ESTIMATES FOR COMMUTATORS OF ORTHOGONAL FOURIER SERIES

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Abstract. In this paper we study weighted norm inequalities for the commutators $[b, S_n]$ where b is a BMO function and S_n denotes the n-th partial sum of the Fourier series relative to a system of orthogonal polynomials on $[-1, 1]$ with respect to general weights. Results about generalized Jacobi and Bessel Fourier series are obtained.

0. Introduction.

Given a linear operator T acting on functions and a function b, and denoting by M_b the operator of pointwise multiplication by $b(x)$, the commutator of this operator and T is defined by

$$[b, T]f(x) = [M_b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

The first results on this commutator were obtained by Coifman, Rochberg and Weiss [CRW]. They proved that if H is the classical Hilbert transform (and also more general singular integrals) and $1 < p < \infty$, then $[b, H]$ is bounded in $L^p(\mathbb{R})$ if and only if $b \in \mathrm{BMO}(\mathbb{R})$. The boundedness of the commutator has been studied among others by Bloom [Bl] involving some weights and where b belongs to an appropriate weighted BMO space and by Segovia and Torrea ([ST 1]), who obtained a vector-valued commutator theorem for operators T including the Hilbert transform, and whose results apply to the Carleson operator, Littlewood-Paley sums, U.M.D. Banach spaces, Parabolic Differential Equations and Approximate Identities (for further references see [ST 2], [ST 3]).

Frequently, the boundedness of the commutator is related to the analytic behaviour of some operator. Let $(X, d\mu)$ be a σ-finite measure space and B a real Banach function space on $(X, d\mu)$. Consider the partial sums of Fourier series relative to the orthonormal polynomials on $L^2(X, d\mu)$, that is, for $f \in L^2(X, d\mu)$ and $x \in X$,

$$S_n f(x) = \int_X K_n(x, y) f(y)\, d\mu(y)$$

where $K_n(x, y)$ is the corresponding kernel. Let $\overline{B}$ be the complexification of B and, for each $b \in \overline{B}$, define $T_n(b) = M_{e^b} S_n M_{e^{-b}}$. Then

$$T_n(b)(f)(x) = \int_X \exp[b(x) - b(y)] K_n(x, y) f(y)\, d\mu(y).$$

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The boundedness of the operators $T_n(b)$ is equivalent to a weighted norm inequality for the operators S_n.

On the other hand, for this particular sequence $\{T_n\}_{n\in\mathbb{N}}$ of operator valued functions, the uniform boundedness in a neighbourhood of $0 \in \overline{B}$ implies the Gâteaux-differentiability (see [CM], [L]) and the Gâteaux-differential of T_n at 0 in the direction $b \in \overline{B}$ is

$$\frac{d}{dz} T_n(zb)\big|_{z=0} = [b, S_n].$$

In particular, these ideas (which came out in [CRW]) show that certain weighted norm inequalities for a basic operator T give information about the commutator $[b, T]$.

The purpose of this paper is to study the uniform boundedness of the commutator of the partial sums of Fourier series with respect to a class of weights which includes, as a particular case, generalized Jacobi Fourier series.

In a first step, we find necessary conditions for the uniform boundedness of the operators $[b, S_n]$ in $L^p(d\mu)$; also, for the uniform weak boundedness $[b, S_n] : L^p(d\mu) \longrightarrow L^{p,\infty}(d\mu)$ or the restricted weak boundedness $[b, S_n] : L^{p,1}(d\mu) \longrightarrow L^{p,\infty}(d\mu)$.

Here, $L^{p,r}(d\mu)$ stands for the classical Lorentz space of all measurable functions f satisfying

$$\|f\|_{L^{p,r}(d\mu)} = \left(\frac{r}{p}\int_0^\infty \left[t^{1/p} f^*(t)\right]^r \frac{dt}{t}\right)^{1/r} < \infty \quad (1 \le p < \infty,\ 1 \le r < \infty),$$
$$\|f\|_{L^{p,\infty}(d\mu)} = \|f\|_{L^p_*(d\mu)} = \sup_{t>0} t^{1/p} f^*(t) < \infty \quad (1 \le p \le \infty),$$

where f^* denotes the nonincreasing rearrangement of f. We refer the reader to [SW] for further information on these topics.

In a second step, we find sufficient conditions for the uniform boundedness of $[b, S_n]$ in $L^p(d\mu)$, which, in many cases, coincide with the necessary conditions previously found. We are concerned with the case $d\mu = w dx$, where w is a positive weight function.

We shall distinguish two cases: firstly, polynomial systems with uniform bounds (the class $\mathcal{H}$ defined below), where we follow the ideas of Coifman, Rochberg and Weiss. This is the case of Jacobi weights $(1-x)^\alpha(1+x)^\beta$ with $\alpha, \beta \ge -1/2$. Fourier Bessel series also fall in this scheme.

Next, we consider a more general setting (the class $\overline{\mathcal{H}}$) where these techniques do not work well and a more detailed examination of the kernels is required. Here, we reduce the problem to the boundedness of $[H, b]$ (where H is the Hilbert transform) in weighted L^p spaces by inserting A_p weights.

We shall need weighted estimates for the partial sums of the Fourier series. The problem of finding conditions on weights u, v such that

$$\|uS_n f\|_{L^p(w)} \le C\|vf\|_{L^p(w)} \quad \forall n \ge 0, \forall f \in L^p(v^p w) \tag{1}$$

has been solved only in some particular cases. For instance, Badkov gives in [B] necessary and sufficient conditions for (1) when $u = v$ and both u, w are generalized Jacobi weights; earlier results can be found in [P 1], [P 2], [W], [M 1]. For the two weight case, see [GPV 1], [GPRV 1], [GPRV 2]. Hermite and Laguerre series have been considered by Askey and Wainger ([AW]) and Muckenhoupt ([M 2]).

Bessel series have been studied by Wing [W], Benedek-Panzone [BP 1], [BP 2] and the authors [GPRV 3].

In this paper we shall see that the boundedness of the commutator for this type of series holds when the partial sums of the Fourier series are bounded.

1. Notations and main results.

Let $d\mu(x) = w(x)dx$, with $w \in L^1(dx)$ and $w > 0$ a.e. in $[-1, 1]$. Let $\{p_n\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to μ.

