MEAN AND WEAK CONVERGENCE OF FOURIER-BESSEL SERIES

by

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ABSTRACT: *We study the uniform boundedness on some weighted L^p spaces of the partial sum operators associated to Fourier-Bessel series, obtaining necessary and sufficient conditions for this boundedness in terms of the weights. On the other hand, we study the weak and restricted weak type of the partial sum operators in the end points of the interval of strong boundedness.*

§0. Introduction.

Let J_α be the Bessel function of order $\alpha > -1$, and $j_n^\alpha(x) = 2^{1/2}|J_{\alpha+1}(\alpha_n)|^{-1}\, J_\alpha(\alpha_n x)$ (where $\{\alpha_n\}$ is the increasing sequence of the zeroes of J_α) the Bessel system of order α, which is orthonormal and complete in $L^2((0,1); xdx)$, and therefore the Fourier series of a function $f \in L^2((0,1); xdx)$ with respect to this system converges to f in the L^2-norm.

The next step is to ask for which $p \in (1,\infty)$, $p \neq 2$ the above convergence is true. The problem can be posed, by the Banach-Steinhauss theorem, in terms of the uniform boundedness on $L^p((0,1); xdx)$ of the partial sum operators

$$S_n^\alpha f(x) = \sum_{k=1}^n c_k j_k^\alpha(x)\ , \quad c_k = c_k(f) = \int_0^1 j_k^\alpha(y) f(y) y dy.$$

The problem was solved by Wing **[13]** (in the case $\alpha \geq -1/2$) and Benedek and Panzone **[1]** (general case). Moreover, these authors obtained some weighted norm estimates of the form

$$\|S_n^\alpha f(x) U(x)\|_p \leq C \|f(x) V(x)\|_p \tag{0.1}$$

where C is a constant independent of n, $U(x) = V(x)$ are functions of radial type x^a and by $\|.\|_p$ we denote the $L^p((0,1); xdx)$-norm.

The first objective of this paper is to extend inequality (0.1) to a wider class of weights $U(x)$ and $V(x)$, not necessarily the same. We obtain sufficient conditions in terms of A_p-conditions (propositions 1 and 2) and we solve completely the problem (theorem 1) when the weights are of the form

$$U(x) = x^a (1-x)^b \prod_{k=1}^m |x - x_k|^{b_k};\ 0 < x_1 < \ldots < x_m < 1;\ a, b, b_k \in \mathbb{R}. \tag{0.2}$$

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The inequality (0.1) is closely related to the problem of the boundedness of the Hilbert transform with two weights. In this sense, the use of pairs of weights in the Muckenhoupt A_p classes (see **[9]**, **[10]**) will be one of the main ingredients in the proofs. We obtain also checkable necessary conditions of integrability on U, V from the inequality (0.1) (theorem 2) in the spirit of Máté, Nevai and Totik's conditions (see **[8]**) for systems of orthogonal polynomials on $[-1, 1]$.

On the other hand, we shall study a problem related to the end points of the interval of mean convergence: by interpolation, the range of p's such that the uniform boundedness (0.1) holds is always an interval. When $U(x) = V(x) = 1$ (for example), this interval is open, say (p_0, p_1), and then the natural question is if S_n^α are "uniformly" of weak type (p_0, p_0) or/and (p_1, p_1). That is, if we denote

$$\|g\|_{L^p_*((0,1);U(x)^p xdx)} = \{\sup_{\lambda>0} \lambda^p \int_{\{x:|g(x)|>\lambda\}} U(x)^p xdx\}^{1/p}$$

then the question is if

$$\|S_n^\alpha f\|_{L^p_*((0,1);U(x)^p xdx)} \leq C\|f(x)V(x)\|_p \tag{0.3}$$

with a constant C independent of n, and $p = p_0$ or p_1.

This "end point problem" has been studied by different authors for another operators (see **[2]**, **[3]**, **[6]**, **[11]**), but in the case of Fourier-Bessel series it seems to be new, even in the unweighted case. In theorem 3, we give necessary conditions on U, V for (0.3) to be true. In the particular case $\alpha \geq -1/2$ and $U(x) = V(x)$ of the form (0.2), we show that (0.3) is false for both end points (theorem 4). Finally, when $\alpha \geq -1/2$ and $U(x) = V(x) = 1$ we show (theorem 5) that S_n^α are "uniformly" of restricted weak type at the end points, i.e., (0.3) is true, but only for functions $f = \chi_E$ where E is a measurable set contained in $(0, 1)$.

