

# ASYMPTOTIC BEHAVIOUR OF ORTHOGONAL POLYNOMIALS RELATIVE TO MEASURES WITH MASS POINTS.

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**Abstract.** *General expressions are found for the orthonormal polynomials and the kernels relative to measures on the real line of the form  $\mu + M\delta_c$ , in terms of those of the measures  $d\mu$  and  $(x-c)^2 d\mu$ . In particular, these relations allow us to obtain that Nevai's class  $M(0, 1)$  is closed for adding a mass point, as well as several bounds for the polynomials and kernels relative to a generalized Jacobi weight with a finite number of mass points.*

## §0. Introduction.

Let  $\mu$  be a positive measure on  $\mathbb{R}$  with infinitely many points of increase and such that all the moments

$$\int_{\mathbb{R}} x^n d\mu \quad (n = 0, 1, \dots)$$

exist. Then, there exists a unique sequence  $\{P_n\}_{n \geq 0}$  of orthonormal polynomials

$$P_n(x) = k_n x^n + \dots, \quad k_n > 0$$

such that

$$\int_{\mathbb{R}} P_n P_m d\mu = \begin{cases} 0, & \text{if } n \neq m; \\ 1, & \text{if } n = m. \end{cases}$$

As usual,  $\{K_n(x, y)\}_{n \geq 0}$  denotes the sequence of kernels associated to  $\mu$ , that is,

$$K_n(x, y) = \sum_{j=0}^n P_j(x) P_j(y).$$

It is well known that the polynomials  $\{P_n\}_{n \geq 0}$  satisfy a three-term recurrence relation

$$xP_n(x) = a_{n+1}P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x), \quad n \geq 0,$$

where  $P_{-1} = 0$ ,  $a_n = k_{n-1}/k_n$  and

$$b_n = \int_{\mathbb{R}} x P_n(x)^2 d\mu(x).$$

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A particularly important class of measures is Nevai's class  $M(0, 1)$  consisting of those measures  $\mu$  for which  $\lim_n a_n = 1/2$  and  $\lim_n b_n = 0$  hold. For such a measure, the polynomials  $\{P_n\}_{n \geq 0}$  have the so called ratio asymptotic property. We refer the reader to [7], [4], [8] for further details on  $M(0, 1)$ . We must remark that every measure  $\mu$  with  $\text{supp } \mu = [-1, 1]$  and  $\mu' > 0$  *a.e.* on  $[-1, 1]$  belongs to  $M(0, 1)$  (see [9], p. 212, or [4], theorem 10). Here,  $\mu'$  denotes the absolutely continuous part of  $\mu$ .

An interesting problem in the theory of orthogonal polynomials is that of finding asymptotic estimates for  $\{P_n\}$ , their leading coefficients  $\{k_n\}$ , the sequence  $\{K_n(x, x)\}$  ( $x \in \text{supp } \mu$ ), etc. (see, for example, [10] and [5] for Jacobi polynomials, [2] and [7] for generalized Jacobi polynomials, [1] and [6] for Laguerre and Hermite; general results can be found in [9], [7], etc.). We will study this problem for orthonormal polynomials associated to modification of measures by mass points.

Let  $M$  be a positive constant and let  $\delta_c$  denote a Dirac measure on a point  $c \in \mathbb{R}$ , that is,

$$\int_{\mathbb{R}} f \delta_c = f(c)$$

for every function  $f$ . Then, associated to the measure  $\nu = \mu + M\delta_c$  there exists a sequence  $\{Q_n\}_{n \geq 0}$  of orthonormal polynomials. We will find expressions which relate the sequences  $\{Q_n\}$  and  $\{P_n\}$  in order to deduce estimates for  $\{Q_n\}$  whenever they are known for  $\{P_n\}$ . A precedent of this type of results is Koornwinder's paper [3], where it is considered the case of Jacobi weights modified by two delta functions at 1 and  $-1$ . Koornwinder obtained an explicit formula which relates the new polynomials with the Jacobi polynomials and their derivatives (which are Jacobi polynomials again). This point of view is useful in order to get second order differential equations satisfied by the new polynomials. However, our main interest is addressed to study the convergence of Fourier series relative to modifications of Jacobi (and more general) weights by a finite number of mass points on all the interval  $[-1, 1]$ . In this sense, it is more useful to find relations which involve the polynomials  $\{P_n\}$ ,  $\{Q_n\}$  and the polynomials orthonormal with respect to the measure  $(x - c)^2 d\mu(x)$ .

