(1/4)}\cos(k\theta+\gamma)[A_n r_n^{\alpha,\beta} + B_n r_{n-1}^{\alpha+2,\beta} + C_n r_{n-2}^{\alpha+4,\beta}] \\ & + [A_n + B_n(x-1) + C_n(x-1)^2]O(n^{-1}) \end{aligned}$$

uniformly for x on compact sets of $(-1,1)$.

From the asymptotic behaviour of the coefficients A_n, B_n and C_n obtained in the previous section, we get

$$q_n^{(\alpha,\beta)}(x) = s_n^{\alpha,\beta}(1-x)^{-(\alpha/2)-(1/4)}(1+x)^{-(\beta/2)-(1/4)}\cos(k\theta+\gamma) + O(n^{-1})$$

and $\lim_n s_n^{\alpha,\beta} = \left(\frac{2}{\pi}\right)^{1/2}$.

Therefore $q_n^{(\alpha,\beta)}(x) = \frac{s_n^{\alpha,\beta}}{r_n^{\alpha,\beta}} p_n^{(\alpha,\beta)}(x) + O(n^{-1})$ and we can write

$$q_n^{(\alpha,\beta)}(x) - p_n^{(\alpha,\beta)}(x) = \left(\frac{s_n^{\alpha,\beta}}{r_n^{\alpha,\beta}} - 1\right) p_n^{(\alpha,\beta)}(x) + O(n^{-1})$$

uniformly for x on compact sets of $(-1,1)$. Thus the result follows. ∎

Remark. From (21) we have $|p_n^{(\alpha,\beta)}(x)| \le C$ for x on compact sets of $(-1,1)$. Then $\lim_n [q_n^{(\alpha,\beta)} - p_n^{(\alpha,\beta)}] = 0$ uniformly on compact sets of $(-1,1)$ could be also deduced applying Theorem 5 in [14] and formula (10) of Lemma 16 in [17].

4. Estimates for the kernels

It is known, (Nevai [17, Lemma 5 on p. 108]), that the kernels associated with Jacobi polynomials satisfy the estimate

$$K_n(x,x) \sim n(1-x+n^{-2})^{-\alpha-(1/2)}(1+x+n^{-2})^{-\beta-(1/2)} \tag{24}$$

uniformly in $|x| \le 1$, $n \ge 1$, where by $f_n(x) \sim g_n(x)$ we mean that there exist some positive constants C_1 and C_2 such that $C_1 f_n(x) \le g_n(x) \le C_2 f_n(x)$ for all $x \in [-1,1]$ and $n \in \mathbb{N}$.

We want to find similar estimates for the new kernels.

Let $L_n(x,y)$ be the kernels relative to the inner product (8). If we consider their expansion in terms of Jacobi orthonormal polynomials, we can deduce, (see [1, p.744]),

$$L_n(x,y) = K_n(x,y) - ML_n(y,1)K_n(x,1) - NL_n^{(0,1)}(y,1)K_n^{(0,1)}(x,1) \tag{25}$$

with

$$L_n(x,1) = D_{n+1}^{-1}\left([1+NK_n^{(1,1)}(1,1)]K_n(x,1) - NK_n^{(0,1)}(1,1)K_n^{(0,1)}(x,1)\right)$$

$$L_n^{(0,1)}(x,1) = D_{n+1}^{-1}\left([1+MK_n(1,1)]K_n^{(0,1)}(x,1) - MK_n^{(0,1)}(1,1)K_n(x,1)\right)$$

Inserting $L_n(x,1)$ and $L_n^{(0,1)}(x,1)$ in (25) and taking $y = x$, we get

$$\begin{aligned} L_n(x,x) = K_n(x,x) - D_{n+1}^{-1}[&M\{1+NK_n^{(1,1)}(1,1)\}K_n(x,1)^2 \\ &- 2MNK_n^{(0,1)}(1,1)K_n(x,1)K_n^{(0,1)}(x,1) + N\{1+MK_n(1,1)\}K_n^{(0,1)}(x,1)^2] \end{aligned} \tag{26}$$

If, as usual, we define the Christoffel function

$$\Lambda_n(x) = \min\{\langle p,p\rangle; \deg p \leq n, p(x) = 1\}$$

it is easy to see that $\Lambda_n(x) = [L_n(x,x)]^{-1}$.

We will use the representation (26) to obtain some bounds for $L_n(x,x)$.

