WEIGHTED L^p-BOUNDEDNESS OF FOURIER SERIES WITH RESPECT TO GENERALIZED JACOBI WEIGHTS.

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Abstract. *Let w be a generalized Jacobi weight on the interval $[-1,1]$ and, for each function f, let $S_n f$ denote the n-th partial sum of the Fourier series of f in the orthogonal polynomials associated to w. We prove a result about uniform boundedness of the operators S_n in some weighted L^p spaces. The study of the norms of the kernels K_n related to the operators S_n allows us to obtain a relation between the Fourier series with respect to different generalized Jacobi weights.*

Let w be a generalized Jacobi weight, that is,

$$w(x) = h(x)(1-x)^{\alpha}(1+x)^{\beta}\prod_{i=1}^{N}|x-t_i|^{\gamma_i}\ ,\ x\in[-1,1]$$

where

a) $\alpha,\beta,\gamma_i > -1$, $t_i \in (-1,1)$, $t_i \neq t_j$ $\forall i \neq j$;

b) h is a positive, continuous function on $[-1,1]$ and $w(h,\delta)\delta^{-1} \in L^1(0,1)$, $w(h,\delta)$ being the modulus of continuity of h.

Let $d\mu = w(x)\,dx$ on $[-1,1]$ and let S_n $(n \geq 0)$ be the n-th partial sum of the Fourier series in the orthonormal polynomials with respect to $d\mu$. The study of the boundedness

$$\|S_n f\|_{L^p(u^p d\mu)} \leq C\|f\|_{L^p(v^p d\mu)}, \tag{1}$$

where $$u(x) = (1-x)^a(1+x)^b\prod_{i=1}^{N}|x-t_i|^{g_i}, \qquad a,b,g_i\in\mathbb{R}$$

and $$v(x) = (1-x)^A(1+x)^B\prod_{i=1}^{N}|x-t_i|^{G_i}, \qquad A,B,G_i\in\mathbb{R}$$

was done by Badkov ([1]) in the case $u = v$ by means of a direct estimation of the kernels $K_n(x,y)$ associated with the polynomials orthogonal with respect to $d\mu$. Later, one of us ([10]) considered the same problem, with u and v not necessarily equal; his method consists of an appropriate use of the theory of A_p weights. He found conditions for (1) which generalized those obtained for $u = v$ by Badkov. However, this result, which we state below, follows only in the case $\gamma_i \geq 0$, $i = 1,\dots,N$.

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Theorem 1. *Let $\gamma_i \geq 0$, $i = 1, \ldots, N$ and $1 < p < \infty$. If the inequalities*

$$\begin{cases} A + (\alpha+1)(\frac{1}{p} - \frac{1}{2}) < \min\{\frac{1}{4}, \frac{\alpha+1}{2}\} \\ B + (\beta+1)(\frac{1}{p} - \frac{1}{2}) < \min\{\frac{1}{4}, \frac{\beta+1}{2}\} \\ G_i + (\gamma_i+1)(\frac{1}{p} - \frac{1}{2}) < \min\{\frac{1}{2}, \frac{\gamma_i+1}{2}\} \quad (i = 1, \ldots, N) \end{cases} \tag{2}$$

$$\begin{cases} a + (\alpha+1)(\frac{1}{p} - \frac{1}{2}) > -\min\{\frac{1}{4}, \frac{\alpha+1}{2}\} \\ b + (\beta+1)(\frac{1}{p} - \frac{1}{2}) > -\min\{\frac{1}{4}, \frac{\beta+1}{2}\} \\ g_i + (\gamma_i+1)(\frac{1}{p} - \frac{1}{2}) > -\min\{\frac{1}{2}, \frac{\gamma_i+1}{2}\} \quad (i = 1, \ldots, N) \end{cases} \tag{3}$$

and

$$A \leq a, \qquad B \leq b, \qquad G_i \leq g_i \tag{4}$$

hold, then

$$\exists C > 0 \;\; \textit{such that} \;\; \|S_n f\|_{L^p(u^p d\mu)} \leq C\|f\|_{L^p(v^p d\mu)} \;\; \forall f \in L^p(v^p d\mu), \; \forall n \in \mathbb{N}.$$