The organization of this paper is as follows: in §1 we obtain algebraic relations among the different sequences of orthogonal polynomials and kernels for general measures. When  $\text{supp } \mu = [-1, 1]$  and  $\mu \in M(0, 1)$ , these relations provide a good information about their asymptotic behaviour. As an application of the previous results, in §2 we obtain several estimates in the particular case of generalized Jacobi weights.

## §1. General results.

The following notation will be used from now on:

$$d\mu^c(x) = (x - c)^2 d\mu(x);$$

$\{P_n^c\}$  is the sequence of orthonormal polynomials relative to  $\mu^c$ ;

$$P_n^c(x) = k_n^c x^n + \dots, \quad k_n^c > 0;$$

$\{K_n^c(x, y)\}$  is the sequence of kernels relative to  $\mu^c$ .

**Lemma 1.** *With the above notation,*

$$K_n(x, c) = \frac{k_n}{k_n^c} P_n(c) P_n^c(x) - \frac{k_{n-1}^c}{k_{n+1}} P_{n+1}(c) P_{n-1}^c(x) \quad \forall n \geq 1.$$

*Proof.* We can put

$$K_n(x, c) = \sum_{j=0}^n \alpha_j P_j^c(x),$$

with

$$\alpha_j = \int_{\mathbb{R}} K_n(x, c) P_j^c(x) (x - c)^2 d\mu(x).$$

Therefore, we only need to show that

$$\text{a) } \alpha_n = \frac{k_n}{k_n^c} P_n(c); \quad \text{b) } \alpha_{n-1} = -\frac{k_{n-1}^c}{k_{n+1}} P_{n+1}(c); \quad \text{c) } \alpha_j = 0, \quad j = 0, 1, \dots, n-2.$$

Part a) can be obtained by looking at the leading coefficients. Part c) is an easy consequence of a well-known property of the kernels  $K_n$ : if  $R_n$  is a polynomial of degree at most  $n$ , then

$$\int_{\mathbb{R}} K_n(x, c) R_n(x) d\mu(x) = R_n(c).$$

In order to prove equation b), we use Christoffel-Darboux formula (see [10], for example) and the orthonormality of  $\{P_n\}$  with respect to  $\mu$ :

$$\begin{aligned} \alpha_{n-1} &= \int_{\mathbb{R}} K_n(x, c) P_{n-1}^c(x) (x - c)^2 d\mu(x) = \int_{\mathbb{R}} [K_n(x, c) (x - c)] [P_{n-1}^c(x) (x - c)] d\mu(x) \\ &= \int_{\mathbb{R}} \frac{k_n}{k_{n+1}} [P_n(c) P_{n+1}(x) - P_{n+1}(c) P_n(x)] [P_{n-1}^c(x) (x - c)] d\mu(x) \\ &= -\frac{k_n}{k_{n+1}} P_{n+1}(c) \int_{\mathbb{R}} P_n(x) [P_{n-1}^c(x) (x - c)] d\mu(x) = -\frac{k_{n-1}^c}{k_{n+1}} P_{n+1}(c) \end{aligned}$$

and the lemma is proved.

In order to find bounds for the orthogonal polynomials and the kernels, it is important to know the size of the coefficients which appear in the formulae we are going to deal with. In the case of measures in  $M(0, 1)$ , we have:

**Lemma 2.** *Assume  $\text{supp } \mu = [-1, 1]$ ,  $\mu \in M(0, 1)$ . Let  $c \in [-1, 1]$ . Then*

$$\lim_{n \rightarrow \infty} \frac{k_n}{k_n^c} = \frac{1}{2}, \quad \lim_{n \rightarrow \infty} \frac{k_{n-1}^c}{k_{n+1}} = \frac{1}{2}, \quad \lim_{n \rightarrow \infty} \frac{k_n}{k_{n-1}^c} = 1.$$

*Proof.* The first limit is a consequence of a result of Máté, Nevai and Totik (see [4], theorem 11), from which it follows

$$\lim_{n \rightarrow \infty} \frac{k_n^c}{k_n} = \exp\left(-\frac{1}{4\pi} \int_0^{2\pi} \log(\cos t - c)^2 dt\right).$$

It is not difficult to see that the integral is equal to  $-4\pi \log 2$ , for every  $c \in [-1, 1]$ .

The other limits can be obtained from the first one, since from our hypothesis it follows that  $k_{n-1}/k_n = a_n \rightarrow 1/2$ . The lemma is proved.