Theorem 6. *Let $(L_n(x,y))$ be the kernels relative to the polynomials $q_n^{(\alpha,\beta)}$. Then there exists a constant C such that for each $x \in [-1,1]$ and $n \geq 1$*

$$|L_n(x,x)| \leq Cn(1-x+n^{-2})^{-\alpha-(1/2)}(1+x+n^{-2})^{-\beta-(1/2)}$$

Proof: From (24) we have for each $x \in [-1,1]$, $n \geq 1$ and $\alpha, \beta > -1$

$$|K_n(x,x)| \leq Cn(1-x+n^{-2})^{-\alpha-(1/2)}(1+x+n^{-2})^{-\beta-(1/2)} \tag{27}$$

Moreover, from (11), (18) and (21),

$$\begin{aligned} |K_n(x,1)| &\leq C|p_n^{(\alpha,\beta)}(1)||p_n^{(\alpha+1,\beta)}(x)| \\ &\leq Cn^{\alpha+(1/2)}(1-x+n^{-2})^{-(\alpha/2)-(3/4)}(1+x+n^{-2})^{-(\beta/2)-(1/4)} \end{aligned} \tag{28}$$

for all $x \in [-1,1]$.

To find a bound for $K_n^{(0,1)}(x,1)$, we will use the formula

$$K_n^{(0,1)}(x,1) = (x-1)K_{n-1}(x,1;w_{\alpha+2,\beta}) + \frac{K_n^{(0,1)}(1,1)}{K_n(1,1)}K_n(x,1) \tag{29}$$

(see [1, Formula (2.9)]), from which, using (28) and Lemma 1, it follows that

$$|K_n^{(0,1)}(x,1)| \leq Cn^{\alpha+(5/2)}(1-x+n^{-2})^{-(\alpha/2)-(3/4)}(1+x+n^{-2})^{-(\beta/2)-(1/4)} \tag{30}$$

for all $x \in [-1,1]$.

Now it suffices to remind that by Lemmas 1 and 2, whenever $MN > 0$

$$MD_{n+1}^{-1}[1 + NK_n^{(1,1)}(1,1)] \leq Cn^{-2\alpha-2}$$

$$2MND_{n+1}^{-1}K_n^{(0,1)}(1,1) \leq Cn^{-2\alpha-4}$$

$$ND_{n+1}^{-1}[1 + MK_n(1,1)] \leq Cn^{-2\alpha-6}$$

and to observe that for each $x \in [-1,1]$, the inequality $n^{-1}(1-x+n^{-2})^{-1} \leq Cn$ holds. For the other values of the parameters M and N, we proceed in a similar way. Thus, the result follows. ∎

This result gives us only upper bounds. Now we want to estimate more accurately $L_n(x,x)$. First, we observe the behaviour of $L_n(x,x)$ at the end points of the interval $[-1,1]$. Evaluating at $x = 1$ the expression of $L_n(x,1)$ given in (25) and using (19), we get

$$L_n(1,1) = D_{n+1}^{-1}[1 + NK_{n-1}(1,1; w_{\alpha+2,\beta})]K_n(1,1)$$

Then, the kernels $L_n(1,1)$ are bounded if $M > 0$, $N \geq 0$ while $L_n(1,1) \cong CK_n(1,1)$ if $M = 0, N \geq 0$. Note that the boundedness of $L_n(1,1)$ depends on the addition of a mass at 1 and not of the term involving derivatives.

Moreover from the expression of $L_n(1,1)$ we can recover the mass M. Indeed, by using Lemmas 1, 2 and 3 it follows that, when $M > 0$ and $N \geq 0$, $\lim_n \Lambda_n(1) = M$. Otherwise, the mass N can be recovered from $L_n^{(1,1)}(1,1)$; since

$$L_n^{(1,1)}(1,1) = D_{n+1}^{-1}[K_n^{(1,1)}(1,1) + MK_{n-1}(1,1; w_{\alpha+2,\beta})K_n(1,1)]$$

when $M \geq 0$ and $N > 0$, we have $\lim_n [L_n^{(1,1)}(1,1)]^{-1} = N$.

Remark. The expressions of the masses M and N, given above, as the limit of $L_n(1,1)^{-1}$ and $L_n^{(1,1)}(1,1)^{-1}$ respectively, can be also obtained from the results of Durán, see [6].

As to $L_n(-1,-1)$, it suffices to take $x = y = -1$ in (25) and we obtain

$$L_n(-1,-1) \cong CK_n(-1,-1) \cong Cn^{2\beta+2}$$

Next, we are going to find uniform estimates for the kernels. When $M > 0$, $L_n(1,1)$ is bounded, so we give uniform estimates on compact sets not containing the mass point 1.