The organization of the paper is as follows: in section 1, we exhibit the main results with the notations which are needed to make them understandable. In section 2, we collect several technical lemmas concerning A_p weights and the boundedness of some operators associated to the problem which we shall use for the proofs of the main results which will be given in section 3.

§1. Main results.

Let $1 < p < \infty$ and $-\infty \leq a < b \leq \infty$. $A_p(a, b)$ will stand for the Muckenhoupt classes (see **[4]**, **[9]**) consisting of those pairs of nonnegative functions (u, v) on (a, b) such that

$$(|I|^{-1} \int_I u)(|I|^{-1} \int_I v^{-p'/p})^{p/p'} \leq C \quad \text{for every interval } I \subseteq (a, b),$$

where $p + p' = pp'$.

We say that $(u, v) \in A_p^\delta(a, b)$ (where $\delta > 1$) if $(u^\delta, v^\delta) \in A_p(a, b)$. It is clear from Holder's inequality that $A_p^\delta(a, b) \subseteq A_p(a, b)$.

We say that a sequence $\{(u_n, v_n)\}_{n=1}^\infty \in A_p^\delta(a, b)$ "uniformly" if $(u_n, v_n) \in A_p^\delta(a, b)$, $\forall n$ and the constant appearing in the definition is independent of n.

In the inequality $\|S_n^\alpha f(x)U(x)\|_p \le C\|f(x)V(x)\|_p$ we shall always mean that C is a constant independent of f and n (and the same for the weak L^p_*-norm).

In our two first results, we distinguish the cases $\alpha \ge -1/2$ and $-1 < \alpha < -1/2$. The reason for that is the different assymptotic behavior of the Bessel functions in each case.

Proposition 1. *(Case $\alpha \ge -1/2$). Let $1 < p < \infty$. Let U, V be weights on $(0,1)$. If $(U(x)^p x^{1-p/2}, V(x)^p x^{1-p/2}) \in A_p^\delta(0,1)$ for some $\delta > 1$ ($\delta = 1$ if $U = V$), then*

$$\|S_n^\alpha f(x)U(x)\|_p \le C\|f(x)V(x)\|_p.$$

Proposition 2. *(Case $-1 < \alpha < -1/2$). Let $1 < p < \infty$ and $M_n = (\alpha_n + \alpha_{n+1})/2$. Let U, V be weights on $(0,1)$. If there exists $\delta > 1$ ($\delta = 1$ if $U = V$) such that*

$$((M_n^{-1} + x)^{-p(\alpha+1/2)} U(x)^p x^{\alpha p+1}, (M_n^{-1} + x)^{-p(\alpha+1/2)} V(x)^p x^{\alpha p+1}) \in A_p^\delta(0,1), \qquad (1.1)$$

$$((M_n^{-1} + x)^{p(\alpha+1/2)} U(x)^p x^{-\alpha p-p+1}, (M_n^{-1} + x)^{p(\alpha+1/2)} V(x)^p x^{-\alpha p-p+1}) \in A_p^\delta(0,1) \quad (1.2)$$

"uniformly", then $\|S_n^\alpha f(x)U(x)\|_p \le C\|f(x)V(x)\|_p$.

Note. When $U = V$ it suffices $\delta = 1$ because from reverse Holder's inequality it follows that $(u,u) \in A_p \Rightarrow (u,u) \in A_p^\delta$ for some $\delta > 1$ (see [**4**]).

Applying both propositions to particular weights of the form (0.2), we have

Theorem 1. *Let $\alpha > -1$, $1 < p < \infty$ and the weights*

$$U(x) = x^a(1-x)^b \prod_{k=1}^m |x - x_k|^{b_k}, \quad V(x) = x^A(1-x)^B \prod_{k=1}^m |x - x_k|^{B_k},$$

where $0 < x_1 < ... < x_m < 1$ and $a, A, b, B, b_k, B_k \in \mathbb{R}$. If the conditions:

$$\begin{array}{llll} B \le b, & pB < p-1, & -1 < pb; & \\ B_k \le b_k, & pB_k < p-1, & -1 < pb_k & (1 \le k \le m); \end{array} \qquad (1.3)$$

$$A \le a, \qquad \left|\frac{1}{p} - \frac{1}{2} + \frac{a}{4} + \frac{A}{4}\right| < \frac{a-A}{4} + \min\{\frac{1}{4}, \frac{\alpha+1}{2}\} \qquad (1.4)$$

are fullfilled, then $\|S_n^\alpha f(x)U(x)\|_p \le C\|f(x)V(x)\|_p$.