**Lemma 3.** Let  $\mu$  be a measure on  $\mathbb{R}$  and  $n \geq 1$ . With the above notation,

$$\int_{\mathbb{R}} P_{n-1}^c(x)(x-c)d\mu(x) = -\frac{k_{n-1}^c}{k_n} \frac{P_n(c)}{K_{n-1}(c,c)}.$$

*Proof.* We can write

$$P_{n-1}^c(x)(x-c) = \sum_{j=0}^n \alpha_j P_j(x).$$

By looking at the leading coefficients,  $\alpha_n = \frac{k_{n-1}^c}{k_n}$ . If  $j = 1, 2, \dots, n-1$ , the orthonormality properties of  $\{P_n\}$  and  $\{P_n^c\}$  yield

$$\begin{aligned} \alpha_j &= \int_{\mathbb{R}} P_{n-1}^c(x)(x-c)P_j(x)d\mu(x) \\ &= \int_{\mathbb{R}} P_{n-1}^c(x) \frac{P_j(x) - P_j(c)}{x-c} (x-c)^2 d\mu(x) + \int_{\mathbb{R}} P_{n-1}^c(x)P_j(c)(x-c)d\mu(x) \\ &= P_j(c) \int_{\mathbb{R}} P_{n-1}^c(x)(x-c)d\mu(x). \end{aligned}$$

For  $\alpha_0$  we obtain the same expression, because  $P_0(x)$  is a constant:

$$\alpha_0 = \int_{\mathbb{R}} P_{n-1}^c(x)(x-c)P_0(x)d\mu(x) = P_0(c) \int_{\mathbb{R}} P_{n-1}^c(x)(x-c)d\mu(x).$$

Therefore

$$\begin{aligned} P_{n-1}^c(x)(x-c) &= \frac{k_{n-1}^c}{k_n} P_n(x) + \sum_{j=0}^{n-1} P_j(c) \left[ \int_{\mathbb{R}} P_{n-1}^c(u)(u-c)d\mu(u) \right] P_j(x) \\ &= \frac{k_{n-1}^c}{k_n} P_n(x) + K_{n-1}(x,c) \int_{\mathbb{R}} P_{n-1}^c(u)(u-c)d\mu(u). \end{aligned}$$

The lemma follows immediately taking  $x = c$  in this equality.

We can now obtain an expression for the polynomials orthonormal with respect to the measure  $\mu + M\delta_c$  in terms of the polynomials  $\{P_n\}$  and  $\{P_n^c\}$ .

**Proposition 4.** Let  $\mu$  be a measure on  $\mathbb{R}$ ,  $c \in \mathbb{R}$ ,  $M > 0$ . Let  $\{Q_n\}_{n \geq 0}$  be the polynomials orthonormal with respect to  $\mu + M\delta_c$ . Then, for each  $n \in \mathbb{N}$  there exist two constants  $A_n, B_n \in (0, 1)$  such that

$$Q_n(x) = A_n P_n(x) + B_n (x-c) P_{n-1}^c(x). \quad (1)$$

Furthermore, if  $\text{supp } \mu = [-1, 1]$ ,  $\mu \in M(0, 1)$  and  $c \in [-1, 1]$ , then

$$\lim_{n \rightarrow \infty} A_n K_{n-1}(c, c) = \frac{1}{\lambda(c) + M}$$

and

$$\lim_{n \rightarrow \infty} B_n = \frac{M}{\lambda(c) + M},$$

where

$$\lambda(c) = \lim_{n \rightarrow \infty} \frac{1}{K_n(c, c)}.$$

*Proof.* We will find firstly a constant  $C_n$  such that  $P_n(x) + C_n(x - c)P_{n-1}^c(x)$  is orthogonal to the polynomials of degree at most  $n - 1$  with respect to the measure  $\mu + M\delta_c$ . We only need to obtain

$$\int_{\mathbb{R}} [P_n(x) + C_n(x - c)P_{n-1}^c(x)](x - c)^j [d\mu(x) + M\delta_c(x)] = 0, \quad j = 0, 1, \dots, n - 1. \quad (2)$$

Let  $j \geq 1$ . Then

$$\begin{aligned} & \int_{\mathbb{R}} [P_n(x) + C_n(x - c)P_{n-1}^c(x)](x - c)^j [d\mu(x) + M\delta_c(x)] \\ &= \int_{\mathbb{R}} [P_n(x) + C_n(x - c)P_{n-1}^c(x)](x - c)^j d\mu(x) \\ &= \int_{\mathbb{R}} P_n(x)(x - c)^j d\mu(x) + C_n \int_{\mathbb{R}} P_{n-1}^c(x)(x - c)^{j-1} d\mu^c(x) = 0. \end{aligned}$$

