A NOTE ON A_p WEIGHTS: PASTING WEIGHTS AND CHANGING VARIABLES

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ABSTRACT. For two weights u, w on $\mathbb{R}^n$, we show that $w \in A_{p,u}$ (the Muckenhoupt class of weights) if and only if $wu \in A_p$ and $wu^{1-p} \in A_p$, under the assumption that $u \in A_r$ for every $r > 1$. We also prove a rather general result on pasting weights on $\mathbb{R}$ that satisfy the A_p condition.

1. INTRODUCTION

$A_p(\mathbb{R}^n)$ weights (see below for an intrinsic definition) were introduced by Muckenhoupt [8]. They are exactly those weight functions on $\mathbb{R}^n$ for which the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy \tag{1}$$

is bounded on $L^p(w)$. Here, the supremum is taken over all the cubes $Q \subseteq \mathbb{R}^n$ containing x and $|Q|$ denotes the Lebesgue measure of Q.

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When another (doubling) measure μ replaces the Lebesgue measure in the definition of the maximal function, then the corresponding $A_{p,\mu}$ weights play the same role (see [1]).

To be precise, let μ be a positive Borel measure on $\mathbb{R}^n$, $1 < p < \infty$ and let p' be the conjugate exponent: $1/p + 1/p' = 1$. If $\Omega \subseteq \mathbb{R}^n$, then $A_{p,\mu}(\Omega)$ denotes the class of weights (i.e., μ-measurable, nonnegative functions defined on Ω) satisfying Muckenhoupt's condition: there exists some positive constant C such that

$$\int_Q w\,d\mu \left(\int_Q w^{-p'/p}\,d\mu\right)^{p/p'} \le C\mu(Q)^p$$

for every cube $Q \subseteq \Omega$. We will write $A_{p,\mu}(\Omega, w)$ for the least constant C.

We write $A_{p,u}(\Omega)$ if $d\mu(x) = u(x)\,dx$, and $A_p(\Omega)$ if $u \equiv 1$, i.e., μ is the Lebesgue measure on Ω. We omit Ω if there is no ambiguity.

The $A_p(\mathbb{R})$ classes also characterize the boundedness of the Hilbert transform on $L^p(w)$, see [4]. The same applies, for instance, to $A_p([0, 2\pi])$ weights and Fourier series, or $A_p([-1, 1])$ weights and Fourier expansions in Chebyshev polynomials (actually, Fourier series on $[0, 2\pi]$ and Fourier expansions in Chebyshev polynomials are closely related via a change of variable). In general, the A_p condition is sufficient for the boundedness of Calderón-Zygmund operators and, in some sense, it is also necessary. We refer the reader to [2, 1] for further details on these topics.

In this context, the relation between different A_p classes is certainly interesting. We refer, for instance, to the relation between "weighted" and "unweighted" clases, i.e., $A_{p,u}$ and A_p. In section 2, we state a result of this type and give some illustrating example; in section 3 we give a very simple proof. In particular, some results of Johnson and Neugebauer [5, 6] follow, relating the A_p conditions for a weight w on $\mathbb{R}$ and the weight $w \circ h$, where h is a given change of variable.

A different, yet also interesting question is the construction of examples of A_p weights. Here, the simplest case is $w(x) = |x|^a$, which belongs to $A_p([0,1])$ if and only if $-1 < a < p-1$. Indeed, this can be checked by simply computing the integrals in the A_p condition. The same holds if we replace $[0,1]$ by $[0,\infty)$ or $\mathbb{R}$. Obviously, the same characterization remains true for power weights $w(x) = |x-b|^a$, but the computations are not so straightforward in the case of

$$w(x) = \prod_{j=1}^{N} |x - t_j|^{a_j},$$

which can be considered essentially as the result of pasting simple power weights, in the sense that w behaves like $|x-t_j|^{a_j}$ near t_j. A contribution on this subject was made by Schröder [10]: if $w \in A_p((a,0])$,

$w \in A_p([0,b))$ and

$$(2) \qquad 0 < \liminf_{\varepsilon \to 0} \frac{\int_0^\varepsilon w(x)\,dx}{\int_{-\varepsilon}^0 w(x)\,dx} \le \limsup_{\varepsilon \to 0} \frac{\int_0^\varepsilon w(x)\,dx}{\int_{-\varepsilon}^0 w(x)\,dx} < \infty,$$

then $w \in A_p((a,b))$. In section 4 we give an elementary proof that under some mild conditions we can paste A_p weights so as to obtain another A_p weight.