The following general result about necessary conditions will imply, in particular, that (1.3), (1.4) are best possible.

Theorem 2. *If the inequality $\|S_n^\alpha f(x)U(x)\|_p \le C\|f(x)V(x)\|_p$ holds, then U, V must satisfy the conditions*

$$\int_0^1 U(x)^p x^{\alpha p+1} dx < +\infty, \qquad \int_0^1 U(x)^p x^{1-p/2} dx < +\infty \qquad (1.5)$$

$$\int_0^1 V(x)^{-p'} x^{\alpha p'+1} dx < +\infty, \qquad \int_0^1 V(x)^{-p'} x^{1-p'/2} dx < +\infty \qquad (1.6)$$

$$U(x) \le C\, V(x) \text{ a.e.} \qquad (1.7)$$

Corollary 1. *With the same notation as in theorem 1, the conditions (1.3) and (1.4) are also necessary for the uniform boundedness $\|S_n^\alpha f(x)U(x)\|_p \le C\|f(x)V(x)\|_p$.*

Regarding weak boundedness, we begin with a version of theorem 2 in this case:

Theorem 3. *Let $\alpha > -1$ and $1 < p < \infty$. If the inequality*

$$\|S_n^\alpha f\|_{L^p_*((0,1);U(x)^p xdx)} \le C\|f(x)V(x)\|_p$$

holds, then U, V must satisfy conditions (1.6), (1.7) and

$$\sup_{\lambda>0} \lambda^p \int_{\{x:x^\alpha>\lambda\}} U(x)^p xdx < +\infty, \quad \sup_{\lambda>0} \lambda^p \int_{\{x:x^{-1/2}>\lambda\}} U(x)^p xdx < +\infty. \qquad (1.5)'$$

From this theorem, straightforward calculations show that the weak boundedness (0.3) fails in the lower end point of the interval of strong boundedness given by (1.3) and (1.4). However, theorem 3 does not work in the upper end point. In this case, we have obtained that the weak boundedness is also false (in the particular case $\alpha \ge -1/2$ and $U(x) = V(x) = x^a(1-x)^b \prod_{k=1}^m |x-x_k|^{b_k}$), but the proof is more intricate, being necessary a more detailed analysis of the partial sum operators. Something very similar occurs when one considers Jacobi-Fourier series (see **[2]**, **[6]**). The result can be stated as follows:

Theorem 4. *Let $\alpha \ge -1/2$, $1 < p < \infty$ and $U(x) = x^a(1-x)^b \prod_{k=1}^m |x-x_k|^{b_k}$. Then, the following are equivalent:*

(i) $\|S_n^\alpha f\|_{L^p_*((0,1);U(x)^p xdx)} \le C\|f(x)U(x)\|_p$

(ii) $|\frac{1}{p} + \frac{a-1}{2}| < \frac{1}{4}$; $-1 < pb < p-1$; $-1 < pb_k < p-1$ $(1 \le k \le m)$.

Finally, following the method of **[2,** theorem **1]**, we establish this (p,p)-restricted weak type:

Theorem 5. *Let $\alpha \ge -1/2$ and $p = 4$ or $p = 4/3$.Then*

$$\|S_n^\alpha(\chi_E)\|_{L^p_*((0,1);xdx)} \le C\|\chi_E\|_p$$

for every measurable set $E \subseteq (0,1)$.

§2. Technical lemmas.