Therefore, all we have to do is to find a constant  $C_n$  for which (2) is verified with  $j = 0$ . In this case, we can calculate the integral in (2):

$$\begin{aligned} & \int_{\mathbb{R}} [P_n(x) + C_n(x - c)P_{n-1}^c(x)] [d\mu(x) + M\delta_c(x)] \\ &= MP_n(c) + \int_{\mathbb{R}} [P_n(x) + C_n(x - c)P_{n-1}^c(x)] d\mu(x) \\ &= MP_n(c) + C_n \int_{\mathbb{R}} (x - c)P_{n-1}^c(x) d\mu(x) = P_n(c) \left[ M - C_n \frac{k_{n-1}^c}{k_n} \frac{1}{K_{n-1}(c, c)} \right], \end{aligned}$$

according to lemma 3. If we take

$$C_n = M \frac{k_n}{k_{n-1}^c} K_{n-1}(c, c),$$

then  $P_n(x) + C_n(x - c)P_{n-1}^c(x)$  is orthogonal to every polynomial of degree at most  $n - 1$ . As  $C_n > 0$ , it is a polynomial of degree  $n$  and leading coefficient positive. Thus, we will obtain the orthonormal polynomial  $Q_n$  by dividing it by its  $L^2(\mu + M\delta_c)$ -norm.

$$\begin{aligned} & \|P_n(x) + C_n(x - c)P_{n-1}^c(x)\|_{L^2(d\mu + M\delta_c)}^2 \\ &= MP_n(c)^2 + \int_{\mathbb{R}} [P_n(x) + C_n(x - c)P_{n-1}^c(x)]^2 d\mu(x) \\ &= MP_n(c)^2 + \int_{\mathbb{R}} P_n(x)^2 d\mu(x) + C_n^2 \int_{\mathbb{R}} P_{n-1}^c(x)^2 (x - c)^2 d\mu(x) \\ &\quad + 2C_n \int_{\mathbb{R}} P_n(x)(x - c)P_{n-1}^c(x) d\mu(x) \\ &= MP_n(c)^2 + 1 + C_n^2 + 2C_n \frac{k_{n-1}^c}{k_n}. \end{aligned}$$

If we denote

$$D_n = [MP_n(c)^2 + 1 + C_n^2 + 2C_n \frac{k_{n-1}^c}{k_n}]^{1/2},$$

then we have

$$Q_n(x) = \frac{1}{D_n}P_n(x) + \frac{C_n}{D_n}(x-c)P_{n-1}^c(x),$$

that is, equation (1) with  $A_n = 1/D_n$  and  $B_n = C_n/D_n$ . From its definition, it is clear that  $D_n > 1$  and  $D_n > C_n$ , so  $A_n, B_n \in (0, 1)$ .

For the second part, let us assume  $\text{supp } \mu = [-1, 1]$ ,  $\mu \in M(0, 1)$  and  $c \in [-1, 1]$ . From the above definitions for  $A_n$ ,  $C_n$  and  $D_n$ , we have

$$\frac{1}{A_n K_{n-1}(c, c)} = [M \frac{P_n(c)^2}{K_{n-1}(c, c)^2} + \frac{1}{K_{n-1}(c, c)^2} + M^2 (\frac{k_n}{k_{n-1}^c})^2 + \frac{2M}{K_{n-1}(c, c)}]^{1/2}. \quad (3)$$

Now, from  $\mu \in M(0, 1)$  it follows (see [7], theorem 3, p. 26, or [8])

$$\lim_{n \rightarrow \infty} \frac{P_n(x)^2}{K_{n-1}(x, x)} = 0 \quad \forall x \in [-1, 1].$$

Since  $K_{n-1}(c, c) \geq P_0^2$  this also implies

$$\lim_{n \rightarrow \infty} \frac{P_n(c)^2}{K_{n-1}(c, c)^2} = 0.$$

From this and lemma 2 we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{A_n K_{n-1}(c, c)} = \lambda(c) + M.$$

Finally,

$$\lim_{n \rightarrow \infty} B_n = \lim_{n \rightarrow \infty} A_n M \frac{k_n}{k_{n-1}^c} K_{n-1}(c, c) = \frac{M}{\lambda(c) + M}$$

and the proposition is completely proved.