2. Change of variables

Proposition 1. *Let u, w be two weights on $\Omega \subseteq \mathbb{R}^n$, $1 < p < \infty$. Then,*

$$wu \in A_p,\ wu^{1-p} \in A_p \Longrightarrow w \in A_{p,u}.$$

Remark 1. Actually, we will prove that that $A_{p,u}(w) \le A_p(wu)A_p(wu^{1-p})$.

Proposition 2. *Let u, w be two weights on $\Omega \subseteq \mathbb{R}^n$, $1 < p < \infty$. Assume that $u \in \cap_{r>1} A_r$. Then,*

$$w \in A_{p,u} \Longrightarrow wu \in A_p,\ wu^{1-p} \in A_p.$$

Remark 2. It follows from the proof that

$$A_p(wu) \le A_r(u)^{\lambda p/(p'\delta')} A_{p,u}(w^\delta)^{1/\delta}, \qquad \lambda = p'\delta' - 1,\ r = 1 + 1/\lambda;$$

$$A_p(wu^{1-p}) \le A_r(u)^{\lambda/\delta'} A_{p,u}(w^\delta)^{1/\delta}, \qquad \lambda = p\delta' - 1,\ r = 1 + 1/\lambda;$$

here, $\delta > 1$ is such that $w^\delta \in A_{p,u}$.

Remark 3. The assumption that $u \in \cap_{r>1} A_r$ in Proposition 2 is necessary in the following sense: let u be a weight on $\Omega \subseteq \mathbb{R}^n$, take some $1 < p < \infty$ and suppose that $wu \in A_p$ for every $w \in A_{p,u}$. Then, $u \in \cap_{r>1} A_r$. Indeed, if M is the (unweighted) Hardy-Littlewood maximal operator (1), we have

$$\int |Mf(x)|^p w(x)u(x)\,dx \le C \int |f(x)|^p w(x)u(x)\,dx, \qquad \forall w \in A_{p,u}$$

(since $wu \in A_p$). Then, Rubio de Francia's extrapolation theorem [9, Theorem 3] gives

$$\int |Mf(x)|^r w(x)u(x)\,dx \le C \int |f(x)|^r w(x)u(x)\,dx, \qquad \forall w \in A_{r,u}$$

for every $1 < r < \infty$. Taking $w \equiv 1$ yields $u \in A_r$.

Corollary 3 (change of variable)**.** *Let Ω_1, Ω_2 be intervals in $\mathbb{R}$, $h : \Omega_1 \longrightarrow \Omega_2$ bijective and absolutely continuous, and let h^{-1} be its inverse function. Let w be a weight on Ω_1, $1 < p < \infty$.*

a) *If $w|h'| \in A_p(\Omega_1)$ and $w|h'|^{1-p} \in A_p(\Omega_1)$, then $w \circ h^{-1} \in A_p(\Omega_2)$.*

b) *Assume that $|h'| \in \cap_{r>1} A_r(\Omega_1)$. If $w \circ h^{-1} \in A_p(\Omega_2)$, then $w|h'| \in A_p(\Omega_1)$ and $w|h'|^{1-p} \in A_p(\Omega_1)$.*

Proof of the corollary. Taking into account that h transforms intervals into intervals, it is straightforward to check that $w \circ h^{-1} \in A_p$ if and only if $w \in A_{p,|h'|}$. We only need to take $u = |h'|$ in propositions 1 and 2. $\square$