We denote by $K_n^\alpha(x,y) = \sum_{k=1}^n j_k^\alpha(x) j_k^\alpha(y)$ the kernel of the operator S_n^α, that is,

$$S_n^\alpha f(x) = \int_0^1 K_n^\alpha(x,y)f(y)ydy.$$

It is known the following decomposition of the kernel (see **[1]**, **[13]**):

$$\begin{aligned} K_n^\alpha(x,y) = {} & J_\alpha(M_n x)J_{\alpha+1}(M_n y)\frac{M_n}{2(y-x)} + J_\alpha(M_n y)J_{\alpha+1}(M_n x)\frac{M_n}{2(x-y)} + \\ & + J_\alpha(M_n x)J_{\alpha+1}(M_n y)\frac{M_n}{2(y+x)} + J_\alpha(M_n y)J_{\alpha+1}(M_n x)\frac{M_n}{2(x+y)} + \\ & + O(1)(xy)^\alpha + O(1)\frac{(xy)^{-1/2}}{2-x-y} = \sum_{i=1}^6 K_{n,i}^\alpha(x,y). \end{aligned}$$

So, we can decompose

$$S_n^\alpha f(x) = \sum_{i=1}^{6} S_{n,i}^\alpha f(x) = \sum_{i=1}^{6} \int_0^1 K_{n,i}^\alpha(x,y) f(y) y dy.$$

We shall use the following notations

$$Hf(x) = p.v. \int_0^1 \frac{f(y)}{x-y} dy; \quad Jf(x) = \int_0^1 \frac{f(y)}{2-x-y} dy; \quad J'f(x) = \int_0^1 \frac{f(y)}{x+y} dy.$$

The main tools for estimating the operators $S_{n,i}^\alpha f$ are:

$$J_\alpha(z) = z^\alpha 2^{-\alpha} \Gamma(\alpha+1)^{-1} + O(z^{\alpha+2}), \quad z \longrightarrow 0 \tag{2.1}$$

$$J_\alpha(z) = 2^{1/2} (\pi z)^{-1/2} \cos(z - \frac{\alpha\pi}{2} - \frac{\pi}{4}) + O(z^{-3/2}) \quad z \longrightarrow \infty \tag{2.2}$$

$$\int_0^1 J_\alpha(\alpha_n x)^2 x \, dx \approx \frac{1}{\pi \alpha_n}, \quad n \longrightarrow \infty \tag{2.3}$$

for $\alpha > -1$ (see **[12]**) and the following results:

Lemma 1 . *Let $1 < p < \infty$. If $(u,v) \in A_p^\delta(0,1)$ for some $\delta > 1$, then*

$$\|Tf(x)\|_{L^p((0,1);u(x)dx)} \le C \|f(x)\|_{L^p((0,1);v(x)dx)}.$$

where $T = H,\ J$ or J'.

Proof.

The result for H is due to Neugebauer **[10]**. In fact, Neugebauer showed that under the hypothesis there exists a $w \in A_p$ (i.e. $(w,w) \in A_p$) and constants C_1, C_2 such that $C_1 u(x) \le w(x) \le C_2 v(x)$ a.e., and then we can apply that H is bounded on $L^p((0,1);w(x)dx)$ if and only if $w \in A_p$ (see **[7]**). The last fact is also true for Mf, being M the Hardy-Littlewood maximal operator (see **[9]**).

Let now $T_1 f(x) = \frac{1}{x}\int_0^x f(y)dy$ the Hardy function and $T_2 f(x) = \int_x^1 \frac{f(y)}{y} dy$ its adjoint. Then, it is easy to see that

$$|J'f(x)| \le T_1(|f|)(x) + T_2(|f|)(x); \quad |T_1 f(x)| \le 2J'(|f|)(x); \quad |T_2 f(x)| \le 2J'(|f|)(x).$$

So, the weighted boundedness for J' is equivalent to the one for T_1 and T_2. The operator T_1 is controlled by M; and for its adjoint T_2 we use the duality property for A_p-classes: $(u,v) \in A_p$ if and only if $(v^{-p'/p}, u^{-p'/p}) \in A_{p'}$. So, J' is controlled by operators which are bounded under the hypothesis.

Finally, we can deal with the operator J in the same way as J' by making the change of variables $X = 1-x, Y = 1-y$ which does not change A_p conditions.

Note. The constants "C" appearing in the weighted norm inequalities depend only on the constants appearing in the definition of the A_p classes. Thus, for sequences of weights belonging to A_p^δ "uniformly", the corresponding weighted norm inequalities are also uniform.