For $f \in L^1(w)$, let $S_n f$ denote the n-th partial sum of the Fourier expansion of f in $\{p_n\}_{n\geq 0}$, i.e.,

$$S_n f(x) = \int_{\mathbb{R}} f(y) K_n(x, y) w(y) dy, \quad K_n(x, y) = \sum_{k=0}^{n} p_k(x) p_k(y)$$

Throughout this paper, C will denote a constant, independent of n, f, but possibly different from line to line.

Let $1 < p < \infty$ and $-\infty \leq a < b \leq \infty$. The class $A_p(a, b)$ consists of those pairs of weights (u, v) such that

$$\left(\frac{1}{|I|}\int_I u(x)\,dx\right)\left(\frac{1}{|I|}\int_I v(x)^{-1/(p-1)}\,dx\right)^{p-1} \leq C$$

where I ranges over all finite intervals $I \subseteq (a, b)$ and $|I|$ stands for the length of the interval I. A weight u is said to belong to A_p if $(u, u) \in A_p$. We refer the reader to [GR] for further details on A_p classes.

We say that $(u, v) \in A_p^\delta(a, b)$ for $\delta > 1$ if $(u^\delta, v^\delta) \in A_p(a, b)$. With this definition we mean that a power of u and v greater than 1 belongs to A_p. We use the same exponent δ although it can change in each occurrence.

We shall take B the space of functions of bounded mean oscillation (BMO) on $[-1, 1]$. If $b \in L^1(dx)$, the mean of b on an interval I is

$$b_I = \frac{1}{|I|}\int_I b(x)\,dx.$$

The function b is said to have bounded mean oscillation on $[-1, 1]$ if

$$\|b\|_* = \sup_I \frac{1}{|I|}\int_I |b(\theta) - b_I|\,d\theta$$

is finite, where the supremum is taken over all intervals $I \subseteq [-1, 1]$. The space BMO of real-valued functions (modulo constants) having bounded mean oscillation on $[-1, 1]$ is a Banach space with $\|\cdot\|_*$ as its norm.

Theorem 1. *Let w be a weight on $[-1, 1]$, $w > 0$ a.e., $\{S_n\}_{n\geq 0}$ the Fourier series relative to $d\mu(x) = w(x)dx$, U, V two weights, $U, V^{-1} > 0$ a.e. Let $b \in$ BMO, $b \notin L^\infty$ and suppose that there exists some constant $C > 0$ with*

$$\|U[b, S_n](V^{-1}f)\|_{L^{p,r}(w)} \leq C\|f\|_{L^{q,s}(w)}$$

for each $n \geq 0$ and $f \in L^{q,s}(w)$ (where $1 < p < \infty$, $1 < q < \infty$; either $r = p$ or $r = \infty$; either $s = q$ or $s = 1$). Then,

$$\|bUw^{-1/2}(1-x^2)^{-1/4}\|_{L^{p,r}(w)} < \infty,$$

and

$$\|bV^{-1}w^{-1/2}(1-x^2)^{-1/4}\|_{L^{q',s'}(w)} < \infty.$$

Let $w(x)$ be a weight function on $[-1,1]$, $p_n(x)$ the corresponding orthonormal polynomials and $q_n(x)$ the orthonormal polynomials with respect to $(1-x^2)w(x)$. We say that w belongs to the class $\mathcal{H}$ of weights if it satisfies

i) $w(x) > 0$ a.e.,
ii) $|p_n(x)| \leq Cw(x)^{-1/2}(1-x^2)^{-1/4}$,
iii) $|q_n(x)| \leq Cw(x)^{-1/2}(1-x^2)^{-3/4}$.

The class $\mathcal{H}$ contains the generalized Jacobi weights

$$w(x) = \varphi(x)(1-x)^\alpha(1+x)^\beta \prod_{i=1}^{N} |x-x_i|^{\gamma_i}$$

where $\alpha, \beta \geq -1/2$, $\gamma_i \geq 0$ $(i = 1, 2, \ldots, N)$, $-1 < x_1 < \cdots < x_n < 1$, φ is positive and continuous on $[-1,1]$ and $\rho(\delta)/\delta \in L^1(0,2)$, ρ being the modulus of continuity of φ (see, e. g., [B]).

Theorem 2. *Let $1 < p < \infty$, $w \in \mathcal{H}$, U and V weights on $[-1,1]$ and $b \in \mathrm{BMO}$. If*

$$\left((1-x^2)^{-p/4}U^p w^{1-p/2}, (1-x^2)^{-p/4}V^p w^{1-p/2}\right) \in A_p^\delta(-1,1),$$

$$\left((1-x^2)^{p/4}U^p w^{1-p/2}, (1-x^2)^{p/4}V^p w^{1-p/2}\right) \in A_p^\delta(-1,1) \tag{2}$$

for some $\delta > 1$ ($\delta = 1$ when $U = V$), then the commutator $[b, S_n]$ is bounded from $L^p(V^p w)$ into $L^p(U^p w)$ uniformly in n.

For generalized Jacobi weights with $\alpha, \beta > -1$, $\gamma_i \geq 0$, the orthogonal polynomials do not have uniform bounds. We extend the class $\mathcal{H}$ of weights and say that a weight w belongs to the class $\overline{\mathcal{H}}$ if $w(x) = (1-x)^\alpha(1+x)^\beta w_1(x)$, where

i) $w(x) > 0$ a.e. and there exist $\varepsilon > 0$ and positive constants C_1 and C_2 such that $C_1 < w_1(x) < C_2$ for all $x \in (1-\varepsilon, 1)$ and $x \in (-1, -1+\varepsilon)$,
ii) $|p_n(x)| \leq C(1-x+a_n)^{-(\alpha/2+1/4)}(1+x+b_n)^{-(\beta/2+1/4)}w_1(x)^{-1/2}$,
iii) $|q_n(x)| \leq C(1-x+a_n)^{-(\alpha/2+3/4)}(1+x+b_n)^{-(\beta/2+3/4)}w_1(x)^{-1/2}$, where $\{a_n\}$ and $\{b_n\}$ are positive sequences such that $\lim\limits_n a_n = \lim\limits_n b_n = 0$.