The objective of this paper is to show that the result remains true without the restriction $\gamma_i \geq 0$ and that conditions (2), (3) and (4) are also necessary for the uniform boundedness:

Theorem 2. *Let $1 < p < \infty$. Then, there exists $C > 0$ such that*

$$\|S_n f\|_{L^p(u^p d\mu)} \leq C\|f\|_{L^p(v^p d\mu)} \quad \forall f \in L^p(v^p d\mu), \; \forall n \in \mathbb{N},$$

if and only if the inequalities (2), (3) *and* (4) *are satisfied.*

For the sake of completeness, we give a brief sketch of the proof of theorem 1 (see also [10]). By using Pollard's decomposition of the kernels $K_n(x, y)$ (see [8], [5]), the uniform boundedness of S_n can be reduced to that of the Hilbert transform with pairs of weights

$$(|P_{n+1}(x)|^p u(x)^p w(x), |Q_n(x)|^{-p}(1 - x^2)^{-p} v(x)^p w(x)^{1-p})$$

and

$$(|Q_n(x)|^p (1 - x^2)^p u(x)^p w(x), |P_{n+1}(x)|^{-p} v(x)^p w(x)^{1-p}),$$

Q_n being the n-th orthonormal polynomial relative to the measure $(1 - x^2)d\mu$. Using now Hunt-Muckenhoupt-Wheeden and Neugebauer results (see [2], [6]), together with some known estimates for generalized Jacobi polynomials (see (8) below), for the above uniform boundedness the following conditions turn out to be sufficient:

$$(u_n^\delta, v_n^\delta) \in A_p((-1, 1))$$

and

$$(\bar{u}_n^\delta, \bar{v}_n^\delta) \in A_p((-1, 1))$$

for some $\delta > 1$, with A_p constants independent of n, where

$$\begin{aligned} u_n(x) =& (1-x)^{ap+\alpha}(1-x+n^{-2})^{-p(2\alpha+1)/4} \\ &\times(1+x)^{bp+\beta}(1+x+n^{-2})^{-p(2\beta+1)/4} \\ &\times\prod_{i=1}^{N}|x-t_i|^{g_ip+\gamma_i}(|x-t_i|+n^{-1})^{-p\gamma_i/2}, \end{aligned}$$

$$\begin{aligned} v_n(x) =& (1-x)^{Ap+\alpha(1-p)+p}(1-x+n^{-2})^{p(2\alpha+3)/4} \\ &\times(1+x)^{Bp+\beta(1-p)+p}(1+x+n^{-2})^{p(2\beta+3)/4} \\ &\times\prod_{i=1}^{N}|x-t_i|^{G_ip+\gamma_i(1-p)}(|x-t_i|+n^{-1})^{p\gamma_i/2} \end{aligned}$$

and similar expressions for $\bar{u}_n$ and $\bar{v}_n$.

These conditions are easy to check using the simpler result (see [10]):

Lemma 3. *Let $\{x_n\}_{n\geq 0}$ be a sequence of positive numbers converging to* 0*. Let $r, s, R, S \in \mathbb{R}$. Then,*

$$(|x|^r(|x|+x_n)^s, |x|^R(|x|+x_n)^S) \in A_p((-1,1))$$

with a constant independent of n if and only if the following inequalities hold:

$$\begin{array}{ccc} r > -1; & R < p-1; & R \leq r; \\ r+s > -1; & R+S < p-1; & R+S \leq r+s. \end{array}$$

At least in the case $u = v$ (thus $g_i = G_i$, $\forall i$), inequality $R \leq r$ requires $\gamma_i \geq 0$ $\forall i$. But, with this assumption, theorem 1 follows.

