CLOSURE OF ANALYTIC POLYNOMIALS IN WEIGHTED JORDAN CURVES

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I. INTRODUCTION AND NOTATIONS

This paper deals with $H^p(\Gamma,\mu)$ space $0 < p < \infty$, where Γ is a rectifiable Jordan curve and $d\mu = w\,ds$ is a finite nonnegative measure on Γ, which is absolutely continuous with respect to arc length ds.

The purpose of this paper is to describe the elements of $H^p(\Gamma,\mu)$ and to obtain some results for this space that have some similarity with those of classical Hardy spaces $H^p(T)$.

We denote by Ω the inside region on Γ, and by ϕ a conformal mapping from $|z|\leq 1$ onto $\overline{\Omega}$ that, without loss of generality, we may suppose ϕ normalized by $\phi(0) = 0$ and $\phi'(0) > 0$. The function ψ will be the inverse function of ϕ. Smirnov domains (ϕ' is an outer function) are the most general domains in which it is possible to develop a consistent theory of Hardy spaces. In this paper we always consider Ω a Smirnov domain. $L^p(\Gamma,\mu)$ is the space of μ-measurable complex functions defined on Γ such that, $\int_\Gamma |f(z)|^p d\mu < \infty$, and $H^p(\Gamma,\mu)$ is the closed subspace of $L^p(\Gamma,\mu)$ generated by the analytic polynomials $P(z) = \sum_0^n a_k z^k$, $z \in \Gamma$. We denote by $L^p(\Gamma) = L^p(\Gamma,ds)$ and $H^p(\Gamma) = H^p(\Gamma,ds)$.

We can associate to measure μ a new measure ν on T (unit circle), given by, $d\nu = w\circ\phi|\phi'|d\theta$, where $d\theta$ is the normalized Lebesgue measure on T. Of course, $f \in L^p(\Gamma,\mu)$ iff $f\circ\phi \in L^p(T,\nu)$. By using Mergelyan's theorem (see [8]), $f \in H^p(\Gamma,\mu)$ if and only if $f\circ\phi \in H^p(T,\nu)$. The subspace S spanned by the polynomials $\{\sum_0^m a_k z^k + \sum_1^n b_k \bar{z}^k\ ;\ z \in \Gamma\}$ is dense in $L^p(\Gamma,\mu)$.

We need that $H^p(\Gamma,\mu) \subsetneq L^p(\Gamma,\mu)$, and by that w and Γ cannot be arbitrary. It is not difficult to see that a neccesary and sufficent condition for $H^p(\Gamma,\mu)$ is a proper subspace of $L^p(\Gamma,\mu)$ is $\log(w\circ\phi|\phi'|)\in L^1(T)$, or, equivalently, $\log(w\circ\phi) \in L^1(T)$, since $\phi' \in H^1$ ([3]). We would like to obtain a equivalent condition on w with rapport to arc length ds. Unfortunately this is not possible and we must restrict the class of curves that we shall consider

Definition 1. Let p be a real number $1 < p < \infty$ and w an integrable nonnegative function of Γ . We say that $w \in A_p(\Gamma)$ (*Muckenhoupt's A_p classes*, [1]) if there exists a constant $C > 0$, such that, for all intervals $J \subseteq \Gamma$

$$\left(\frac{1}{s(J)}\int_J w\,ds\right)\left(\frac{1}{s(J)}\int_J w^{-1/p-1}ds\right)^{p-1} \leq C$$

where $s(J)$ is the arc length of J. The A_∞ class is the union of all A_p classes and the A_1 class is the limit of A_p classes

Definition 2. Let Γ be a rectifiable Jordan curve. Γ is said to be a *chord-arc curve* if there is a constant $C > 0$, such that, for all points $z_1, z_2 \in \Gamma$, $s(z_1,z_2) \leq C|z_1-z_2|$, where $s(z_1,z_2)$ is the arc length of the shorter arc along Γ with endpoints z_1 and z_2.