Theorem 7. *a) Suppose $M > 0, N \geq 0$. Let $\varepsilon > 0$, then*

$$L_n(x,x) \sim n(1 - x + n^{-2})^{-\alpha-(1/2)}(1 + x + n^{-2})^{-\beta-(1/2)}$$

uniformly on $[-1, 1-\varepsilon]$, $n \geq 1$.

b) Suppose $M = 0, N \geq 0$. Then

$$L_n(x,x) \sim n(1 - x + n^{-2})^{-\alpha-(1/2)}(1 + x + n^{-2})^{-\beta-(1/2)}$$

uniformly on $|x| \leq 1$, $n \in \mathbb{N}$.

Proof: Because of Theorem 6, it suffices to prove that, for n large enough

$$L_n(x,x) \geq Cn(1-x+n^{-2})^{-\alpha-(1/2)}(1+x+n^{-2})^{-\beta-(1/2)}$$

uniformly on $[-1, 1-\varepsilon]$ when $M > 0$ and on $[-1,1]$ when $M = 0$.

For the sake of simplicity, we write

$$d(x,n) = n(1-x+n^{-2})^{-\alpha-(1/2)}(1+x+n^{-2})^{-\beta-(1/2)}$$

a) Let $N > 0$. Using Lemmas 1 and 2 and formulas (28) and (30), we obtain that the three last summands in (26) are bounded by $Cd(x,n)n^{-2}(1-x+n^{-2})^{-1}$. Thus, taking into account (24), the result follows. For $N = 0$, we handle in a similar way.

b) For $N = 0$ the result is obvious because of $L_n(x,x) = K_n(x,x)$. Suppose $N > 0$, as $D_{n+1} = 1 + NK_n^{(1,1)}(1,1)$, from (26) we have

$$L_n(x,x) \geq ND_{n+1}^{-1}[K_n^{(1,1)}(1,1)K_n(x,x) - K_n^{(0,1)}(x,1)^2]$$

and using, again, the estimates for the kernels and (29) we can deduce the result. ∎

Now we consider the analogue of the Szegő extremum problem for the inner product (8).

The generalized Szegő extremum problem, associated with a finite positive Borel measure on the real line, consists of finding $\lim_n \lambda_n(x;\mu)$ with $\lambda_n(x;\mu)$ the Christoffel functions corresponding to μ. It is known that, for $\mu = w_{\alpha,\beta}$,

$$\lim_n n\lambda_n(x) = \pi w_{\alpha,\beta}(x)(1-x^2)^{1/2}$$

uniformly for x on compact sets of $(-1,1)$, see [17, Theorem 35 on p. 94]. A solution of this problem, when μ belongs to the Szegő class of the interval $[-1,1]$, has been given in [16, Theorem 5] by proving that $\lim_n n\lambda_n(x;\mu) = \pi\mu'(x)(1-x^2)^{1/2}$ for almost every $x \in [-1,1]$, where μ' is almost everywhere the Radon-Nikodym derivative of μ.

Theorem 8. *Let Λ_n be the Christoffel functions associated with (8). Then*

$$\lim_n n\Lambda_n(x) = \pi w_{\alpha,\beta}(x)(1-x^2)^{1/2}$$

uniformly for x on compact sets of $(-1,1)$.

Proof: We only need to prove $\lim_n n^{-1}L_n(x,x) = \lim_n n^{-1}K_n(x,x)$, uniformly for x on compact sets of $(-1,1)$. Thus, by (26), it suffices to deduce

$$\lim_n MD_{n+1}^{-1}[1+NK_n^{(1,1)}(1,1)]K_n(x,1)^2 = 0$$

$$\lim_n MND_{n+1}^{-1}K_n^{(0,1)}(1,1)K_n(x,1)K_n^{(0,1)}(x,1) = 0$$

$$\lim_n ND_{n+1}^{-1}[1+MK_n(1,1)]K_n^{(0,1)}(x,1)^2 = 0$$

uniformly for x on compact sets of $(-1,1)$ and this follows by considering (18), (23), and (29). ∎

From the results of Section 2 and formulas (28) and (30), the following bounds for $L_n(x,1)$ and $L_n^{(0,1)}(x,1)$ can also be obtained:

Theorem 9. *There exists a constant C such that for each $x \in [-1,1]$ and $n \geq 1$*

$$
\begin{array}{rlll}
|L_n(x,1)| \leq C(1+x+n^{-2})^{-(\beta/2)-(1/4)} & & \text{if} & M>0 \\
|L_n(x,1)| \leq Cn^{2\alpha+4}(1+x+n^{-2})^{-(\beta/2)-(1/4)} & & \text{if} & M=0 \\
|L_n^{(0,1)}(x,1)| \leq C(1+x+n^{-2})^{-(\beta/2)-(1/4)} & & \text{if} & N>0 \\
|L_n^{(0,1)}(x,1)| \leq Cn^{2\alpha+4}(1+x+n^{-2})^{-(\beta/2)-(1/4)} & & \text{if} & N=0
\end{array}
$$

Notice that the bounds above for $L_n(x,1)$ when $M>0$ and for $L_n^{(0,1)}(x,1)$ when $N>0$ are, respectively, smaller than the ones for $K_n(x,1)$ and $K_n^{(0,1)}(x,1)$ (see formulas (28) and (30)).