Remark 4. If $w_1, w_2 \in A_p$ and $0 \le \lambda \le 1$, then $w_1^{\lambda} w_2^{1-\lambda} \in A_p$, by Hölder's inequality. Hence, under the hypothesis of Proposition 2, $wu^{\alpha} \in A_p$ for $1 - p \le \alpha \le 1$. In terms of a change of variable in $\mathbb{R}$, we have as a corollary:

$$v \in A_p(\Omega_2) \Longrightarrow (v \circ h) \cdot |h'|^{\alpha} \in A_p(\Omega_1), \qquad 1 - p \le \alpha \le 1.$$

This result was proved by Johnson and Neugebauer in [5, Theorem 2.7] (for the case $0 \le \alpha \le 1$) and [6, Corollaries 3.1 and 3.4] (on the full range $1 - p \le \alpha \le 1$). In fact, our proof of Proposition 2 and the discussion on the necessity of $u \in \cap_{r>1} A_r$ are a simplified version of the proof of [5, Theorem 2.7]. Also, we must remark that in the case $n = 1$ our Proposition 2 could be deduced from [6, Corollaries 3.1 and 3.4], since for each weight function u on $\mathbb{R}$ there is some h with $u = |h'|$.

Example (maximal operator of Fourier-Jacobi series)**.** Let us take $\alpha, \beta \ge -1/2$ and consider the Fourier-Jacobi series associated to the measure $d\mu^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}\, dx$. In other words, this is the Fourier expansion associated to the Jacobi polynomials of order (α, β), which are orthogonal on $(-1, 1)$ with respect to $\mu^{(\alpha,\beta)}$.

Let us write $\mu'(x) = (1-x)^\alpha(1+x)^\beta$, let u be a weight on $(-1,1)$ and take

$$w(t) = u(\cos t)(1-\cos t)^{(2-p)(2\alpha+1)/4}(1+\cos t)^{(2-p)(2\beta+1)/4}.$$

Following some results of J. E. Gilbert, it was proved in [3] that under condition $w \in A_p((0,\pi))$ the maximal operator $S^*_{\alpha,\beta}$ of the Fourier-Jacobi series is bounded on $L^p(ud\mu^{(\alpha,\beta)})$. Now, we can traslate this A_p condition into the interval $(-1,1)$: apply Corollary 3 to the weight

$$V(x) = u(x)(1-x)^{(2-p)(2\alpha+1)/4}(1+x)^{(2-p)(2\beta+1)/4},$$

with $h(x) = \arccos x$, $h : (-1,1) \longrightarrow (0,\pi)$. A direct proof that $|h'(x)| = (1-x^2)^{-1/2}$ satisfies the A_r hypothesis can be given, but either Schröder's result or our Proposition 4 below can be successfully used, as well. Then, Corollary 3 yields

$$w \in A_p(0,\pi) \iff u(x)(1-x^2)^{\pm p/4}(\mu')^{1-p/2} \in A_p(-1,1).$$

Thus, the two A_p conditions on the right are sufficient for the boundedness of the maximal operator $S^*_{\alpha,\beta}$. Actually, they are also necessary even for the uniform boundedness of the Fourier-Jacobi series, at least for power-like weights (see [7]).

3. Proof of Propositions 1 and 2

Proof of Proposition 1. Let Q be a cube, $Q \subseteq \Omega$. By the hypothesis,

$$\int_Q wu \left(\int_Q w^{-p'/p} u^{-p'/p} \right)^{p/p'} \leq A_p(wu)|Q|^p,$$

$$\int_Q wu^{1-p} \left(\int_Q w^{-p'/p} u \right)^{p/p'} \leq A_p(wu^{1-p})|Q|^p,$$

where $|Q|$ is the Lebesgue measure of Q. Let $C = A_p(wu)A_p(wu^{1-p})$. It follows that