Let us introduce now some notation: $\{P_n(x)\}$, $\{k_n\}$ and $\{K_n(x,y)\}$ will be, respectively, the orthonormal polynomials, their leading coefficients and the kernels relatives to $d\mu$; if $c \in (-1,1)$, $\{P_n^c(x)\}$, $\{k_n^c\}$ and $\{K_n^c(x,y)\}$ will be the corresponding to $(x-c)^2 d\mu$. Then, it is not difficult to establish $\forall n \in \mathbb{N}$ the relations

$$K_n(x,y) = (x-c)(y-c)K_{n-1}^c(x,y) + \frac{K_n(x,c)K_n(c,y)}{K_n(c,c)}; \tag{5}$$

$$K_n(x,c) = \frac{k_n}{k_n^c}P_n(c)P_n^c(x) - \frac{k_{n-1}^c}{k_{n+1}}P_{n+1}(c)P_{n-1}^c(x). \tag{6}$$

It can be also shown (see [4], theorems 10 and 11, and [9], pag. 212) that

$$\lim_{n\to\infty}\frac{k_n}{k_n^c} = \lim_{n\to\infty}\frac{k_{n-1}^c}{k_{n+1}} = \frac{1}{2}. \tag{7}$$

If we define

$$d(x,n) = (1-x+n^{-2})^{-(2\alpha+1)/4}(1+x+n^{-2})^{-(2\beta+1)/4}\prod_{i=1}^{N}(|x-t_i|+n^{-1})^{-\gamma_i/2},$$

it is known ([1]) that there exists a constant C such that $\forall x \in [-1,1]$, $\forall n \in \mathbb{N}$

$$|P_n(x)| \le C d(x,n). \tag{8}$$

There are also some well-known estimates for the kernels, one of them being this ([7], pag. 4 and pag. 119, theorem 25): if $c \in (-1,1)$ and the factor $|x-c|$ occurs in w with an exponent γ, there exist some positive constants C_1 and C_2, depending on c, such that $\forall n \in \mathbb{N}$

$$C_1 n^{\gamma+1} \le K_n(c,c) \le C_2 n^{\gamma+1}. \tag{9}$$

From now on, all constants will be denoted C, so by C we will mean a constant, possibly different in each occurrence. Using (6), (7) and (8) we obtain the following result:

Proposition 4. *Let $1 < p < \infty$, $1/p + 1/q = 1$ and suppose the inequality (3) holds. Let $-1 < c < 1$ and let γ and g be the exponents of $|x-c|$ in w and u, respectively. Then, there exists a positive constant C such that $\forall n \ge 0$:*

$$\|K_n(x,c)\|_{L^p(u^p w)} \le \begin{cases} C n^{(\gamma+1)/q - g} & \text{if } g < (\gamma+1)(1/2 - 1/p) + 1/2 \\ C n^{\gamma/2} (\log n)^{1/p} & \text{if } g = (\gamma+1)(1/2 - 1/p) + 1/2 \\ C n^{\gamma/2} & \text{if } (\gamma+1)(1/2 - 1/p) + 1/2 < g \end{cases}$$

Proof. From (8) it follows that $|P_n(c)| \le C n^{\gamma/2}$. Since $\{P_n^c\}$ is the sequence associated with $(x-c)^2 d\mu$, it also follows from (8) that

$$|P_n^c(x)| \le C(|x-c| + n^{-1})^{-1} d(x,n).$$

Now, from (6) and (7) we get:

$$|K_n(x,c)| \le C n^{\gamma/2} (|x-c| + n^{-1})^{-1} d(x,n). \tag{10}$$

Let us take $\varepsilon > 0$ such that $|t_i - c| > \varepsilon$ for all $t_i \ne c$. We can write:

$$\|K_n(x,c)\|^p_{L^p(u^p w)}$$

$$= \int_{|x-c| \ge \varepsilon} |K_n(x,c)|^p u(x)^p w(x) dx + \int_{|x-c| < \varepsilon} |K_n(x,c)|^p u(x)^p w(x) dx$$