Remark. Some of the previous results about the kernels appear in [10] for w a generalized Jacobi weight and $N=0$.

Finally, it is worth observing that if in the product (1) μ is the Jacobi measure and we take $c=-1$, since Jacobi polynomials satisfy $p_n^{(\alpha,\beta)}(-x) = (-1)^n p_n^{(\beta,\alpha)}(x)$, we get the same results as above but exchanging α and β.

References

[1] M. Alfaro, F. Marcellán, M.L. Rezola and A. Ronveaux, On orthogonal polynomials of Sobolev type: Algebraic properties and zeros, *SIAM J. Math. Anal. 23 (1992) 737-757.*

[2] M. Alfaro, F. Marcellán, M.L. Rezola and A. Ronveaux, Sobolev-type orthogonal polynomials: The nondiagonal case, *J. Approx. Theory 83 (1995) 266-287.*

[3] H. Bavinck and H.G. Meijer, Orthogonal polynomials with respect to a symmetric inner product involving derivatives, *Applicable Analysis 33 (1989) 103-117.*

[4] H. Bavinck and H.G. Meijer, On orthogonal polynomials with respect to an inner product involving derivatives: zeros and recurrence relations, *Indag. Math. (N.S.) 1 (1990) 7-14.*

[5] C. Bernardi, Y. Maday, *Approximations spectrales de problèmes aux limites elliptiques*, Mathèmatiques et Applications, 10. Springer Verlag, Paris 1992.

[6] A.J. Durán, On orthogonal polynomials with respect to a positive definite matrix of measures, *Canad. J. Math. 47 (1995) 82-112.*

[7] W.N. Everitt, L.L. Littlejohn, Orthogonal Polynomials and Spectral Theory: a survey, in *Orthogonal Polynomials and their applications,* (C. Brezinski et al. eds.), IMACS Annals on Computational and Applied Mathematis, Vol 9, J.C. Baltzer, Basel 1991, 21-55.

[8] A. Foulquié, F. Marcellán and B. Osilenker, Estimates for polynomials orthogonal with respect to some Gegenbauer-Sobolev type inner product. Submitted

[9] F.A. Grünbaum, L. Heine, Bispectral Darboux Transformations: An extension of the Krall Polynomials, *International Mathematics Research Notices 8 (1997) 359-392.*

[10] J.J. Guadalupe,M. Pérez, F.J. Ruiz and J.L. Varona, Asymptotic behaviour of orthogonal polynomials relative to measures with mass points, *Mathematika 40 (1993) 331-278.*

[11] A. Iserles, P.E. Koch, S.P. Nørsett, J.M. Sanz-Serna, Orthogonality and Approximation in a Sobolev Space, in *Algorithms for Approximations,* (J.C. Mason and M.G. Cox, eds.), Chapman and Hall, London 1990, 117-124.

[12] D.C. Lewis, Polynomial least square approximations, *Amer.J.Math. 69 (1947) 273-344.*

[13] G. López, F. Marcellán and W. Van Assche, Relative asymptotics for polynomials orthogonal with respect to a discrete Sobolev inner product, *Constr. Approx. 11 (1995) 107-137.*

[14] F. Marcellán and W. Van Assche, Relative asymptotics for orthogonal polynomials with a Sobolev inner product, *J. Approx. Theory 72 (1992) 192-209.*

[15] F. Marcellán and B. Osilenker, Estimates for polynomials orthogonal with respect to some Legendre-Sobolev type inner product. To appear in Math. Notes.

[16] A. Máté, P.G. Nevai and V. Totik, Szegő's extremum problem on the unit circle, *Ann. Math. 134 (1991) 433-453.*

[17] P.G. Nevai, *Orthogonal Polynomials*, Memoirs Amer. Math. Soc. 213, Amer. Math. Soc., Providence, RI, 1979.

[18] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Pub. 23, Amer. Math. Soc., Providence, RI, 1975 (4th edition).

[19] G.N. Watson, *A Teatrise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, 1966 (2nd edition).

Manuel Alfaro and María Luisa Rezola
Dpto. de Matemáticas
Universidad de Zaragoza
50009 Zaragoza, Spain

Francisco Marcellán
Dpto. de Matemáticas
Escuela Politécnica Superior
Universidad Carlos III de Madrid
28911 Leganés, Madrid, Spain