$$\begin{aligned}
&\int_Q wu \left(\int_Q w^{-p'/p} u \right)^{p/p'} \\
&\quad \leq C|Q|^{2p} \left(\int_Q w^{-p'/p} u^{-p'/p} \right)^{-p/p'} \left(\int_Q wu^{1-p} \right)^{-1} \\
&\quad = C \left(\int_Q u \right)^p \left[\frac{|Q|}{\left(\int_Q u \right)^{1/2} \left(\int_Q w^{-p'/p} u^{-p'/p} \right)^{1/2p'} \left(\int_Q wu^{-p/p'} \right)^{1/2p}} \right]^{2p} \\
&\quad \leq C \left(\int_Q u \right)^p,
\end{aligned}$$

where the last inequality follows from the three function Hölder's inequality applied to

$$1 = u^{1/2} \cdot [w^{-p'/p} u^{-p'/p}]^{1/2p'} \cdot [wu^{-p/p'}]^{1/2p}. \qquad \square$$

Proof of Proposition 2. Since $u \in \cap_{r>1} A_r$, for each $r > 1$ and each cube Q we have

$$\int_Q u \left(\int_Q u^{-1/(r-1)} \right)^{r-1} \leq A_r(u)|Q|^r.$$

Let us take $\lambda = 1/(r-1)$, that is: $r = 1 + 1/\lambda$; for each $\lambda > 0$ we have

$$\left(\int_Q u\right)^{\lambda} \int_Q u^{-\lambda} \leq A_r(u)^{\lambda} |Q|^{\lambda+1}. \tag{3}$$

a) Let us prove that $wu \in A_p$. Let $\delta > 1$ be such that $w^{\delta} \in A_{p,u}$ (see [1, 2]). Take $1/\delta + 1/\delta' = 1$. Let Q be any cube contained in Ω. By Hölder's inequality,

$$\int_Q wu \leq \left(\int_Q w^{\delta} u\right)^{1/\delta} \left(\int_Q u\right)^{1/\delta'},$$

$$\int_Q w^{-p'/p} u^{-p'/p} = \int_Q w^{-p'/p} u^{-p'} u \leq \left(\int_Q w^{-p'\delta/p} u\right)^{1/\delta} \left(\int_Q u^{1-p'\delta'}\right)^{1/\delta'}.$$

Taking this into account and the fact that $w^{\delta} \in A_{p,u}$,

$$\begin{aligned}
&\int_Q wu \left(\int_Q w^{-p'/p} u^{-p'/p}\right)^{p/p'} \\
&\quad \leq \left[\int_Q w^{\delta} u \left(\int_Q (w^{\delta})^{-p'/p} u\right)^{p/p'}\right]^{1/\delta} \left(\int_Q u\right)^{1/\delta'} \left(\int_Q u^{1-p'\delta'}\right)^{p/(p'\delta')} \\
&\quad \leq A_{p,u}(w^{\delta})^{1/\delta} \left(\int_Q u\right)^{p/\delta + 1/\delta'} \left(\int_Q u^{1-p'\delta'}\right)^{p/(p'\delta')} \\
&\quad = A_{p,u}(w^{\delta})^{1/\delta} \left[\left(\int_Q u\right)^{p'\delta'-1} \int_Q u^{1-p'\delta'}\right]^{p/(p'\delta')} \\
&\quad \leq A_{p,u}(w^{\delta})^{1/\delta} A_r(u)^{\lambda p/(p'\delta')} |Q|^p,
\end{aligned}$$

where in the last inequality we use (3) with $\lambda = p'\delta' - 1$ and for the previous step

$$(p'\delta' - 1)\frac{p}{p'\delta'} = p - \frac{p}{p'\delta'} = \frac{p}{\delta} + \frac{p}{\delta'}\left(1 - \frac{1}{p'}\right) = \frac{p}{\delta} + \frac{1}{\delta'}.$$

b) Let us now prove that $wu^{1-p} \in A_p$. Part (a) can be conveniently modified so as to get a direct proof. Alternatively, the elementary fact that for any v, μ, $1 < s < \infty$