Theorem 3. *Let $1 < p < \infty$, $w \in \overline{\mathcal{H}}$, $U(x) = (1-x)^a(1+x)^b u(x)$, $V(x) = (1-x)^A(1+x)^B v(x)$ with $u > 0$ a.e., $v > 0$ a.e. and such that $C_1 < u(x), v(x) < C_2$ for $x \in (1-\varepsilon, 1)$ and $x \in (-1, -1+\varepsilon)$. If $b \in \mathrm{BMO}$,*

$$\left|(\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{a+A}{2}\right| < \frac{a-A}{2} + \min\left\{\frac{1}{4}, \frac{\alpha+1}{2}\right\}, \quad A \leq a,$$

$$\left|(\beta+1)\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{b+B}{2}\right|<\frac{b-B}{2}+\min\left\{\frac{1}{4},\frac{\beta+1}{2}\right\},\quad B\le b$$

and

$$\left(u^p w_1^{1-p/2}, v^p w_1^{1-p/2}\right)\in A_p^\delta(-1,1)$$

for some $\delta>1$ *(*$\delta=1$ *when* $u=v$*), then the commutator* $[b,S_n]$ *is bounded from* $L^p(V^p w)$ *into* $L^p(U^p w)$ *uniformly in* n*.*

As a consequence of these results for generalized Jacobi weights, we obtain

Corollary 1. *Let* $1<p<\infty$*,*

$$w(x)=(1-x)^\alpha(1+x)^\beta\prod_{i=1}^{N}|x-x_i|^{\gamma_i}$$

with $x_i\in(-1,1)$*,* $x_i\neq x_j$ $\forall i\neq j$*,* $\alpha,\beta>-1$*,* $\gamma_i\ge 0$ $\forall i$ *and*

$$U(x)=(1-x)^a(1+x)^b\prod_{i=1}^{N}|x-x_i|^{g_i}.$$

Then the commutator $[b,S_n]$ *is uniformly bounded from* $L^p(U^p w)$ *into* $L^p(U^p w)$ *for each* $b\in\mathrm{BMO}$ *if and only if*

$$\left|a+(\alpha+1)\left(\frac{1}{p}-\frac{1}{2}\right)\right|<\min\left\{\frac{1}{4},\frac{\alpha+1}{2}\right\}\tag{3}$$

$$\left|b+(\beta+1)\left(\frac{1}{p}-\frac{1}{2}\right)\right|<\min\left\{\frac{1}{4},\frac{\beta+1}{2}\right\}\tag{4}$$

and

$$\left|g_i+(\gamma_i+1)\left(\frac{1}{p}-\frac{1}{2}\right)\right|<\min\left\{\frac{1}{2},\frac{\gamma_i+1}{2}\right\},\quad i=1,2,\dots,N.\tag{5}$$

Corollary 2. *With the same notation, inequalities (3), (4), (5) are also necessary for the weak and restricted weak* $(p,p)-$*boundedness of the conmutator* $[b,S_n]$ *for each* $b\in\mathrm{BMO}$*.*

Remark. Notice that, in contrast to this situation, the operators S_n are of restricted weak type when w is a Jacobi weight and p is an endpoint of the open interval determined by (3), (4), (5) (see [GPV 2]).

2. Proofs of the theorems.

Proof of Theorem 1. For each $0<L<K<\infty$, let us define

$$\mathcal{P}(L)=\{x\in[-1,1];\ |b(x)|<L\},$$
$$\mathcal{G}(K)=\{x\in[-1,1];\ K<|b(x)|\}.$$

Then, for each $x\in\mathcal{G}(K)$, $y\in\mathcal{P}(L)$ we have:

- $*$) $\operatorname{sgn}(b(x)-b(y))=\operatorname{sgn} b(x)$;
- $*$) $|b(y)|<L<\frac{L}{K}|b(x)|$, so that

$$|b(x)-b(y)|\ge|b(x)|-|b(y)|>\frac{K-L}{K}|b(x)|.$$

From the hypothesis it follows

$$\|u[b, S_n - S_{n-1}](v^{-1}f)\|_{L^{p,r}(w)} \le C\|f\|_{L^{q,s}(w)};$$

$$[b, S_n - S_{n-1}](v^{-1}f)(x) = p_n(x)\int_{-1}^{1}[b(x) - b(y)]p_n(y)v(y)^{-1}f(y)w(y)dy,$$

where $\{p_n\}$ are the orthonormal polynomials with respect to $w(x)dx$. Now, take $0 < L < K < \infty$ and

$$f(y) = [\operatorname{sgn} p_n(y)]\chi_{\mathcal{P}(L)}(y)|h(y)|,$$

h being any function in $L^{q,s}(w)$. Here and in the sequel, χ_A denotes the characteristic function on a measurable set A. For each $x \in \mathcal{G}(K)$,

$$\big|[b, S_n - S_{n-1}](v^{-1}f)(x)\big|$$

$$= |-p_n(x)\operatorname{sgn} b(x)| \int_{-1}^{1} |b(y) - b(x)|\ |p_n(y)|\ v(y)^{-1}\chi_{\mathcal{P}(L)}(y)\ |h(y)|\ w(y)dy$$

$$\ge \frac{K-L}{K}\ |p_n(x)|\ |b(x)|\ \|p_n v^{-1}\chi_{\mathcal{P}(L)}h\|_{L^1(w)}.$$

Thus,

$$\|u[b, S_n - S_{n-1}](v^{-1}f)\|_{L^{p,r}(w)}$$

$$\ge \frac{K-L}{K}\|\chi_{\mathcal{G}(K)}bup_n\|_{L^{p,r}(w)}\ \|p_n v^{-1}\chi_{\mathcal{P}(L)}h\|_{L^1(w)}$$

and therefore

$$\frac{K-L}{K}\|\chi_{\mathcal{G}(K)}bup_n\|_{L^{p,r}(w)}\ \|p_n v^{-1}\chi_{\mathcal{P}(L)}h\|_{L^1(w)} \le C\|f\|_{L^{q,s}(w)} \le C\|h\|_{L^{q,s}(w)}$$

for each $h \in L^{q,s}(w)$. By duality,

$$\frac{K-L}{K}\|\chi_{\mathcal{G}(K)}bup_n\|_{L^{p,r}(w)}\ \|p_n v^{-1}\chi_{\mathcal{P}(L)}\|_{L^{q',s'}(w)} \le C.$$

Also,

$$\frac{K-L}{KL}\|\chi_{\mathcal{G}(K)}bup_n\|_{L^{p,r}(w)}\ \|\chi_{\mathcal{P}(L)}bv^{-1}p_n\|_{L^{q',s'}(w)} \le C. \tag{6}$$