**Remark.** If  $P_n(c) = 0$ , it is easy to show directly that  $Q_n = P_n$ . This is not in contradiction with our proposition, since in this case it can also be proved that  $P_n(x) = (x-c)P_{n-1}^c(x)$  and  $A_n + B_n = 1$ .

**Corollary 5.** Let  $\text{supp } \mu = [-1, 1]$ ,  $c \in [-1, 1]$ ,  $M > 0$ . Then,  $\mu \in M(0, 1)$  if and only if  $\mu + M\delta_c \in M(0, 1)$ .

*Proof.* a) If  $\mu \in M(0, 1)$ , from [4], theorem 11, we have  $\mu^c \in M(0, 1)$ . Now, from (1) and the fact that  $\lim_n (A_n + B_n) = 1$ , it is easy to deduce that  $\mu + M\delta_c \in M(0, 1)$ .

b) From [4], theorem 11 again, if  $\mu + M\delta_c \in M(0, 1)$  then  $(x-c)^2[d\mu + M\delta_c] = (x-c)^2 d\mu \in M(0, 1)$  and this implies  $\mu \in M(0, 1)$ .

We can also find some relations which involve the kernels.

**Proposition 6.** Let  $\mu$  be a measure on  $\mathbb{R}$ ,  $c \in \mathbb{R}$  and  $M > 0$ . Let  $\{L_n\}_{n \geq 0}$  be the kernels relative to  $\mu + M\delta_c$ . Then, for each  $n \in \mathbb{N}$

$$L_n(x, y) = \frac{1}{1 + MK_n(c, c)} K_n(x, y) + \frac{MK_n(c, c)}{1 + MK_n(c, c)} (x - c)(y - c) K_{n-1}^c(x, y).$$

*Proof.* If  $y \in \mathbb{R}$ , it is a well-known fact that the kernels  $\{K_n^c\}$  verify

$$\int_{\mathbb{R}} R_n(x) K_n^c(x, y) (x - c)^2 d\mu(x) = R_n(y)$$

for every polynomial  $R_n$  of degree at most  $n$ . Actually, this property characterizes the kernels relative to any measure.

If we write

$$(x - c)(y - c) K_{n-1}^c(x, y) = \sum_{j=0}^n \alpha_j(y) P_j(x), \quad (4)$$

then it is easy to show for  $j \geq 1$  that

$$\begin{aligned} \alpha_j(y) &= (y - c) \int_{\mathbb{R}} K_{n-1}^c(x, y) \frac{P_j(x) - P_j(c)}{x - c} (x - c)^2 d\mu(x) \\ &\quad + (y - c) P_j(c) \int_{\mathbb{R}} K_{n-1}^c(x, y) (x - c) d\mu(x). \end{aligned}$$

By the above property, we obtain

$$\alpha_j(y) = P_j(y) - P_j(c) + (y - c) P_j(c) \int_{\mathbb{R}} K_{n-1}^c(x, y) (x - c) d\mu(x)$$

and it is immediate to see that  $\alpha_0$  also verifies this formula.

From this formula and (4) it follows

$$\begin{aligned} &(x - c)(y - c) K_{n-1}^c(x, y) \\ &= K_n(x, y) - K_n(x, c) + (y - c) K_n(x, c) \int_{\mathbb{R}} K_{n-1}^c(u, y) (u - c) d\mu(u). \end{aligned}$$

If we let  $x = c$ , we obtain

$$(y - c) \int_{\mathbb{R}} K_{n-1}^c(x, y) (x - c) d\mu(x) = 1 - \frac{K_n(c, y)}{K_n(c, c)}$$

and, replacing this equation into the previous one,

$$\begin{aligned} &\frac{1}{1 + MK_n(c, c)} K_n(x, y) + \frac{MK_n(c, c)}{1 + MK_n(c, c)} (x - c)(y - c) K_{n-1}^c(x, y) \\ &= K_n(x, y) - \frac{MK_n(c, y)}{1 + MK_n(c, c)} K_n(x, c). \end{aligned}$$

Therefore, it will be enough to prove that

$$\int_{\mathbb{R}} [K_n(x, y) - \frac{MK_n(c, y)}{1 + MK_n(c, c)} K_n(x, c)] R_n(x) [d\mu(x) + M\delta_c(x)] = R_n(y)$$

whenever  $R_n$  is a polynomial of degree at most  $n$ . This is an easy consequence of the fact that the kernels  $\{K_n\}$  verify the analogous property with respect to the measure  $\mu$ . The proposition is proved.