If Γ is chord-arc, then Ω is Smirnov's (see [9]).

II. A DESCRIPTION OF $H^p(\Gamma,\mu)$

In the sequel Γ will be a chord-arc curve and then $|\phi'| \in A_q$ for some $q \in (1,\infty)$, ([9]).

Fixed a q of them, it follows

Theorem 1. *If w is a weight on Γ such that $\log w \in L^q(\Gamma)$ and $0 < p < \infty$, we have:*

i) $H^p(\Gamma,\mu) \subsetneq L^p(\Gamma,\mu)$.

ii) $H^p(\Gamma,\mu) = K_p H^p(\Gamma)$, where $K_p = \left(\frac{w}{c_1}\right)^{-1/p} \exp\left[\frac{-i}{p}\left(\log \frac{w}{c_1}\right)^{\sim}\right]$, $c_1 = \exp \int_T \log(w\circ\phi)\,d\theta$ and $\sim$ denotes the conjugation operator defined on Γ by $\tilde{f} = (f\circ\phi)^{\sim}\circ\psi$.

Proof. i) As $|\phi'|^{-q'/q} \in L^1(T)$ and $(\log(w\circ\phi))^q \in L^1(T,|\phi'|d\theta)$, by applying Hölder's inequality, we obtain

$$\int_T |\log w\circ\phi|\,d\theta = \left(\int_T |\log w\circ\phi|^q |\phi'|\,d\theta\right)^{1/q}\left(\int_T |\phi'|^{-q'/q}\right)^{1/q'} < \infty$$

and, then, as we said before, it holds.

ii) Let $d\nu = w\circ\phi|\phi'|d\theta$ the image measure of μ on T. By [5], we have $H^p(T,\nu) = K_{p,\nu} H^p(T)$, $0 < p < \infty$, where

$$K_{p,\nu} = \left(\frac{w\circ\phi|\phi'|}{c}\right)^{-1/p} \exp\left[\frac{-i}{p}\left(\log \frac{w\circ\phi|\phi'|}{c}\right)^{\sim}\right]$$

and $c = \exp \int_T \log(w\circ\phi|\phi'|)\,d\theta$. Using the same result for $d\alpha = w\circ\phi\,d\theta$ and $d\beta = |\phi'|d\theta$, we obtain ii), denoting $K_p = K_{p,\alpha}\circ\psi$. #

Remarks.

1) The condition $\log w \in L^q(\Gamma)$ is sharp, because, if $\log w \in L^r(\Gamma)$ always implies $H^p(\Gamma,\mu) \subsetneq L^p(\Gamma,\mu)$, then, $|\phi'|^{-r'/r} \in L^1$ (or, equivalently, $\int_T g|\phi'|^{-1/r} < \infty$ for all $g \in L^r$).

2) The sequence of polynomials $\{P_n(z)\}$, obtained by orthonormalization of the sequence $\{z^n\}$ on $H^2(\Gamma)$, is a basis of $H^2(\Gamma)$, (see [3]),

applying the preceding theorem $\{K_2P_n(z)\}$ is an orthonormal basis of $H^2(\Gamma,d\mu)$.

Corollary 1. *$H^r(\Gamma,\mu) = H^p(\Gamma,\mu) \cdot H^q(\Gamma,\mu)$ whenever $1/p+1/q = 1/r$.*

Proof. It suffices to show that $K_r = K_p . K_q$ and $H^r(\Gamma) = H^p(\Gamma).H^q(\Gamma)$ which is trivial.

III. BASIS IN $H^p(\Gamma,\mu)$ AND ITS DUAL SPACE

Let X be a Banach space and let X* be its dual space. The sequences $\{x_i\}$ in X and $\{x_i^*\}$ in X* form a biorthogonal system if $<x_1,x_i^*> = \delta_{i,j}$, $i,j \in N$. The sequence $\{x_i\}$ is a basis of X if and only if $\{x_i\}$ and $\{x_i^*\}$ form a biorthogonal system and for every $x \in X$ the serie $\sum_1^\infty <x,x_i^*> x_i$ converges to x (in the norm sense). Moreover, if X is reflexive, then, $\{x_i^*\}$ is a basis of X*, (see [7]).