$$v \in A_{s,\mu} \iff v^{-s'/s} \in A_{s',\mu}$$

with $A_{s',\mu}(v^{-s'/s}) = A_{s,\mu}(v)^{s'/s}$, together with part (a) gives

$$w \in A_{p,u} \iff w^{-p'/p} \in A_{p',u} \Longrightarrow w^{-p'/p}u \in A_{p'} \iff wu^{1-p} \in A_p,$$

and the appropriate relation for the A_p constants follows as well. $\square$

4. Pasting A_p weights

In this section $n = 1$, i.e., μ is a Borel measure on $\mathbb{R}$ and we deal with weights defined on a measurable subset of $\mathbb{R}$.

Remark 5. Assume that J is an interval, $\mu(J) < \infty$, $w \in A_{p,\mu}(J)$ and $w \not\equiv 0$, i.e., w is not (μ almost everywhere) the null weight on J. Then,

$$\int_A w \, d\mu > 0$$

for every measurable subset $A \subseteq J$ of positive measure, since otherwise we would have $w = 0$ μ-almost everywhere on A,

$$\int_J w^{-p'/p} \, d\mu = +\infty,$$

and the $A_{p,\mu}$ condition on the whole interval J would yield $w \equiv 0$ on J.

Proposition 4. *Let Ω be an open interval on $\mathbb{R}$, μ a Borel measure on Ω with $\operatorname{supp}\mu = \Omega$, and w a weight on Ω. Assume that there exist some open intervals $J_0, J_1, \ldots, J_N$ such that*

(a) $\Omega = \cup_{k=0}^{N} J_k$;

(b) $J_0, J_1, \ldots, J_{N-1}$ *have finite measure;*

(c) $w \in A_{p,\mu}(J_k)$, *for every* $k = 0, 1, \ldots, N$;

(d) $w \not\equiv 0$ *on* J_k, *for every* $k = 0, 1, \ldots, N-1$.

Then, $w \in A_{p,\mu}(\Omega)$.

Remark 6. Obviously, the intervals J_k cannot be disjoint, rather they overlap. But the notation $J_0, J_1, \ldots, J_N$ means no particular order. Regarding condition (d), it makes the proof easier at some point, but actually it is not necessary. Indeed, if we take Remark 5 into account and the fact that the J_k overlap, omitting condition (d) essentially leads to the following situation:

$$\Omega = J_1 \cup J_2 \cup J_3,$$

$$J_1 = (a, b), \qquad J_2 = (b - \delta, c + \delta), \qquad J_3 = (c, d),$$

$$w \equiv 0 \text{ on } J_1 \cup J_3, \qquad w \in A_{p,\mu}(J_2),$$

$$\mu((b, b + \varepsilon)) = \infty, \ \forall \varepsilon > 0,$$

$$\mu((c - \varepsilon, c)) = \infty, \ \forall \varepsilon > 0.$$

It is then immediate that $w \in A_{p,\mu}(\Omega)$.

Remark 7. If μ is the Lebesgue measure on an interval $\Omega \subseteq \mathbb{R}$, then condition (b) yields $\Omega \neq \mathbb{R}$. This condition cannot be just omitted, as the following example shows: consider

$$w(x) = \begin{cases} (1+x)^a, & \text{if } x \geq 0 \\ (1-x)^b, & \text{if } x < 0 \end{cases}$$

with $-1 < a < p-1$, $-1 < b < p-1$ and $a < b$. It is easy to check that $w \in A_p((-1/2, \infty))$ and $w \in A_p((-\infty, 1/2))$. However, $w \notin A_p(\mathbb{R})$: for the interval $I = (-n, n)$, easy computations yield

$$\int_I w \sim n^{1+b},$$
$$\int_I w^{-p'/p} \sim n^{1-a/(p-1)},$$

so that

$$\int_I w \left(\int_I w^{-p'/p} \right)^{p/p'} \sim n^{p+b-a}$$

and the A_p condition fails.