Using (10), we obtain for the first term

$$\int_{|x-c| \ge \varepsilon} |K_n(x,c)|^p u(x)^p w(x) dx \le C n^{p\gamma/2} \int_{|x-c| \ge \varepsilon} (|x-c| + n^{-1})^{-p} d(x,n)^p u(x)^p w(x) dx$$

$$\le C n^{p\gamma/2} \int_{-1}^{1} d(x,n)^p u(x)^p w(x) dx.$$

It is easy to deduce from (3) that this last integral is bounded by a constant which does not depend on n, so

$$\int_{|x-c|\geq\varepsilon} |K_n(x,c)|^p u(x)^p w(x)dx \leq Cn^{p\gamma/2}. \tag{11}$$

Let us take now the second term; since for $|x-c|<\varepsilon$ there exists a constant C such that $\forall n\ d(x,n) \leq C(|x-c|+n^{-1})^{-\gamma/2}$, $u(x) \leq C|x-c|^g$ and $w(x) \leq C|x-c|^\gamma$, we have

$$\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)dx \leq Cn^{p\gamma/2} \int_{|x-c|<\varepsilon} (|x-c|+n^{-1})^{-p} d(x,n)^p u(x)^p w(x)dx$$

$$\leq Cn^{p\gamma/2} \int_{|x-c|<\varepsilon} (|x-c|+n^{-1})^{-p(1+\gamma/2)}|x-c|^{gp+\gamma}dx$$

$$\leq Cn^{p\gamma/2} \int_0^1 (y+n^{-1})^{-p(1+\gamma/2)} y^{gp+\gamma}dy$$

$$= Cn^{p\gamma/2+p(1+\gamma/2)-gp-\gamma-1} \int_0^1 (ny+1)^{-p(1+\gamma/2)}(ny)^{gp+\gamma}ndy$$

$$= Cn^{p\gamma/2+p(1+\gamma/2)-gp-\gamma-1} \int_0^n (r+1)^{-p(1+\gamma/2)} r^{gp+\gamma}dr.$$

Taking into account that $p(1+\gamma/2)-gp-\gamma-1 = p[(\gamma+1)(1/2-1/p)-g+1/2]$ and there exist some constants C_1 and C_2 such that $C_1 \leq r+1 \leq C_2$ on $[0,1]$ and $C_1 r \leq r+1 \leq C_2 r$ on $[1,n]$, we finally get the inequality

$$\begin{aligned} \int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)dx &\leq Cn^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]} \int_0^1 r^{gp+\gamma}dr \\ &+ Cn^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]} \int_1^n r^{-p[(\gamma+1)(1/2-1/p)-g+1/2]-1}dr. \end{aligned} \tag{12}$$

Since (3) implies $gp+\gamma > -1$, the first term is bounded by

$$Cn^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]} \int_0^1 r^{gp+\gamma}dr \leq Cn^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]}. \tag{13}$$

For the second term, let us consider separately the three cases in the statement.

a) If $g < (\gamma+1)(1/2-1/p)+1/2$, then $-p[(\gamma+1)(1/2-1/p)-g+1/2]-1 < -1$. Thus

$$\int_1^n r^{-p[(\gamma+1)(1/2-1/p)-g+1/2]-1}dr \leq C.$$

In this case, (12) and (13) imply:

$$\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)dx \leq Cn^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]}.$$

Since $p[(\gamma+1)(1/2-1/p)-g+1/2]>0$, from this inequality and (11) we obtain

$$\|K_n(x,c)\|^p_{L^p(u^pw)} \leq Cn^{p\gamma/2+p[(\gamma+1)(1/2-1/p)-g+1/2]}$$

$$= Cn^{p[(\gamma+1)(1-1/p)-g]} = Cn^{p[(\gamma+1)/q-g]},$$

as we had to prove.