Denote by $\hat{K}_{\nu,p} = K_{\nu,p}||K_{\nu,p}||^{-1}_{L^p(T,\nu)}$ and by $K_p = \hat{K}_{\nu,p} o \psi$. We start in this section with the following theorem.

Theorem 2. *$\{K_p\psi^n\}_0^\infty$ is a basis of $H^p(\Gamma,\mu)$ and $\{|K_p|^{p-2}K_p\psi^n\}$ is a basis of $H^p(\Gamma,\mu)^*$, $1 < p < \infty$.*

Proof. Since $H^p(\Gamma,\mu)$ is reflexive, we only have to show that $\{K_p\psi^n\}$ and $\{|K_p|^{p-2}K_p\psi^n\}$ form a biorthogonal system and that $\sum_0^n (\int_{pT}|K_p|^{p-2}\bar{K}_p\bar{\psi}^i d\mu)\psi_i$ converge, in $L^p(\Gamma,\mu)$ sense, to f, for all $f \in H^p(\Gamma,\mu)$. In order to prove this we have

$$\int_\Gamma K_p\psi^n|K_p|^{p-2}\bar{K}_p\bar{\psi}^m d\mu = \int_T e^{i(n-m)\theta}|\hat{K}_{\nu,p}|^p d\nu = \delta_{n,m}$$

Moreover, if $f \in H^p(\Gamma,\mu)$, $f o \phi = \hat{K}_{p,\nu}.h$, where $h \in H^p(T)$ and, then,

$$a_m = \int_\Gamma f|K_p|^{p-2}\bar{K}_p\bar{\psi}^m d\mu = \int_T he^{-im\theta}d\theta = \hat{h}(m) .$$

Hence, $||f- \sum_0^n a_m\psi^m K_p||_{L^p(\Gamma,\mu)} = ||h- \sum_0^n \hat{h}(m)e^{im\theta}||_{L^p(T)}$ and the theorem holds, because of convergence of Fourier series in L^p-norm if $1 < p < \infty$. #

This result permits us to offer a representation of dual space $H^p(\Gamma,\mu)^*$ in the following sense

Theorem 3. *If $1 < p < \infty$, there exists an isomorphism $T:H^p(\Gamma,\mu)^* \longrightarrow H^{p'}(\Gamma,\mu)$ which transforms the basis $\{\psi^n|K_p|^{p-2}K_p\}$ of $H^p(\Gamma,\mu)^*$ into the basis $\{K_{p'}\psi^n\}$ of $H^{p'}(\Gamma,\mu)$, $p^{-1}+(p')^{-1} = 1$.*

Proof. We only must prove that there exists a constant $C > 0$, such that

$$C^{-1}\|\sum_{0}^{n} a_m K_{p'}\psi^m\|_{H^{p'}(\Gamma,\mu)} \leq \|\sum_{0}^{n} a_m\psi^m|K_p|^{p-2}K_p\|_{H^p(\Gamma,\mu)^*} \leq$$

$$\leq C\|\sum_{0}^{n} a_m K_{p'}\psi^m\|_{H^{p'}(\Gamma,\mu)}$$

for all finite sequence $a_1,\ldots,a_n$ of complex numbers.