In a similar way, taking

$$f(y) = [\operatorname{sgn} b(y)]\operatorname{sgn} p_n(y)\chi_{\mathcal{G}(K)}(y)|h(y)|,$$

and $x \in \mathcal{P}(L)$, we obtain

$$\frac{K-L}{KL}\|\chi_{\mathcal{P}(L)}bup_n\|_{L^{p,r}(w)}\ \|\chi_{\mathcal{G}(K)}bv^{-1}p_n\|_{L^{q',s'}(w)} \le C. \tag{7}$$

Now, by a result of Máté, Nevai and Totik (see [MNT 2]),

$$C\|gw^{-1/2}(1-x^2)^{-1/4}\|_{L^p(w)} \leq \liminf_n \|gp_n\|_{L^p(w)}$$

for any measurable function g. A similar property holds in $L^{p,\infty}(w)$ (see [GPV 2]). Then, taking $\liminf$ in (6) and (7) we have

$$\|\chi_{\mathcal{G}(K)} buw^{-1/2}(1-x^2)^{-1/4}\|_{L^{p,r}(w)}\ \|\chi_{\mathcal{P}(L)} bv^{-1}w^{-1/2}(1-x^2)^{-1/4}\|_{L^{q',s'}(w)} < \infty,$$

$$\|\chi_{\mathcal{P}(L)} buw^{-1/2}(1-x^2)^{-1/4}\|_{L^{p,r}(w)}\ \|\chi_{\mathcal{G}(K)} bv^{-1}w^{-1/2}(1-x^2)^{-1/4}\|_{L^{q',s'}(w)} < \infty$$

for each $0 < L < K < \infty$. Since $b \notin L^\infty$ we have $\chi_{\mathcal{G}(K)} b \neq 0$ for every $K > 0$ and there exists some $L_0 > 0$ such that $\chi_{\mathcal{P}(L)} b \neq 0$ for every $L > L_0$. Now, $u, v^{-1} > 0$ almost everywhere, so that for $L_0 < L < K < \infty$ the above norms cannot vanish and as a consequence they cannot be ∞ as well. This proves the theorem. □

Proof of Theorem 2. Write

$$u(x) = (1-x^2)^{-p/4}U(x)^p w(x)^{1-p/2}, \quad v(x) = (1-x^2)^{-p/4}V(x)^p w(x)^{1-p/2},$$

$$\overline{u}(x) = (1-x^2)^{p/4}U(x)^p w(x)^{1-p/2}, \quad \overline{v}(x) = (1-x^2)^{p/4}V(x)^p w(x)^{1-p/2}.$$

The following lemmas will be proved below.

Lemma 1. *Assume that $w \in \mathcal{H}$ and let U and V be as above and satisfying*

$$(u, v) \in A_p^\delta \quad \textit{and} \quad (\overline{u}, \overline{v}) \in A_p^\delta$$

for some $\delta > 1$. Then

$$\|US_n f\|_{p,w} \leq C\|Vf\|_{p,w}$$

where C depends only on the A_p constants of (u, v) and $(\overline{u}, \overline{v})$.

Lemma 2. *Let $(u_1, v_1) \in A_p^\delta$ for some $\delta > 1$ and $b \in$ BMO. Then, there exist $\delta > 1$ and $\gamma > 0$ such that $(e^{sb}u_1, e^{sb}v_1) \in A_p^\delta$ for all s with $|s| < \gamma$, and the A_p constant is independent of s.*

Now, for a fixed function $b \in$ BMO and $n \in \mathbb{N}$, put

$$T_z f = e^{zb} S_n(e^{-zb} f), \quad z \in \mathbb{C}.$$

Let us show the analyticity of this operator-valued function. From the hypothesis and Lemma 2 it follows

$$(e^{sb}u, e^{sb}v) \in A_p^\delta \quad \text{and} \quad (e^{sb}\overline{u}, e^{sb}\overline{v}) \in A_p^\delta$$

for all s such that $|s| < \gamma$. Then, by Lemma 1, we have

$$\|e^{sb}US_n f\|_{p,w} \leq C\|e^{sb}Vf\|_{p,w}.$$

Therefore, for $|z| < \gamma$

$$\|UT_z f\|_{p,w} \leq C\|Vf\|_{p,w}.$$

Then, for $|z| < \gamma$, $T_z \in \mathcal{L}(L^p(V^p w), L^p(U^p w))$. Moreover, the constant C in the last inequality is independent of z, $|z| < \gamma$. So, the application T_z is bounded (with the operator norm) in $|z| < \gamma$. Then, in order to prove the analyticity in $|z| < \gamma$ it is enough to show that the mapping $z \mapsto \langle T_z f, g\rangle$ is holomorphic for every f in a dense subspace of $L^p(V^p w)$ and every g in a dense subspace of the dual of $L^p(U^p w)$ (see [K, p. 365]).

If f, g are bounded functions we can differentiate the expression

$$\langle T_z f, g\rangle = \int_{-1}^{1}\int_{-1}^{1} e^{z(b(x)-b(y))} K_n(x,y) f(x) g(y) U(x)^p w(x) w(y)\, dx\, dy$$

by differentiating under the integral sign, since the derivative of the integrand can be dominated by

$$C e^{\gamma|b(x)-b(y)|}|b(x)-b(y)|\ |K_n(x,y)| U(x)^p w(x) w(y)$$

which is integrable on $[-1,1]\times[-1,1]$. This follows from a suitable handling of the hypothesis (integrability conditions which are implicit in the A_p^δ conditions (2), $b \in \mathrm{BMO}$ and $w \in \mathcal{H}$).

Besides, this process shows that

$$\frac{d}{dz} T_z\Big|_{z=0} = [b, S_n].$$

Therefore, $[b, S_n]$ is a bounded operator from $L^p(V^p w)$ into $L^p(U^p w)$. Moreover, by Cauchy's integral theory, the norm of $[b, S_n]$ is controlled by the maximum of the norms of T_z (which are independent of n) when z ranges in a circle and hence the norms of $[b, S_n]$ are independent of n. This concludes the proof of Theorem 2. □

Proof of Lemma 1.