b) If $(\gamma+1)(1/2-1/p)+1/2<g$, then $-p[(g+1)(1/2-1/p)-g+1/2]-1>-1$. Therefore

$$\int_1^n r^{-p[(\gamma+1)(1/2-1/p)-g+1/2]-1}dr \leq Cn^{-p[(\gamma+1)(1/2-1/p)-g+1/2]}.$$

By (12) and (13), it follows

$$\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)dx \leq Cn^{p\gamma/2}$$

and

$$\|K_n(x,c)\|^p_{L^p(u^pw)} \leq Cn^{p\gamma/2}.$$

c) If $g=(\gamma+1)(1/2-1/p)+1/2$

$$\int_1^n r^{-p[(\gamma+1)(1/2-1/p)-g+1/2]-1}dr = \log n;$$

hence,

$$\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x)dx \leq Cn^{p\gamma/2}\log n$$

and

$$\|K_n(x,c)\|^p_{L^p(u^pw)} \leq Cn^{p\gamma/2}\log n.$$

This concludes the proof of the proposition.

Corollary 5. *Let $1<p<\infty$, $1/p+1/q=1$ and suppose the inequality (2) holds. Let $-1<c<1$ and γ and G be the exponents of $|x-c|$ in w and v, respectively. Then, there exists a positive constant C such that $\forall n\in\mathbb{N}$*

$$\|K_n(x,c)\|_{L^q(v^{-q}w)} \leq \begin{cases} Cn^{\gamma/2} & \text{if } G<(\gamma+1)(1/2-1/p)+1/2 \\ Cn^{\gamma/2}(\log n)^{1/q} & \text{if } G=(\gamma+1)(1/2-1/p)+1/2 \\ Cn^{(\gamma+1)/p+G} & \text{if } (\gamma+1)(1/2-1/p)+1/2<G \end{cases}$$

Proof. Just apply proposition 4 to the weight v^{-1} and keep in mind the equality $1/2-1/p=1/q-1/2$.

The following result is just what we need to extend theorem 1 to the general case $\gamma_i>-1$.

Corollary 6. *Let $1 < p < \infty$, $1/p + 1/q = 1$. Suppose the inequalities* (2), (3) *and* (4) *hold. Let $-1 < c < 1$. Then, there exists a positive constant C such that $\forall n \geq 0$:*

$$\|K_n(x,c)\|_{L^p(u^p w)} \|K_n(x,c)\|_{L^q(v^{-q} w)} \leq C K_n(c,c).$$

Proof. It is a simple consequence of proposition 4, corollary 5 and the estimate (9). The only thing we must do is to consider each case in these results separately.

Note. Although it will not be used in what follows, corollary 6 also holds when $c = \pm 1$. The proof is similar: starting from other expressions for $K_n(x, \pm 1)$, analogous results to proposition 4 and corollary 5 can be obtained, and then corollary 6 follows.

We are now ready to prove our main result:

Proof of theorem 2. a) Let us assume first that the inequalities (2), (3) and (4) hold. We prove that the operators S_n are uniformly bounded by induction on the number of negative exponents γ_i. If $\gamma_i \geq 0$ $\forall i$, the result is true, as we saw before (theorem 1). Now, suppose there exist k negative exponents γ_i, with $k > 0$, and the result is true for $k-1$. Let $c \in (-1, 1)$ be a point with a negative exponent γ. Let us remember the formula (5):

$$K_n(x,y) = (x-c)(y-c)K^c_{n-1}(x,y) + \frac{K_n(x,c)K_n(c,y)}{K_n(c,c)}.$$

We define the operators:

$$T_n f(x) = \int_{-1}^{1} \frac{K_n(x,c)K_n(c,y)}{K_n(c,c)} f(y)w(y)dy,$$

$$R_n f(x) = \int_{-1}^{1} (x-c)(y-c)K^c_{n-1}(x,y) f(y)w(y)dy.$$

Then, $S_n = T_n + R_n$. We are going to study firstly the operators T_n:

$$T_n f(x) = \frac{K_n(x,c)}{K_n(c,c)} \int_{-1}^{1} K_n(c,y) f(y)w(y)dy;$$

thus

$$\|T_n f\|_{L^p(u^p w)} \leq \frac{\int_{-1}^{1} |K_n(c,y)| v(y)^{-1} |f(y)| v(y) w(y) dy}{K_n(c,c)} \; \|K_n(x,c)\|_{L^p(u^p w)}$$

$$\leq \frac{\|K_n(x,c)\|_{L^p(u^p w)} \|K_n(x,c) v(x)^{-1}\|_{L^q(w)}}{K_n(c,c)} \; \|fv\|_{L^p(w)}$$

$$= \frac{\|K_n(x,c)\|_{L^p(u^p w)} \|K_n(x,c)\|_{L^q(v^{-q} w)}}{K_n(c,c)} \; \|f\|_{L^p(v^p w)}.$$

From corollary 6 it follows

$$\|T_n f\|_{L^p(u^p d\mu)} \leq C \|f\|_{L^p(v^p d\mu)} \;\forall f \in L^p(v^p d\mu), \;\forall n \in \mathbb{N}.$$

So, we only need to prove the same bound for the operators R_n. But, if we denote by S_n^c the partial sums of the Fourier series with respect to the measure $(x-c)^2 w(x)dx$, it turns out that

$$R_n f(x) = (x-c)\int_{-1}^{1}(y-c)K_{n-1}^c(x,y)f(y)w(y)dy = (x-c)S_{n-1}^c(\frac{f(y)}{y-c},x),$$

whence

$$\begin{aligned}
&\|R_n f\|_{L^p(u^p w)} \le C\|f\|_{L^p(v^p w)}\ \forall f\in L^p(v^p w),\ \forall n\in\mathbb{N}\\
\iff\ &\|(x-c)S_{n-1}^c(\frac{f(y)}{y-c},x)\|_{L^p(u^p w)} \le C\|f\|_{L^p(v^p w)}\ \forall f\in L^p(v^p w),\ \forall n\in\mathbb{N}\\
\iff\ &\|(x-c)S_{n-1}^c g(x)\|_{L^p(u^p w)} \le C\|(x-c)g\|_{L^p(v^p w)}\ \forall g\in L^p(|x-c|^p v^p w),\ \forall n\in\mathbb{N}\\
\iff\ &\|S_{n-1}^c g(x)\|_{L^p(|x-c|^p u^p w)} \le C\|g\|_{L^p(|x-c|^p v^p w)}\ \forall g\in L^p(|x-c|^p v^p w),\ \forall n\in\mathbb{N}\\
\iff\ &\|S_{n-1}^c g(x)\|_{L^p(\tilde{u}^p (x-c)^2 w)} \le C\|g\|_{L^p(\tilde{v}^p (x-c)^2 w)}\ \forall g\in L^p(\tilde{v}^p (x-c)^2 w),\ \forall n\in\mathbb{N},
\end{aligned}$$

where $\tilde{u}(x) = |x-c|^{1-2/p}u(x)$ and $\tilde{v}(x) = |x-c|^{1-2/p}v(x)$.

Therefore, we must prove the boundedness of the partial sums S_n^c with the pair of weights $(\tilde{u},\tilde{v})$. But the Fourier series we are considering now corresponds to the Jacobi generalized weight $(x-c)^2 w(x)$, which has only $k-1$ negative exponents γ_i, since on the point c the exponent is $\gamma+2>1$. By hypothesis, the theorem holds in this case and we only have to see that the conditions in the statement hold for the weights $(x-c)^2 w(x)$, $|x-c|^{1-2/p}u(x)$ and $|x-c|^{1-2/p}v(x)$.