## §2. Generalized Jacobi weights with mass points.

Let  $w$  be a generalized Jacobi weight, that is:

$$w(x) = h(x)(1-x)^\alpha(1+x)^\beta \prod_{i=1}^N |x-t_i|^{\gamma_i}, \quad x \in [-1, 1]$$

where:

a)  $\alpha, \beta, \gamma_i > -1$ ,  $t_i \in (-1, 1)$ ,  $t_i \neq t_j \forall i \neq j$ ;

b)  $h$  is a positive, continuous function on  $[-1, 1]$  and  $w(h, \delta)\delta^{-1} \in L^1(0, 2)$ ,  $w(h, \delta)$  being the modulus of continuity of  $h$ .

If we define

$$d(x, n) = (1-x+n^{-2})^{-\alpha/2-1/4}(1+x+n^{-2})^{-\beta/2-1/4} \prod_{i=1}^N (|x-t_i|+n^{-1})^{-\gamma_i/2},$$

then the polynomials  $\{P_n\}$  orthonormal with respect to the measure  $w(x)dx$  on the interval  $[-1, 1]$  verify the estimate

$$|P_n(x)| \leq Cd(x, n) \quad \forall x \in [-1, 1], \quad \forall n \geq 1, \quad (5)$$

where  $C$  is a constant independent of  $n$  and  $x$  (see [2]). In the sequel  $C$  will denote a constant independent of  $n$  and  $x$ , but possibly different in each occurrence.

As to the kernels, it can be shown (see [7], p. 120 and p. 4) that

$$K_n(x, x) \sim n(1-x+n^{-2})^{-\alpha-1/2}(1+x+n^{-2})^{-\beta-1/2} \prod_{i=1}^N (|x-t_i|+n^{-1})^{-\gamma_i} \quad (6)$$

uniformly in  $|x| \leq 1$ ,  $n \geq 1$ , where by  $f \sim g$  in a domain  $D$  we mean that there exist some positive constants  $C_1$  and  $C_2$  such that  $C_1 f(y) \leq g(y) \leq C_2 f(y) \forall y \in D$ .

Our aim is to prove similar bounds for the polynomials and the kernels relative to a measure which consists of a generalized Jacobi weight and a finite number of mass points on the interval  $[-1, 1]$ . So, let  $k \in \mathbb{N}$ ,  $a_i \in [-1, 1]$  and  $M_i > 0$ ,  $i = 1, \dots, k$ . We will denote

$$d\nu = w(x)dx + \sum_{i=1}^k M_i \delta_{a_i}$$

on the interval  $[-1, 1]$ . By  $\{Q_n\}$  and  $\{L_n\}$  we mean, respectively, their orthonormal polynomials and kernels. Without loss of generality we can assume  $a_i \in \{1, -1, t_1, \dots, t_N\}$ , since in the definition of  $w$  we can allow some of the exponents to be 0. Furthermore, for every  $t \in [-1, 1]$  we can speak of its exponent in  $w$ , referring to the exponent of the factor  $|x-t|^\gamma$  in  $w$ . Obviously, there are only finitely many points with an exponent different from 0.

With this notation, we can deduce some bounds from the results of the previous section. Notice that  $w > 0$  a.e. on  $[-1, 1]$ , so that the measure  $w(x)dx$  belongs to  $M(0, 1)$ .



**Proposition 7.** *There exists a constant  $C$  such that for each  $n \geq 1$ ,  $x \in [-1, 1]$*

$$\begin{aligned} \text{a) } |Q_n(x)| &\leq C(1-x+n^{-2})^{-\alpha/2-1/4}(1+x+n^{-2})^{-\beta/2-1/4} \prod_{i=1}^N (|x-t_i|+n^{-1})^{-\gamma_i/2}; \\ \text{b) } |L_n(x, x)| &\leq Cn(1-x+n^{-2})^{-\alpha-1/2}(1+x+n^{-2})^{-\beta-1/2} \prod_{i=1}^N (|x-t_i|+n^{-1})^{-\gamma_i}. \end{aligned}$$

*Proof.* a) We are going to prove the bound for  $Q_n$  by induction on the number  $k$  of mass points. If  $k = 0$ , the measure is a generalized Jacobi weight and we already know the formula (5). Let  $k > 0$  and assume the property holds for  $k - 1$  mass points.