The main idea of this proof comes out in [P 1] (see also [GPV 1]). We use Pollard's decomposition of the kernels $K_n(x,y)$,

$$K_n(x,y) = r_n T_{1,n}(x,y) + s_n T_{2,n}(x,y) + s_n T_{3,n}(x,y)$$

where

$$T_{1,n}(x,y) = p_{n+1}(x) p_{n+1}(y),$$

$$T_{2,n}(x,y) = (1-y^2)\frac{p_{n+1}(x) q_n(y)}{x-y},$$

$$T_{3,n}(x,y) = (1-x^2)\frac{p_{n+1}(y) q_n(x)}{y-x}$$

and $\{r_n\}$, $\{s_n\}$ are bounded sequences. In fact, for any measure $d\mu$ on $[-1,1]$ with $\mu' > 0$ a.e.,

$$\lim_n r_n = -1/2, \quad \lim_n s_n = 1/2$$

(this can be deduced from [P 1] and either [R] or [MNT 1]). Therefore, we can write

$$S_n f = r_n W_{1,n} f + s_n W_{2,n} f - s_n W_{3,n} f$$

where

$$W_{1,n}f(x) = p_{n+1}(x)\int_{-1}^{1} p_{n+1}fw,$$

$$W_{2,n}f(x) = p_{n+1}(x)H((1-y^2)q_n fw, x)$$

and

$$W_{3,n}f(x) = (1-x^2)q_n(x)H(p_{n+1}fw, x),$$

H being the Hilbert transform on the interval $[-1,1]$. Thus, the study of S_n can be reduced to that of $W_{i,n}$ $(i = 1, 2, 3)$.

$i = 1$): By using the uniform estimates for p_n and q_n and Hölder's inequality with $1/p + 1/p' = 1$, we have

$$\|UW_{1,n}f\|_{p,w} = \|Up_{n+1}\|_{p,w}\left|\int_{-1}^{1} p_{n+1}(y)f(y)w(y)\,dy\right|$$

$$\leq C\|U(x)w(x)^{-1/2}(1-x^2)^{-1/4}\|_{p,w}\|V(x)^{-1}w(x)^{-1/2}(1-x^2)^{-1/4}\|_{p',w}\|Vf\|_{p,w}.$$

From the A_p conditions in the hypothesis it follows

$$U(x)^p w(x)^{1-p/2}(1-x^2)^{-p/4} \in L^1(dx)$$

and

$$(V(x)^p w(x)^{1-p/2}(1-x^2)^{p/4})^{-p'/p} \in L^1(dx),$$

that is,

$$\|U(x)w(x)^{-1/2}(1-x^2)^{-1/4}\|_{p,w} < \infty$$

and

$$\|V(x)^{-1}w(x)^{-1/2}(1-x^2)^{-1/4}\|_{p',w} < \infty.$$

Therefore

$$\|UW_{1,n}f\|_{p,w} \leq C\|Vf\|_{p,w}.$$

$i = 2$): Since

$$\Big((1-x^2)^{-p/4}w^{1-p/2}U^p, (1-x^2)^{-p/4}w^{1-p/2}V^p\Big) \in A_p^\delta(-1,1)$$

for some $\delta > 1$, the Hilbert transform is bounded from $L^p((1-x^2)^{-p/4}w^{1-p/2}V^p)$ into $L^p((1-x^2)^{-p/4}w^{1-p/2}U^p)$ (it is a consequence of Theorem 3 in [N]).

Write $g(y) = (1-y^2)q_n(y)f(y)w(y)$. Then

$$\|UW_{2,n}f\|_{p,w} = \|Up_{n+1}Hg\|_{p,w}$$

$$\leq C\|U(1-x^2)^{-1/4}w^{-1/2}Hg\|_{p,w}$$

$$\leq C\|V(1-x^2)^{-1/4}w^{-1/2}g\|_{p,w} \leq C\|Vf\|_{p,w}.$$

$i = 3$): It can be done in a similar way using the second A_p^δ-condition. $\square$

Proof of Lemma 2.

For any interval I, write

$$I(f) = \frac{1}{|I|}\int_I f(x)\,dx.$$

The condition $(u_1, v_1) \in A_p^\delta$ can be written

$$I(u_1^\delta) I(v_1^{-\delta/(p-1)})^{p-1} \le C \quad \text{for each interval.}$$

It is known [N] that there exists $\delta > 1$ such that $(u_1^\delta, v_1^\delta) \in A_p$ if and only if there is some $\sigma \in A_p$ with $C_1 u_1 \le \sigma \le C_2 v_1$, where δ and the A_p constants depend on each other. In order to prove that $(e^{sb}u_1, e^{sb}v_1) \in A_p^\delta$ it is enough to show that $e^{sb}\sigma \in A_p$ uniformly in s.

Since $\sigma \in A_p$, by the reverse Hölder's inequality there exists $\varepsilon > 1$ such that $\sigma^\varepsilon \in A_p$. As $b \in \mathrm{BMO}$ (and also $-b \in \mathrm{BMO}$), by the John-Nirenberg inequality there exists $\lambda > 0$ small enough such that $e^{sb} \in A_p$ for $|s| < \lambda$ uniformly in s, that is, with an A_p constant independent of s (see [GR]).

By Hölder's inequality with $1/\varepsilon + 1/\varepsilon' = 1$,

$$I(e^{sb}\sigma) \le I(\sigma^\varepsilon)^{1/\varepsilon} I(e^{\varepsilon' sb})^{1/\varepsilon'}$$

and

$$I((e^{sb}\sigma)^{-1/(p-1)}) \le I(\sigma^{-\varepsilon/(p-1)})^{1/\varepsilon} I(e^{-\varepsilon' sb/(p-1)})^{1/\varepsilon'}.$$

Therefore

$$\begin{aligned} &I(e^{sb}\sigma) I((e^{sb}\sigma)^{-1/(p-1)})^{p-1} \\ &\le \left[I(\sigma^\varepsilon) I(\sigma^{-\varepsilon/(p-1)})\right]^{1/\varepsilon} \left[I(e^{\varepsilon' sb}) I(e^{-\varepsilon' sb/(p-1)}\right]^{1/\varepsilon'} \le C \end{aligned}$$

for every s such that $|s| < \lambda/\varepsilon'$. □

Lemma 3. *Let $R, S \in \mathbb{R}$, $a_n > 0$, $\lim_n a_n = 0$, $t \in [-1, 1]$. Then:*

a) $|x-t|^R(|x-t| + a_n)^S \in A_p(-1, 1)$ *uniformly in n if and only if* $-1 < R < p-1$, $-1 < R + S < p - 1$*;*
b) *for a product of terms of this type, these conditions are applied separately to each factor.*

For the proof of this lemma, see [GPV 2].