For the proof of theorem 1, we need to analyze when certain sequences of weights of radial type belong to A_p classes "uniformly". For that, we give the

Lemma 2. *Let* $r, s, R, S \in \mathbb{R}$, $1 < p < \infty$ *and let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of positive numbers such that* $\lim x_n = 0$. *Then*

$$(x^r(x+x_n)^s, x^R(x+x_n)^S) \in A_p^{\delta}(0,1) \quad \text{for some} \quad \delta > 1 \Longleftrightarrow$$
$$-1 < r; \quad -1 < r+s; \quad R < p-1; \quad R+S < p-1; \quad R \le r; \quad R+S \le r+s \tag{2.4}$$

Proof.

Straightforward calculations show that for $0 \le a < b \le 1$, $0 \le c \le 1$, $\gamma > -1$, $\gamma + \mu > -1$, there exists a constant K, depending only on γ, μ, such that

$$\int_a^b x^{\gamma}(x+c)^{\mu}dx \le \begin{cases} Kb^{\gamma+\mu}(b-a), & \text{if } c \le b; \\ Kb^{\gamma}c^{\mu}(b-a), & \text{if } b \le c. \end{cases}$$

and then (2.4) implies

$$(\int_a^b x^r(x+c)^s dx)(\int_a^b (x^R(x+c)^S)^{-p'/p}dx)^{p/p'} \le K(b-a)^p,$$

that is, $(x^r(x+c)^s, x^R(x+c)^S) \in A_p(0,1)$ "uniformly" on c. Now, these pairs of weights belong to $A_p^{\delta}(0,1)$, for some $\delta > 1$ because the inequalities $-1 < r$; $-1 < r+s$; $R < p-1$; $R+S < p-1$ are strict.

Reciprocally, the A_p condition implies that $x^r(x+x_n)^s$ and $(x^R(x+x_n)^S)^{-1/p-1}$ are integrables on $(0,1)$ and then $-1 < r$, $R < p-1$. Taking limits on n we get the integrability of x^{r+s} and $(x^{R+S})^{-1/p-1}$ (even more, the pair $(x^{r+s}, x^{R+S}) \in A_p(0,1)$).This gives $-1 < r+s$ and $R+S < p-1$. Finally, by using that if $(u,v) \in A_p(0,1)$ then $u \le v$ *a.e.*, we have $R \le r$, $R+S \le r+s$.

On the other hand, we also need to deal with products of weights in A_p classes. In this sense it is not difficult to prove

Lemma 3. *Let* $\{u_n(x)\}_{n=0}^{\infty}$, $\{v_n(x)\}_{n=0}^{\infty}$, $\{U_n(x)\}_{n=0}^{\infty}$, $\{V_n(x)\}_{n=0}^{\infty}$ *be sequences of weights on a finite interval* (a,b). *Let* $c \in (a,b)$ *and* $\epsilon > 0$. *Assume that there exist positive constants* $C_i, (i=1,2)$ *such that* $C_1 \le U_n(x) \le C_2$, $C_1 \le V_n(x) \le C_2$ *on* $(a, c+\epsilon)$ *and* $C_1 \le u_n(x) \le C_2$, $C_1 \le v_n(x) \le C_2$ *on* $(c-\epsilon, b)$, $\forall n$. *If* $\{(u_n, v_n)\} \in A_p(a,c)$ *"uniformly" and* $\{(U_n, V_n)\} \in A_p(c,b)$ *"uniformly" then* $\{(u_nU_n, v_nV_n)\} \in A_p(a,b)$ *"uniformly".*

In particular, the iterated use of lemma 3 and the fact that $(|x-c|^{\alpha}, |x-c|^{\beta}) \in A_p^{\delta}(a,b)$ for some $\delta > 1$ (where $a \le c \le b$) if and only if $-1 < \alpha$, $\beta < p-1$ and $\beta \le \alpha$ gives

Lemma 4. *Let* $\alpha > -1$ *and the weights*

$$U(x) = x^a(1-x)^b \prod_{k=1}^{m} |x-x_k|^{b_k}, \quad V(x) = x^A(1-x)^B \prod_{k=1}^{m} |x-x_k|^{B_k}$$

where $0 < x_1 < \dots < x_m < 1$ *and* $a, A, b, B, b_k, B_k \in \mathbb{R}$. *Then,* $(U,V) \in A_p^{\delta}(0,1)$ *if and only if*

$$\begin{aligned} &A \le a, \qquad -1 < a, \qquad A < p-1; \\ &B \le b, \qquad -1 < b, \qquad B < p-1; \\ &B_k \le b_k, \qquad -1 < b_k, \qquad B_k < p-1 \qquad (1 \le k \le m). \end{aligned}$$

The following lemma will be crucial in the study of necessary conditions for the mean and weak convergence of Fourier Bessel series. It is parallel to the one given by Máté, Nevai and Totik [**8,** theorem **2**] in the context of orthogonal polynomials.