Let  $\{P_n\}$  be the orthonormal polynomials with respect to the measure

$$d\mu = w(x)dx + \sum_{i=1}^{k-1} M_i \delta_{a_i}$$

so that, according to the notation we used in section 1,  $\{P_n^{a_k}\}$  are the polynomials orthonormal with respect to

$$(x - a_k)^2 d\mu(x) = (x - a_k)^2 w(x)dx + \sum_{i=1}^{k-1} (a_i - a_k)^2 M_i \delta_{a_i}.$$

Since  $d\nu = d\mu + M_k \delta_{a_k}$ , from proposition 4 it follows

$$Q_n(x) = A_n P_n(x) + B_n (x - a_k) P_{n-1}^{a_k}(x),$$

with  $A_n, B_n \in (0, 1)$ . Taking into account that both  $d\mu$  and  $(x - a_k)^2 d\mu(x)$  are generalized Jacobi weights with  $k - 1$  mass points, they satisfy the boundedness in the statement. Now, it is easy to see that  $Q_n$  satisfies that boundedness.

Therefore, part a) is proved. As to b), proposition 6 yields

$$L_n(x, x) = C_n K_n(x, x) + (1 - C_n)(x - a_k)^2 K_{n-1}^{a_k}(x, x) \quad (7)$$

with  $C_n \in (0, 1)$ . Similar arguments and formula (6) lead to this bound and the proposition is completely proved.

The previous result establishes only upper bounds, which sometimes is not enough. In some applications (for example, in the study of the convergence of the Fourier series) it is necessary to estimate more exactly the rate of growth of  $L_n(x, x)$ , at least at some points. In the case of a generalized Jacobi weight, with no point masses, we have even uniform estimates (formula (6)). These estimates cannot hold when the measure has mass points, since there the kernels  $L_n(x, x)$  are bounded (see [7], p. 4, for example). However, we can obtain such estimates uniformly on compact sets not containing mass points.

**Proposition 8.** *Let  $\varepsilon > 0$ . Then,*

$$L_n(x, x) \sim n(1 - x + n^{-2})^{-\alpha-1/2}(1 + x + n^{-2})^{-\beta-1/2} \prod_{i=1}^N (|x - t_i| + n^{-1})^{-\gamma_i}$$

uniformly in  $|x - a_i| \geq \varepsilon$  ( $i = 1, \dots, k$ ),  $|x| \leq 1$ ,  $n \in \mathbb{N}$ .

*Proof.* We only need to prove that

$$L_n(x, x) \geq Cn(1 - x + n^{-2})^{-\alpha-1/2}(1 + x + n^{-2})^{-\beta-1/2} \prod_{i=1}^N (|x - t_i| + n^{-1})^{-\gamma_i}$$

uniformly in  $|x - a_i| \geq \varepsilon$  ( $i = 1, \dots, k$ ),  $|x| \leq 1$  and  $n$  large enough. This follows by induction, using formula (7) again and having in mind that  $0 < C_n < 1$ .

As an application of the results of section 1, some bounds for  $L_n(x, a_i)$  can also be obtained.

**Proposition 9.** *a) Let  $1 \leq i \leq k$  and suppose  $a_i \neq \pm 1$ . Then there exists a constant  $C$  such that for each  $x \in [-1, 1]$  and  $n \geq 1$*

$$|L_n(x, a_i)| \leq C(1 - x + n^{-2})^{-\alpha/2-1/4}(1 + x + n^{-2})^{-\beta/2-1/4} \prod_{t_j \neq a_i} (|x - t_j| + n^{-1})^{-\gamma_j/2}.$$

*b) If 1 is a mass point, there exists a constant  $C$  such that for each  $x \in [-1, 1]$  and  $n \geq 1$*

$$|L_n(x, 1)| \leq C(1 + x + n^{-2})^{-\beta/2-1/4} \prod_{i=1}^N (|x - t_i| + n^{-1})^{-\gamma_i/2}.$$

*c) If -1 is a mass point, there exists a constant  $C$  such that for each  $x \in [-1, 1]$  and  $n \geq 1$*

$$|L_n(x, -1)| \leq C(1 - x + n^{-2})^{-\alpha/2-1/4} \prod_{i=1}^N (|x - t_i| + n^{-1})^{-\gamma_i/2}.$$

*Proof.* a) Assume  $1 \leq i \leq k$  and  $a_i \neq \pm 1$ . Let  $\gamma$  be the exponent of  $|x - a_i|$  in  $w$ . If we denote

$$d\mu = w(x)dx + \sum_{j=1, j \neq i}^k M_j \delta_{a_j},$$

then  $\nu = \mu + M_i \delta_{a_i}$ . Let  $\{P_n\}$  and  $\{K_n\}$  be the orthonormal polynomials and the kernels relative to  $\mu$  and  $k_n$  the leading coefficient of  $P_n$ . Analogously,  $\{P_n^{a_i}\}$ ,  $\{K_n^{a_i}\}$  and  $\{k_n^{a_i}\}$  with respect to  $(x - a_i)^2 d\mu$ .