Proof of Theorem 3.

Coming back again to Pollard's decomposition we have

$$[b, S_n] = \sum_{i=1}^{3} [b, W_{i,n}]$$

where

$$\begin{aligned} [b, W_{1,n}]f &= bp_{n+1}\int_{-1}^{1} p_{n+1}fw - p_{n+1}\int_{-1}^{1} p_{n+1}fbw, \\ [b, W_{2,n}]f &= p_{n+1}[b, H]((1-y^2)q_n fw), \\ [b, W_{3,n}]f &= (1-x^2)q_n[b, H](p_{n+1}fw). \end{aligned}$$

We consider each operator separately.

i) Boundedness of $[b, W_{2,n}]$.

Write

$$\lambda_n = U^p |p_{n+1}|^p w \quad \text{and} \quad \mu_n = V^p |q_n|^{-p} (1-x^2)^{-p} w^{1-p}.$$

Now

$$\|U[b, W_{2,n}]f\|_{p,w} \le C\|Vf\|_{p,w}$$

if and only if

$$\|[b, H]g\|_{p,\lambda_n} \le C\|g\|_{p,\mu_n}$$

with some constant C independent of n. In order to prove this last inequality, the idea consists in inserting weights ϕ_n, that is, finding functions ϕ_n such that

$$C_1\lambda_n \le \phi_n \le C_2\mu_n \quad \text{and} \quad \phi_n \in A_p \text{ uniformly },$$

that is, with an A_p-constant independent of n. By using the estimates for p_n and q_n we have

$$\lambda_n \le Cu^p w_1^{1-\frac{p}{2}} (1-x)^{ap+\alpha} (1+x)^{bp+\beta} (1-x+a_n)^{-p(\frac{\alpha}{2}+\frac{1}{4})} (1+x+b_n)^{-p(\frac{\beta}{2}+\frac{1}{4})},$$

$$\mu_n \ge Cv^p w_1^{1-\frac{p}{2}} (1-x)^{Ap-\alpha+\alpha(1-p)} (1+x)^{Bp-\beta+\beta(1-p)} \\ \times (1-x+a_n)^{p(\frac{\alpha}{2}+\frac{3}{4})} (1+x+b_n)^{p(\frac{\beta}{2}+\frac{3}{4})}.$$

It is not difficult to see, from the hypothesis, that we can take a real number R such that

$$Ap - p + \alpha(1-p) \le R \le ap + \alpha,$$

$$-1 < R < p - 1$$

and choose S such that

$$Ap - p + \alpha(1-p) + p(\alpha/2 + 3/4) \le R + S \le ap + \alpha - p(\alpha/2 + 1/4),$$

$$-1 < R + S < p - 1.$$

Now, it is a straightforward calculation to verify that

$$C(1-x)^{ap+\alpha}(1-x+a_n)^{-p(\frac{\alpha}{2}+\frac{1}{4})} \le (1-x)^R (1-x+a_n)^S$$

$$\le C(1-x)^{Ap-p+\alpha(1-p)}(1-x+a_n)^{p(\frac{\alpha}{2}+\frac{3}{4})}.$$

We can also take $\widetilde{R}$ and $\widetilde{S}$ such that

$$Bp - p + \beta(1-p) \le \widetilde{R} \le bp + \beta,$$

$$Bp - p + \beta(1-p) + p(\beta/2 + 3/4) \le \widetilde{R} + \widetilde{S} \le bp + \beta - p(\beta/2 + 1/4),$$

$$-1 < \widetilde{R} < p - 1,$$

$$-1 < \widetilde{R} + \widetilde{S} < p - 1,$$

so that

$$C(1+x)^{bp+\beta}(1+x+b_n)^{-p(\frac{\beta}{2}+\frac{1}{4})} \le (1+x)^{\widetilde{R}}(1+x+b_n)^{\widetilde{S}}$$
$$\le C(1+x)^{Bp-p+\beta(1-p)}(1+x+b_n)^{p(\frac{\beta}{2}+\frac{3}{4})}.$$

If we write

$$\alpha_n(x) = (1-x)^R(1-x+a_n)^S,$$
$$\beta_n(x) = (1+x)^{\widetilde{R}}(1+x+b_n)^{\widetilde{S}}$$

we have

$$C\lambda_n \le u^p w_1^{1-p/2}\alpha_n\beta_n,$$
$$v^p w_1^{1-p/2}\alpha_n\beta_n \le C\mu_n.$$

As

$$(u^p w_1^{1-p/2}, v^p w_1^{1-p/2}) \in A_p^\delta$$

then, there exists a positive function ϕ satisfying

$$C_1 u^p w_1^{1-p/2} \le \phi \le C_2 v^p w_1^{1-p/2} \quad \text{and} \quad \phi^p w_1^{1-p/2} \in A_p.$$

Besides, there are positive constants C_1 and C_2 such that

$$C_1 \le \phi(x) \le C_2 \quad \text{for all} \quad x \in (-1,-1+\varepsilon) \quad \text{and} \quad x \in (1-\varepsilon, 1).$$

On the other hand, having in mind that $a_n > 0$, $\lim_n a_n = 0$ and

$$-1 < R < p-1, \qquad -1 < R+S < p-1,$$

from Lemma 3 it follows

$$\alpha_n = (1-x)^R(1-x+a_n)^S \in A_p \quad \text{uniformly}.$$

Also, it is clear that α_n is bounded below and above by positive constants on the interval $[-1, 1-\varepsilon]$. In a similar way we obtain that $\beta_n \in A_p$ uniformly and there exist positive constants C_1 and C_2 such that $C_1 < \beta(x) < C_2$ for all $x \in [-1+\varepsilon, 1]$. Then, splitting in pieces the integrals appearing in the A_p condition it can be shown that

$$\phi_n = \phi^p w_1^{1-p/2}\alpha_n\beta_n \in A_p \quad \text{uniformly}.$$

Since the commutator of the Hilbert transform with a function $b \in \mathrm{BMO}$ is bounded with A_p weights (see [Bl]), then

$$\|[b,H]g\|_{p,\lambda_n} \le C\|[b,H]g\|_{p,\phi_n} \le C_1\|g\|_{p,\phi_n} \le C_2\|g\|_{p,\mu_n}$$

and the boundedness of $[b, W_{2,n}]$ follows.

ii) Boundedness of $[b, W_{3,n}]$.