$H^p(T)^*$ is isomorphic (no isometric) to $H^{p'}(T)$ ([3]) and there is a constant $\lambda > 0$ such that

$$\lambda\|h\|_{H^{p'}(T)} \leq \|h\|_{H^p(T)^*} \leq \|h\|_{H^{p'}(T)}$$

for all $h \in H^{p'}(T)$. Denote, $f_n = \sum_{0}^{n} a_m\psi^m|K_p|^{p-2}K_p$. If $g \in H^p(\Gamma,\mu)$ $g \circ \phi = \widehat{K}_{p,\nu}.h$ with $h \in H^p(T)$ and $\|g\circ\phi\|_{H^p(T,\nu)} = \|h\|_{H^p(T)}$

$$<g,f_n> = \int_\Gamma g(\sum_{0}^{n} \bar{a}_m\bar{\psi}^m|K_p|^{p-2}\bar{K}_p)d\mu = \sum_{0}^{n} \bar{a}_m\int_T e^{-im\theta}h\,d\theta$$

Then, we have

$$\|f_n\|_{H^p(\Gamma,\mu)^*} = \sup\{|\sum_{0}^{n} a_m\hat{h}(m)| ; h \in H^p(T) , \|h\|_p \leq 1\} =$$

$$=\|\sum_{0}^{n} a_m e^{im\theta}\|_{H^p(T)^*}$$

But

$$\lambda\|\sum_{0}^{n} a_m e^{im\theta}\|_{H^{p'}(T)} \leq\|\sum_{0}^{n} a_m e^{im\theta}\|_{H^p(T)^*} \leq \|\sum_{0}^{n} a_m e^{im\theta}\|_{H^{p'}(T)}$$

and $\|\sum_{0}^{n} a_m\psi^m K_{p'}\|_{H^{p'}(\Gamma,\mu)} = \|\sum_{0}^{n} a_m e^{im\theta}\|_{H^{p'}(T)}$

Hence, the inequalities hold trivially. #

Now, we consider the subspace of $L^p(\Gamma,\mu)$, $\tilde{H}^p(\Gamma,\mu)$, spanned by the conjugate of polynomials $P(z) = \sum_{0}^{n} a_m z^m$. For this subspace, we may do a similar development to that of $H^p(\Gamma,\mu)$ and so we shall obtain $\tilde{H}^p(\Gamma,\mu) = \bar{K}_p\tilde{H}^p(\Gamma,ds)$. Since the sequences $\{K_p\psi^n\}_{-\infty}^{\infty}$ and $\{|K_p|^{p-2}K_p\psi^n\}_{-\infty}^{+\infty}$ are basis of $L^p(\Gamma,\mu)$ and $L^{p'}(\Gamma,\mu)$, respectively, it follows.

<u>Corollary 2</u>. *Let f a function of $L^p(\Gamma,\mu)$; f belongs to $H^p(\Gamma,\mu)$ if and only if*

$$\int_\Gamma f|K_p|^{p-2}\bar{K}_p\bar{\psi}^n d\mu = 0 \quad \textit{for all } n < 0 .$$

When $w = 1$ and $\Gamma = T$ we obtain the classical result $\hat{f}(n) = 0$ for all $n < 0$. #

IV. THE CONJUGATION OPERATOR

If f is a μ-measurable complex function defined on Γ and $f \circ \phi \in L^1(T)$ we may define $\tilde{f} = (f \circ \phi)^{\sim} \circ \psi$, where $(f \circ \phi)^{\sim}$ is the conjugation operator of $f \circ \phi$. If $P(z)$ is a polynomial and $f(z) = \mathrm{Re}(P(z)/\Gamma)$, then $\tilde{f} = \mathrm{Im}(P(z)/\Gamma)$. The conjugate function operator, defined on Γ is bounded from $L^p(\Gamma)$ into $L^p(\Gamma)$ if and only if $|\phi'| \in A_p(T)$, $1 < p < \infty$. Jones and Zinsmeister (see [6]) have proved that for every p there is a chord-arc curve Γ such that $|\phi'| \notin A_p$. Then, we must restrict our class of curves because we went that the conjugation operator is bounded for a fixed p. In order to do it, we consider a well known special class of curves.

Definition 3. Let Γ be a rectifiable Jordan curve. Γ is said *quasiregular* if for each $\varepsilon > 0$ there is a $\eta > 0$ such that if $z_1, z_2 \in \Gamma$ verifying $|z_1 - z_2| \leq \eta$, then $s(z_1, z_2) \leq (1+\varepsilon)|z_1 - z_2|$.