Lemma 5. *Let $\alpha > -1$. Let $h(x)$ be a Lebesgue measurable nonnegative function on $[-1,1]$, $\{r_n\}$ a sequence such that $\lim_{n\to\infty} r_n = +\infty$ and $1 \le p < \infty$. Then*

$$\lim_{n\to\infty} \int_0^1 |r_n^{1/2} J_\alpha(r_n x)|^p h(x)dx \ge M \int_0^1 h(x)\, x^{-p/2}\, dx$$

where M is a positive constant independent of $h(x)$ and $\{r_n\}$. In particular, (with $r_n = \alpha_n$)

$$\lim_{n\to\infty} \int_0^1 |j_n^\alpha(x)|^p h(x)dx \ge M \int_0^1 h(x)\, x^{-p/2}\, dx. \tag{2.5}$$

Proof.

It can be seen in [**5**]. There, the proof is given only in the case $r_n = \alpha_n$ but the arguments are the same for any sequence $\{r_n\}$ such that $\lim_{n\to\infty} r_n = +\infty$.

§3. Proofs of the main results.

Proof of proposition 1.

We use the estimate

$$|J_\alpha(z)| \le C\, z^{-1/2}, \quad z \in (0,\infty), \quad \alpha \ge -1/2 \tag{3.1}$$

which can be deduced from (2.1) and (2.2). We are going to prove that all the operators $S^\alpha_{n,i}$ ($i = 1$ to 6) are uniformly bounded.

$$\int_0^1 |S^\alpha_{n,1} f(x) U(x)|^p x dx =$$

$$= \int_0^1 |\int_0^1 \frac{f(y)y^{1/2}(yM_n)^{1/2}J_{\alpha+1}(yM_n)}{2(y-x)}dy|^p |(xM_n)^{1/2}J_\alpha(xM_n)|^p U(x)^p x^{1-p/2} dx \le$$

$$\le C \int_0^1 |\int_0^1 \frac{f(y)y^{1/2}(yM_n)^{1/2}J_{\alpha+1}(yM_n)}{2(y-x)}dy|^p U(x)^p x^{1-p/2} dx \le$$

$$\le C \int_0^1 |f(x)x^{1/2}(xM_n)^{1/2}J_{\alpha+1}(xM_n)|^p V(x)^p x^{1-p/2} dx \le C \int_0^1 |f(x)V(x)|^p x dx,$$

where we have used (3.1), the hypothesis on U, V and lemma 1 for the Hilbert transform.

The boundedness of $S^\alpha_{n,2}$ is completely similar and the operators $S^\alpha_{n,3}$, $S^\alpha_{n,4}$ can be treated as the previous operators but using the operator J' instead of the Hilbert transform.

For the boundedness of $S^\alpha_{n,5}$ we only need Holder's inequality and the fact that u and $v^{-1/(p-1)}$ are integrable when $(u,v) \in A_p(0,1)$. Finally, it is immediate that

$$\|S^\alpha_{n,6} f(x) U(x)\|_p \le C \|f(x)V(x)\|_p$$

if and only if

$$\|Jf(x)\|_{L^p((0,1);U(x)^p x^{1-p/2}dx)} \le C\|f(x)\|_{L^p((0,1);V(x)^p x^{1-p/2}dx)}$$

and then we can apply lemma 1 for J.

Proof of proposition 2.