We can prove that there are positive constants C_1, C_2 and weights ψ_n uniformly in A_p, such that

$$C_1 U(x)^p(1-x^2)^p|q_n(x)|^p w(x) \le \psi_n(x) \le C_2 V(x)^p |p_{n+1}(x)|^{-p} w(x)^{1-p},$$

$$\psi_n \in A_p \quad \text{uniformly}$$

and we proceed like before.

iii) Boundedness of $[b, W_{1,n}]$.

$$[b, W_{1,n}]f = A_n f + B_n f,$$

where

$$A_n f = (b - b_Q) p_{n+1} \int_{-1}^{1} p_{n+1} f w,$$

$$B_n f = p_{n+1} \int_{-1}^{1} (b - b_Q) p_{n+1} f w$$

and Q stands for the interval $[-1, 1]$.

$$\|U A_n f\|_{p,w} = \|(b - b_Q) U p_{n+1}\|_{p,w} \left| \int_{-1}^{1} p_{n+1} f w \right|$$

$$\leq \|(b - b_Q) U p_{n+1}\|_{p,w} \; \|p_{n+1} V^{-1}\|_{p',w} \; \|V f\|_{p,w}.$$

Let $\delta > 1$ satisfying the A_p hypothesis, $\varepsilon > 0$, and $\frac{1}{p} = \frac{1}{s} + \frac{1}{p\delta} + \frac{1}{p(1+\varepsilon)}$. From the definitions of λ_n, α_n, β_n and Hölder's inequality we have

$$\|(b - b_Q) U p_{n+1}\|_{p,w} = \|(b - b_Q) \lambda_n^{1/p}\|_p$$

$$\leq \|(b - b_Q)[u^p w_1^{1-p/2}]^{1/p} \alpha_n^{1/p} \beta_n^{1/p}\|_p$$

$$\leq \|(b - b_Q)\|_s \; \|[u^p w_1^{1-p/2}]^\delta\|_1^{1/(p\delta)} \; \|\alpha_n \beta_n\|_{1+\varepsilon}^{1/p}.$$

From the A_p hypothesis,

$$\|[u^p w_1^{1-p/2}]^\delta\|_1^{1/(p\delta)} < C.$$

Now, $\varepsilon > 0$ can be taken small enough so that

$$\|\alpha_n \beta_n\|_{1+\varepsilon}^{1/p} < C.$$

Finally, from the John-Nirenberg theorem, there exists some C such that

$$\|(b - b_Q)\|_s \leq C \|b\|_*.$$

Putting these inequalities together, it follows

$$\|(b - b_Q) U p_{n+1}\|_{p,w} < C.$$

In an analogous way

$$\|p_{n+1} V^{-1}\|_{p',w} < C.$$

Thus

$$\|U A_n f\|_{p,w} \leq C \|V f\|_{p,w}.$$

The operators $B_n f$ can be handled the same as before. $\square$

Proof of Corollaries 1 and 2.

a) If $r \in \mathbb{R}$ and $pr+\alpha+1=0$, from the definition of $L^{p,\infty}(x^\alpha)$, it is not difficult to see

$$\|x^r \chi_{(0,\lambda)}(x)\|_{L^{p,\infty}(x^\alpha)} = C,$$

for some constant $C>0$ independent of $\lambda>0$. Therefore,

$$\|x^r \log \frac{1}{|x|} \chi_{(0,1)}(x)\|_{L^{p,\infty}(x^\alpha)} \geq C \log \frac{1}{\lambda},$$

so that

$$\|x^r \log \frac{1}{|x|} \chi_{(0,1)}(x)\|_{L^{p,\infty}(x^\alpha)} = \infty.$$

Now, if the restricted weak boundedness

$$[b, S_n] : L^{p,1}(w) \longrightarrow L^{p,\infty}(w)$$

holds uniformly in n for each $b \in \mathrm{BMO}$, from Theorem 1 we have

$$\| \log \frac{1}{|x-t|} u w^{-1/2} (1-x^2)^{-1/4} \|_{L^{p,\infty}(w)} < \infty,$$

and

$$\| \log \frac{1}{|x-t|} v^{-1} w^{-1/2} (1-x^2)^{-1/4} \|_{L^{p',\infty}(w)} < \infty$$

for each $t \in [-1,1]$, since $b(x) = \log|x-t|^{-1} \in \mathrm{BMO}$. This leads to (3), (4), (5), which proves Corollary 2 and, as a consequence, the only if part of Corollary 1.

b) Suppose now that (3), (4), (5) hold. From Lemma 3 and the fact that generalized Jacobi polynomials belong to the class $\mathcal{H}$ (if $\alpha, \beta \geq -1/2$, $\gamma_i \geq 0$) or the class $\overline{\mathcal{H}}$ (for any $\alpha, \beta > -1$, $\gamma_i \geq 0$), it is easy to show that the hypothesis of Theorem 2 or Theorem 3 also hold. $\square$

3. Fourier-Bessel series.

Let us now consider the Bessel function J_α of order $\alpha > -1$ and let $\{\alpha_n\}_{n=1}^\infty$ be the increasing sequence of the zeros of J_α. The Bessel system of order α, $\{j_n^\alpha\}_{n=1}^\infty$, where

$$j_n^\alpha(x) = 2^{1/2} |J_{\alpha+1}(\alpha_n)|^{-1} J_\alpha(\alpha_n x), \quad n \geq 1,$$

is orthogonal and complete in $L^2((0,1), x\,dx)$. Let $S_n^\alpha f$ denote the n-th partial sum operators

$$S_n^\alpha f(x) = \sum_{k=1}^n c_k j_k^\alpha(x), \qquad c_k = c_k(f) = \int_0^1 j_k^\alpha(y) f(y) y\,dy.$$

Theorem 1′. *Let U, V be two weights on $(0,1)$. If there exists some constant $C>0$ such that*

$$\|U[b,S_n^\alpha](V^{-1}f)\|_{L^{p,r}(xdx)} \le C\|f\|_{L^{q,s}(xdx)}$$

for each $n\ge 0$, $f\in L^{q,s}(xdx)$ (where $1<p<\infty$, $1<q<\infty$; either $r=p$ or $r=\infty$; either $s=q$ or $s=1$), then,

$$\|\log\frac{1}{|x-a|}Ux^{-1/2}\|_{L^{p,r}(xdx)}<\infty,$$

$$\|\log\frac{1}{|x-a|}V^{-1}x^{-1/2}\|_{L^{q',s'}(xdx)}<\infty$$

for each $a\in[-1,1]$.