Remark 8. Proposition 4 implies Schröder's result, since under condition (2) it follows that $w \in A_p((a, \varepsilon))$ and $w \in A_p((-\varepsilon, b))$ for some $\varepsilon > 0$.

Proof of Proposition 4. Let I be a nonempty interval, $I \subseteq \Omega$. We must prove that there is some constant C, independent of I, such that

$$\int_I w\,d\mu \left(\int_I w^{-p'/p}\,d\mu\right)^{p/p'} \leq C\mu(I)^p. \tag{4}$$

If $I \subseteq J_k$ for some k, we are done, by hypothesis (obviously, a common constant can be chosen for all the $A_{p,\mu}$ conditions). We can therefore suppose now that for every $k \in \{0, 1, \ldots, N\}$, $I \not\subseteq J_k$. There must be some $m \in \{1, 2, \ldots, N\}$ such that

$$I \subseteq \cup_{k=0}^{m} J_k, \qquad I \not\subseteq \cup_{k=0}^{m-1} J_k.$$

Now, let us show that (4) holds with some constant which depends on m, but not on I (then, the biggest constant will work for every interval). We claim that

$$\int_I w\,d\mu \leq C \int_{I\cap J_m} w\,d\mu \tag{5}$$

and

$$\int_I w^{-p'/p}\,d\mu \leq C \int_{I\cap J_m} w^{-p'/p}\,d\mu, \tag{6}$$

with some constants depending on m, but not on I. If this is true (it will be proved below), then our result follows immediately:

$$\begin{aligned}\int_I w\,d\mu \left(\int_I w^{-p'/p}\,d\mu\right)^{p/p'} &\leq C \int_{I\cap J_m} w\,d\mu \left(\int_{I\cap J_m} w^{-p'/p}\,d\mu\right)^{p/p'}\\ &\leq C|I\cap J_m|^p\\ &\leq C|I|^p,\end{aligned}$$

where in the second inequality we use that $w \in A_{p,\mu}(J_m)$ and at each occurrence C denotes a different constant which depends only on m.

Thus, only (5) and (6) remain to be proved. Now, for every $k = 0, 1, \ldots, m-1$,

$$\int_{I\cap J_k} w\,d\mu \leq \int_{J_k} w\,d\mu < \infty. \tag{7}$$

The fact that the second integral is finite follows from the hypothesis that $w \in A_{p,\mu}(J_k)$ when applied to the whole J_k, which has finite measure.

On the other hand, since I and the J_k are intervals and

$$I \subseteq \cup_{k=0}^{m} J_k, \qquad I \not\subseteq \cup_{k=0}^{m-1} J_k, \qquad I \not\subseteq J_m,$$

it follows that there is some $n \leq m-1$ with $\emptyset \neq J_n \cap J_m \subseteq I \cap J_m$. Then,

$$\int_{I\cap J_m} w\,d\mu \geq \int_{J_n\cap J_m} w\,d\mu > 0. \tag{8}$$

The fact that the second integral cannot vanish follows from Remark 5 (with $J = J_n$), together with the trivial property that every open interval contained in $\Omega = \operatorname{supp}\mu$ has positive measure. Let us take

$$C_m = \min\left\{ \int_{J_n \cap J_m} w\, d\mu : \emptyset \neq J_n \cap J_m \right\}.$$

Then (7) and (8) yield

$$\int_{I\cap J_k} w\, d\mu \leq \frac{\int_{J_k} w\, d\mu}{C_m} \int_{I \cap J_m} w\, d\mu.$$

Summing up in $k = 0, 1, \dots, m-1$, we obtain

$$\int_I w\, d\mu \leq \int_{I\cap J_m} w\, d\mu + \sum_{k=0}^{m-1} \int_{I\cap J_k} w\, d\mu \leq C \int_{I\cap J_m} w\, d\mu,$$

where

$$C = 1 + \frac{1}{C_m} \sum_{k=0}^{m-1} \int_{J_k} w\, d\mu.$$

This proves inequality (5). For the proof of (6), just replace w by $w^{-p'/p}$. $\square$

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