If we write

$$\Psi_n(x) = (1 - x + n^{-2})^{-\alpha/2-1/4}(1 + x + n^{-2})^{-\beta/2-1/4} \prod_{t_j \neq a_i} (|x - t_j| + n^{-1})^{-\gamma_j/2}$$

what we have to prove is  $|L_n(x, a_i)| \leq C\Psi_n(x)$ . Now, from proposition 6 and lemma 1 we obtain

$$L_n(x, a_i) = \frac{k_n}{k_n^{a_i}} \frac{P_n(a_i)}{1 + M_i K_n(a_i, a_i)} P_n^{a_i}(x) - \frac{k_{n-1}^{a_i}}{k_{n+1}} \frac{P_{n+1}(a_i)}{1 + M_i K_n(a_i, a_i)} P_{n-1}^{a_i}(x).$$

We only need to estimate the right hand side. From proposition 7 we get

$$\begin{aligned} |P_n(a_i)| &\leq Cn^{\gamma/2}; \\ |P_{n+1}(a_i)| &\leq Cn^{\gamma/2}; \\ |P_n^{a_i}(x)| &\leq C(|x - a_i| + n^{-1})^{-(\gamma+2)/2} \Psi_n(x); \\ |P_{n-1}^{a_i}(x)| &\leq C(|x - a_i| + n^{-1})^{-(\gamma+2)/2} \Psi_n(x). \end{aligned}$$

Since  $a_i$  is not a mass point for  $\mu$ , proposition 8 yields

$$K_n(a_i, a_i) \sim n^{1+\gamma}.$$

Finally, by lemma 2

$$\lim_{n \rightarrow \infty} \frac{k_n}{k_n^{a_i}} = \frac{1}{2}; \quad \lim_{n \rightarrow \infty} \frac{k_{n-1}^{a_i}}{k_{n+1}} = \frac{1}{2}.$$

It is now easy to deduce

$$|L_n(x, a_i)| \leq Cn^{-1-\gamma/2} (|x - a_i| + n^{-1})^{-1-\gamma/2} \Psi_n(x) \leq C\Psi_n(x).$$

b) Assume 1 is a mass point. We define now

$$d\mu = w(x)dx + \sum_{i=1, a_i \neq 1}^k M_i \delta_{a_i},$$

so  $d\nu = d\mu + M\delta_1$ ,  $M > 0$ . If, according to our usual notation,  $\{P_n\}$ ,  $\{K_n\}$  and  $\{k_n\}$  refer to  $d\mu$  and  $\{R_n\}$  are the orthonormal polynomials relative to the measure  $(1-x)d\mu$ ,  $\{r_n\}$  being their leading coefficients, it is not difficult to show that

$$K_n(x, 1) = \frac{k_n}{r_n} P_n(1) R_n(x)$$

(only standard properties of  $K_n(x, 1)$  are needed). Thus, proposition 6 leads to

$$L_n(x, 1) = \frac{k_n}{r_n} \frac{P_n(1)}{1 + M K_n(1, 1)} R_n(x).$$

We proceed now analogously to part a), since  $d\mu$  and  $(1-x)d\mu$  are generalized Jacobi weights with masses at points different from 1. Notice that, by Hölder's inequality

$$\begin{aligned} \frac{k_n}{r_n} &= \int_{-1}^1 R_n(x) P_n(x) (1-x) d\mu(x) \\ &\leq \left( \int_{-1}^1 R_n(x)^2 (1-x)^2 d\mu(x) \right)^{1/2} \left( \int_{-1}^1 P_n(x)^2 d\mu(x) \right)^{1/2} \\ &\leq \sqrt{2} \left( \int_{-1}^1 R_n(x)^2 (1-x) d\mu(x) \right)^{1/2} \left( \int_{-1}^1 P_n(x)^2 d\mu(x) \right)^{1/2} = \sqrt{2}. \end{aligned}$$

Part c) is similar to b). Thus, the result is proved.

**Remark.** As it was pointed out in the introduction, the main application of this kind of estimates would be in the study of the convergence of Fourier series. This will be considered in a forthcoming paper.

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