The proof is similar to that of Theorem 1, if we replace the mentioned results of [MNT 2] and [GPV 2] by the analogous results for Fourier-Bessel series (see [GPRV 2, Lemma 2] and [GPRV 3, proof of Theorem 3]).

In a similar way to the case of weights in the class $\mathcal{H}$ we obtain

Theorem 2′. *Let $1<p<\infty$, $\alpha\ge -1/2$, U and V weights on $(0,1)$ and $b\in$ BMO. If*

$$\left(x^{1-p/2}U(x)^p, x^{1-p/2}V(x)^p\right)\in A_p^\delta(0,1)$$

for some $\delta>1$ ($\delta=1$ if $u=v$), then the commutator $[b,S_n^\alpha]$ is bounded from $L^p(V^px)$ into $L^p(U^px)$.

The proof can be done in a similar way to Theorem 2, using Proposition 1 in [GPRV 3] instead of Lemma 1. Also, for $-1<\alpha<-1/2$ a result analogous to Theorem 3 can be stated. Finally, theorems 1′ and 2′ give the following result.

Corollary. *Let $1<p<\infty$, $\alpha\ge -1/2$, and*

$$U(x)=x^a(1-x)^b\prod_{k=1}^m|x-x_k|^{b_k}.$$

($a,b,b_k\in\mathbb{R}$). Then, the following conditions are equivalent:

a) $\|US_n^\alpha(U^{-1}f)\|_{L^p(xdx)}\le C\|f\|_{L^p(xdx)}$ *for each* $f\in L^p(xdx)$
b) $\|US_n^\alpha(U^{-1}f)\|_{L^{p,\infty}(xdx)}\le C\|f\|_{L^p(xdx)}$ *for each* $f\in L^p(xdx)$
c) $\|US_n^\alpha(U^{-1}f)\|_{L^{p,\infty}(xdx)}\le C\|f\|_{L^{p,1}(xdx)}$ *for each* $f\in L^{p,1}(xdx)$
d) $\left|\frac{1}{p}+\frac{a-1}{2}\right|<\frac{1}{4}$, $-1<pb<p-1$, $-1<pb_k<p-1$ $(1\le k\le m)$.

References

[AW] R. Askey and S. Wainger, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. **87** (1965), 695–708.

[B] V. M. Badkov, *Convergence in the mean and almost everywhere of Fourier series in polynomials orthogonal on an interval*, Math. USSR Sb. **24** (1974), 223–256.

[BP 1] A. I. Benedek and R. Panzone, *On mean convergence of Fourier-Bessel of negative order*, Studies in App. Math. **50** (1971), 281–292.

[BP 2] ______, *Mean convergence of series of Bessel functions*, Rev. Un. Mat. Arg. **26** (1972), 42–61.

[Bl] S. Bloom, *A commutator theorem and weighted* BMO, Trans. Amer. Math. Soc. **292** (1985), 103–122.

[CM] R. R. Coifman and M. A. M. Murray, *Uniform analyticity of orthogonal projections*, Trans. Amer. Math. Soc. **312** (1989), 779–817.

[CRW] R. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635.

[GR] J. García Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, Nederlands, 1985.

[GPRV 1] J. J. Guadalupe, M. Pérez, F. J. Ruiz and J. L. Varona, *Weighted L^p-boundedness of Fourier series with respect to generalized Jacobi weights*, Publicacions Matemàtiques **35** (1991), 449–459.

[GPRV 2] ______, *Two notes on convergence and divergence a.e. of Fourier series with respect to some orthogonal systems*, Proc. Amer. Math. Soc. **116** (1992), 457–464.

[GPRV 3] ______, *Mean and weak convergence of Fourier-Bessel series*, J. Math. Anal. Appl. **173** (1993), 370–389.

[GPV 1] J. J. Guadalupe, M. Pérez and J. L. Varona, *Mean and weak convergence of some orthogonal Fourier expansions by using A_p theory*, Lect. Notes Pure Appl. Math. (J. Vinuesa, ed.) (Proc. Int. Congr. Laredo/Spain 1987), vol. 117, Marcel Dekker, New York, 1989, pp. 161–169.

[GPV 2] ______, *Weighted weak behaviour of Fourier-Jacobi series*, Math. Nachr. **158** (1992), 161–174.

[K] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1976.

[L] E. Laeng, *Analytic dependence of orthogonal polynomials*, Rev. Mat. Iberoamericana **7** (1991), 287–312.

[MNT 1] A. Máté, P. Nevai and V. Totik, *Asymptotics for the ratio of the leading coefficients of orthogonal polynomials on the unit circle*, Constr. Approx. **1** (1985), 63–69.

[MNT 2] ______, *Necessary conditions for weighted mean convergence of Fourier series in orthogonal polynomials*, J. Approx. Theory **46** (1986), 314–322.

[M 1] B. Muckenhoupt, *Mean convergence of Jacobi series*, Proc. Amer. Math. Soc. **23** (1969), 306–310.

[M 2] ______, *Mean convergence of Hermite and Laguerre series.* II, Trans. Amer. Math. Soc. **147** (1970), 433–460.

[N] C. J. Neugebauer, *Inserting A_p weights*, Proc. Amer. Math. Soc. **87** (1983), 644–648.

[P 1] H. Pollard, *The mean convergence of orthogonal series.* II, Trans. Amer. Math. Soc. **63** (1948), 355–367.

[P 2] ______, *The mean convergence of orthogonal series.* III, Duke Math. J. **16** (1949), 189–191.

[R] E. A. Rahmanov, *On the asymptotics of the ratio of orthogonal polynomials.* II, Math. USSR. Sb. **46** (1983), 105–117.

[ST 1] C. Segovia and J. L. Torrea, *Vector-valued commutators and applications*, Indiana Univ. Math. J. **38** (1989), 959–971.

[ST 2] ______, *Weighted inequalities for commutators of fractional and singular integrals*, Publicacions Matemàtiques **35** (1991), 209–235.

[ST 3] ______, *Higher order commutators for vector-valued Calderón-Zygmund operators*, Trans. Amer. Math. Soc. **336** (1993), 537–556.

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, New Jersey, 1975.

[W] G. M. Wing, *The mean convergence of orthogonal series*, Amer. J. Math. **72** (1950), 792–808.

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