Here, we shall use that there exists a constant C such that $\forall z \in (0,\infty)$

$$|z^{1/2}J_\alpha(z)| \le C\,(1+z^{\alpha+1/2}), \quad |z^{1/2}J_{\alpha+1}(z)| \le C\,(1+z^{\alpha+1/2})^{-1} \tag{3.2}$$

which follows from (2.1) and (2.2). For $\alpha < -1/2$, we have

$$1+(xM_n)^{\alpha+1/2} \approx (xM_n)^{\alpha+1/2}\,(1+xM_n)^{-\alpha-1/2}$$

and so, condition (1.1) is equivalent to

$$((1+(xM_n)^{\alpha+1/2})^p U(x)^p x^{1-p/2}, (1+(xM_n)^{\alpha+1/2})^p V(x)^p x^{1-p/2}) \in A_p^\delta(0,1)$$

"uniformly". Now, applying (3.2) and lemma 1 we get

$$\int_0^1 |S_{n,1}^\alpha f(x)U(x)|^p x dx =$$

$$= \int_0^1 |\int_0^1 \frac{f(y)y^{1/2}(yM_n)^{1/2}J_{\alpha+1}(yM_n)}{2(y-x)}dy|^p |(xM_n)^{1/2}J_\alpha(xM_n)|^p U(x)^p x^{1-p/2}dx \le$$

$$\le C\int_0^1 |\int_0^1 \frac{f(y)y^{1/2}(yM_n)^{1/2}J_{\alpha+1}(yM_n)}{2(y-x)}dy|^p (1+(xM_n)^{\alpha+1/2})^p U(x)^p x^{1-p/2}dx \le$$

$$\le C\int_0^1 |f(x)x^{1/2}(xM_n)^{1/2}J_{\alpha+1}(xM_n)|^p (1+(xM_n)^{\alpha+1/2})^p V(x)^p x^{1-p/2}dx \le$$

$$\le C\int_0^1 |f(x)V(x)|^p x dx.$$

The boundedness of $S_{n,2}^\alpha$ is similar since (1.2) is equivalent to

$$((1+(xM_n)^{\alpha+1/2})^{-p} U(x)^p x^{1-p/2}, (1+(xM_n)^{\alpha+1/2})^{-p} V(x)^p x^{1-p/2}) \in A_p^\delta(0,1).$$

The boundedness of $S_{n,3}^\alpha$ and $S_{n,4}^\alpha$ is analogous, taking the operator J' instead of H. From conditions (1.1) and (1.2) (taking $n=1$) it follows that

$$\int_0^1 U(x)^p x^{\alpha p+1}dx < +\infty \quad , \quad \int_0^1 V(x)^{-p'} x^{\alpha p'+1}dx < +\infty$$

and this fact together with Holder's inequality leads to the boundedness of $S_{n,5}^\alpha$.

Finally, taking limits on n in condition (1.1) we get $(U(x)^p x^{1-p/2}, V(x)^p x^{1-p/2}) \in A_p^\delta(0,1)$ and then (see the proof of proposition 1), $S_{n,6}^\alpha$ is bounded.

Proof of theorem 1.

For $\alpha \geq -1/2$, the result follows from proposition 1 and lemma 4. Let $-1 < \alpha < -1/2$; we must check conditions (1.1) and (1.2). If we put $U(x) = x^a U_1(x)$, $V(x) = x^A V_1(x)$, then U_1 and V_1 are constant-like near 0. Thus, lemma 3 tells us that we just need to verify the conditions

$$(U_1(x)^p, V_1(x)^p) \in A_p^\delta(0,1),$$

$$((x+\frac{1}{M_n})^{p(\alpha+1/2)} x^{-\alpha p-p+1+ap}, (x+\frac{1}{M_n})^{p(\alpha+1/2)} x^{-\alpha p-p+1+Ap}) \in A_p^\delta(0,1),$$

$$((x+\frac{1}{M_n})^{-p(\alpha+1/2)} x^{\alpha p+1+ap}, (x+\frac{1}{M_n})^{-p(\alpha+1/2)} x^{\alpha p+1+Ap}) \in A_p^\delta(0,1).$$

With the help of lemma 4 we obtain the first condition and the two last ones are an easy consequence of lemma 2.

Proof of theorem 2.

Let $T_n f = S_n f - S_{n-1} f = c_n(f) j_n^\alpha$. From the hypothesis, we have

$$|c_n(f)| \, \|j_n^\alpha(x) U(x)\|_p \leq C \|f(x)V(x)\|_p$$

with constant independent of n. Moreover, by applying duality to the operator $f \in L^p((0,1); V(x)^p x dx) \longrightarrow c_n(f)$ we obtain

$$\|j_n^\alpha(x) U(x)\|_p \, \|j_n^\alpha(x) V(x)^{-1}\|_{p'} \leq C \tag{3.3}$$

“uniformly” on n. Now, by using (2.5) in this inequality we get the two second conditions in (1.5) and (1.6).