Γ is quasiregular if and only if $\log \phi' \in VMOA(D) = H^1(D) \cap VMO(T)$, where VMO(T) is the span of trigonometric polynomials in BMO(T). Particulary, if Γ is quasiregular, Γ is chord-arc and $|\phi'| \in A_p(T)$ for all $p > 1$ (see [9]).

Lemma 1. *If Γ is quasiregular and $w \in A_p(\Gamma)$, then, $w \circ \phi |\phi'| \in A_p(T)$.*

Proof. Let J be an arc of Γ and $\psi(J) = I$ the corresponding arc of T. Since $w \in A_p(\Gamma)$, $w \in A_{p-\varepsilon}(\Gamma)$ for some $\varepsilon > 0$, and by using Hölder's inequality, we have

$$(\frac{1}{|I|}\int_I w \circ \phi|\phi'|)(\frac{1}{|I|}\int_I (w \circ \phi|\phi'|)^{-1/p-1})^{p-1} \leq$$

$$\leq (\frac{1}{|I|}\int_I (w \circ \phi).|\phi'|).(\frac{1}{|I|}\int_I (w \circ \phi)^{-1/p-\varepsilon-1}.|\phi'|)^{p-\varepsilon-1}$$

$$(\frac{1}{|I|}\int_I |\phi'|^{-(p-\varepsilon)\varepsilon})^{\varepsilon} \leq$$

$$\leq (\frac{1}{|I|}\int_I w \circ\phi|\phi'|)(\frac{1}{|I|}\int_I (w \circ \phi)^{-1/p-\varepsilon-1}|\phi'|)^{p-\varepsilon-1} (\frac{|I|}{s(J)})^{p-\varepsilon} C \leq$$

$$\leq (\frac{1}{s(J)}\int_J w)(\frac{1}{s(J)}\int_J w^{-1/p-\varepsilon-1})^{p-\varepsilon-1} C'$$

and the conclusion of Lemma is proved.

Lemma 2. *Let f be a real function on T and $w = \exp(f)$. The following are equivalent:*

i) $f \in \overline{L^\infty(T)}_{BMO}$ (BMO-closure of $L^\infty(T)$).

ii) If $q > 1$, w and w^{-1} satisfy the reverse Hölder inequality (w and $w^{-1} \in RHI(q)$), i.e., there exists a constant $C_q > 0$ such that

$$\left(\frac{1}{|I|}\int_I w^q\right)^{1/q} \leq \frac{C_q}{|I|}\int_I w \quad \textit{for all intervals} \quad I \subseteq T$$

Proof. ii) $\Rightarrow$ i) .

Since w and $w^{-1} \in RHI(q)$ for all $q > 1$, is not difficult to prove that w^q and $w^{-q} \in A_\infty(T)$ for all $q > 1$ or equivalentely $w, w^{-1} \in A_p(T)$ $\forall p > 1$ or f belongs to BMO-closure of $L^\infty(T)$.

i) $\Rightarrow$ ii) .

As VMO(T) is in the closure in BMO of L^∞, for each $\varepsilon > 0$ we can put $f = f_1 + f_o$, where $f_1 \in L^\infty$, $f_o \in BMO$ and $||f_o||_* < \varepsilon$. Thus, $w = e^{f_1}.e^{f_o}$ and w is equivalent to $e^{f_o} = w_o$. By using John-Niremberg's inequality and Garnett-Jones theorems ([4]) there is a fixed constant C such that if $g \in BMO$ with $||g||_* < \varepsilon$, then exp (g) $\in A_2(T)$ with constant (smaller or equal to C) and therefore exp(g) $\in RHI(1+\delta)$ with $\delta > 0$. Particulary, $w_o \in RHI(1+\delta)$ and, also, $w \in RHI(1+\delta)$. By applying the same argument to the function qf(q > 1) , we get $w^q \in RHI(1+\delta)$. Choosing $q = 1+\delta$, we obtain $w \in RHI(1+\delta)^2$ and by iterating this argument, we conclude $w \in RHI(q)$, for all $q > 1$.