Except for the point c, these weights have the same exponents as w, u and v. Thus, those conditions are the same and therefore they are satisfied. At the point c, the exponents are, respectively: $\gamma+2$, $g+1-2/p$, $G+1-2/p$.

So, we have to check the inequalities

$$(G+1-\frac{2}{p}) + (\gamma+2+1)(\frac{1}{p}-\frac{1}{2}) < \min\{\frac{1}{2},\frac{\gamma+2+1}{2}\},$$

$$(g+1-\frac{2}{p}) + (\gamma+2+1)(\frac{1}{p}-\frac{1}{2}) > -\min\{\frac{1}{2},\frac{\gamma+2+1}{2}\}$$

and

$$G+1-\frac{2}{p} \le g+1-\frac{2}{p}.$$

It is clear, from our hypothesis, that they are satisfied. Consequently, we have

$$\|S_{n-1}^c g(x)\|_{L^p(\tilde{u}^p (x-c)^2 w)} \le C\|g\|_{L^p(\tilde{v}^p (x-c)^2 w)}\ \forall g\in L^p(\tilde{v}^p (x-c)^2 w),\quad \forall n\in\mathbb{N}.$$

Thus,

$$\|R_n f\|_{L^p(u^p w)} \le C\|f\|_{L^p(v^p w)}\ \forall f\in L^p(v^p w),\quad \forall n\in\mathbb{N}$$

and

$$\|S_n f\|_{L^p(u^p \mu)} \le C\|f\|_{L^p(v^p \mu)}\ \forall f\in L^p(v^p \mu),\quad \forall n\in\mathbb{N}.$$

Therefore, the result is true for k negative exponents γ_i. By induction, it is true in general and the first part of the theorem is proved.

b) Now, assume that the operators S_n are uniformly bounded. Let us prove that (2), (3) and (4) are satisfied.

From a result of Máté, Nevai and Totik ([3], theorem 1), it follows

$$\begin{aligned} u &\in L^p(d\mu); \\ v^{-1} &\in L^q(d\mu); \\ w(x)^{-1/2}(1-x^2)^{-1/4}u(x) &\in L^p(w(x)dx); \\ w(x)^{-1/2}(1-x^2)^{-1/4}v(x)^{-1} &\in L^q(w(x)dx). \end{aligned}$$

These conditions are equivalent to (2) and (3). Thus, we only need to prove (4), that is:

$$\exists C > 0 \ \ such\ that \ \ u \le Cv \ \ \mu - a.e.$$

In fact, we are going to show that the same C of the hypothesis works. First of all, let us note that from the hypothesis it follows

$$\|R\|_{L^p(u^p d\mu)} \le C\|R\|_{L^p(v^p d\mu)} \tag{14}$$

for every polynomial R, since $S_n R = R$ if n is big enough.

It is clear that there exists a polynomial Q such that both $|Q|^p u^p$ and $|Q|^p v^p$ are μ-integrable. Let us denote $u' = |Q|^p u^p$ and $v' = |Q|^p v^p$. Then, for every $f \in L^p(u'd\mu) \cap L^p(v'd\mu)$ there exists a sequence of polynomials R_n such that

$$\lim_{n\to\infty} \int_{-1}^{1} |f - R_n|^p (u' + v') d\mu = 0.$$

From this and (14) we obtain

$$\int_{-1}^{1} |f|^p u' d\mu = \lim_{n\to\infty} \int_{-1}^{1} |R_n Q|^p u^p d\mu \le C^p \lim_{n\to\infty} \int_{-1}^{1} |R_n Q|^p v^p d\mu = C^p \int_{-1}^{1} |f|^p v' d\mu.$$

Taking now $E = \{x \in [-1,1]; u(x) > Cv(x)\}$ and f the characteristic function on E, we deduce $\mu(E) = 0$.

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