The same reasons work for -f and the result holds also for w^{-1}.

Theorem 4. *If Γ is quasiregular then, the conjugation operator is bounded on $L^p(\Gamma, wds)$ $(1 < p < \infty)$ if and only if $w \in A_p(\Gamma)$.*

Proof. Since the conjugate function operator is bounded on $L^p(\Gamma, w\circ\phi|\phi'|d\theta)$ if and only if $w\circ\phi.|\phi'| \in A_p(T)$ the "if part" of the theorem is an inmediate consequence of lemma 1.

For the converse we suppose that $w\circ\phi|\phi'| \in A_p(T)$ and then, for some $\varepsilon > 0$, $w\circ\phi.|\phi'| \in A_{p-\varepsilon}(T)$. Since Γ is quasiregular then $|\phi'|, |\phi'|^{-1} \in RHI(q)$ for all $q > 1$ (lemma 2). Thus

$$\left(\frac{1}{s(J)}\int_J w\right)\left(\frac{1}{s(J)}\int_J w^{-1/p-1}\right)^{p-1} \leq \left(\frac{1}{|I|}\int_I w\circ\phi.|\phi'|\right).$$

$$\left(\frac{1}{|I|}\int_I (w\circ\phi|\phi'|)^{-1/p-\varepsilon-1}\right)^{p-\varepsilon-1}\left(\frac{1}{|I|}\int_I |\phi'|^{p/\varepsilon}\right)^{\varepsilon}\left(\frac{|I|}{s(J)}\right)^p \leq C$$

In a similar way as in the case $|z| = 1$,

Corollary 3. *If Γ is quasiregular, then $w \in A_p(\Gamma)$ iff*

$$L^p(\Gamma,\mu) = H^p(\Gamma,\mu) \oplus \tilde{H}^p_o(\Gamma,\mu) \quad \textit{where} \quad \tilde{H}^p_o(\Gamma,\mu) = \bar{z}\,\tilde{H}^p(\Gamma,\mu) .$$

Remark 1. In the proof of preceding theorem we only use $\log|\phi'| \in VMO$, therefore, $|\phi'|, |\phi'|^{-1} \in A_p$ for all $p > 1$. Quasiregular curves verify this condition and also every curve which is transformed of a quasiregular curve by a conformal mapping with bounded derivate (they are not neccesarily quasiregular). The class of curves (boundaries of Smirnov domains) for which the conjugate function operator is bounded, strictly contains the quasiregular curves.

Remark 2. Let $Tf(z) = P.V.\int_\Gamma \frac{f(w)}{w-z}\, ds(w)$ a singular integral on Γ.

For $\Gamma = T$, it is known that T is bounded on $L^p(T)$ if and only if $\sim$ is bounded on $L^p(T)$. We could ask if the same is true for general Γ's, the answer is no. Indeed, T is bounded on $L^2(\Gamma)$ iff Γ is regular (see [2]) and if $\sim$ is bounded then Γ is regular (see [10]), but the converse is not true (see [6]).

Abounding in these reasons we have: if we denote

$H^2(\Omega_1)$ the closure on $L^2(\Gamma)$ of the polynomials in z and

$H^2(\Omega_2)$ the closure on $L^2(\Gamma)$ of the polynomials in z^{-1},

then T is bounded on $L^2(\Gamma)$ iff $L^2(\Gamma) = H^2(\Omega_1) \oplus H^2(\Omega_2)$ ([2])

while $\sim$ is bounded on $L^2(\Gamma)$ iff $L^2(\Gamma) = H^2(\Gamma) \oplus \bar{z}\,\tilde{H}^2(\Gamma)$.

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