On the other hand, taking $n = 1$ in (3.3) and using that $J_\alpha(\alpha_1 x) \geq C x^\alpha$ when $x \in (0, 1/2)$ (which follows from (2.1)), we find the necessary conditions

$$\int_0^{1/2} U(x)^p x^{\alpha p+1} dx < +\infty, \quad \int_0^{1/2} V(x)^{-p'} x^{\alpha p'+1} dx < +\infty$$

which combined with the two already obtained conditions gives (1.5) and (1.6).

Finally, we take the function $f(x) = \min\{U(x), V(x)^{-1}, U(x)V(x)^{-1}\} j_0(x)$. By using Holder's inequality and (3.3) it is easy to see that $f \in L^p((0,1); V(x)^p x dx) \cap L^2((0,1); x dx)$. For every measurable set $E \subset (0,1)$, let $g(x) = f(x)\chi_E(x)$. Since $S_n g \longrightarrow g$ in $L^2(x dx)$, there exists a subsequence such that $S_{n_j} g \longrightarrow g$ a.e.. Then, by using Fatou's lemma and the uniform boundedness of $\{S_{n_j}\}$ from $L^p((0,1); V(x)^p x dx)$ into $L^p((0,1); U(x)^p x dx)$ we obtain

$$\int_E |f(x)|^p U(x)^p x dx = \int_0^1 |g(x)|^p U(x)^p x dx \leq \liminf \int_0^1 |S_{n_j} g(x)|^p U(x)^p x dx \leq$$

$$\le C \int_0^1 |g(x)|^p V(x)^p x dx = C \int_E |f(x)|^p V(x)^p x dx$$

$\forall E \subset (0,1)$, which implies (1.7) easily.

Proof of theorem 3.

As in the proof of theorem 2, by applying the uniform boundedness of T_n we get

$$\|j_n^\alpha(x)\|_{L_*^p((0,1);U(x)^p x dx)} \|j_n^\alpha(x) V(x)^{-1}\|_{p'} \le C \tag{3.4}$$

with C independent of n.

We shall use the following "Kolmogorov's inequality":

$$\|g\|_{L_*^p(h(x)dx)} \le \sup_E \|g\chi_E\|_{L^r(h(x)dx)} \|\chi_E\|^{-1}_{L^s(h(x)dx)} \le \left(\frac{p}{p-r}\right)^{1/r} \|g\|_{L_*^p(h(x)dx)}$$

where $h(x)dx$ is a Borel measure, $0 < r < p < \infty$, $1/s = 1/r - 1/p$ and the supremum is taken over all measurable sets E of finite, positive Lebesgue measure (see [**4,** lemma **V.2.8,** p. **485**]).

This result and (2.5) imply the existence of a positive constant C independent of h and n such that

$$\liminf_{n\to\infty} \|j_n^\alpha(x)\|_{L_*^p(h(x)dx)} \ge C \|x^{-1/2}\|_{L_*^p(h(x)dx)}.$$

By applying this condition and (2.5) again in (3.4) we obtain the second conditions in (1.5)' and (1.6).

In order to obtain the two other conditions in (1.5)' and (1.6), we observe firstly that it suffices to prove

$$\|x^\alpha \chi_{(0,1/2)}\|_{L_*^p((0,1);U(x)^p x dx)} < \infty, \qquad \int_0^{1/2} V(x)^{-p'} x^{\alpha p'+1} dx < +\infty \tag{3.5}$$

since the conditions already obtained imply that we do not have any problems of integrability at 1.

Now, the inequalities (3.5) are easily obtained from (3.4) (with $n = 1$) and the estimate (2.1) in a very similar way as it was done in the proof of theorem 2.

In order to verify (1.7), we take the function

$$f_\lambda(x) = \lambda^{1/2} \chi_{\{W(x) > \lambda^{1/2}\}}(x) \lambda^{1/2} \chi_{\{|j_0(x)| > \lambda^{1/2}\}}(x)$$

for each $\lambda > 0$, where $W(x) = \min\{U(x), V(x)^{-1}, U(x)V(x)^{-1}\}$. By Holder's inequality and (3.4), f_λ belongs to $L^p((0,1); V(x)^p x dx) \cap L^2((0,1); x dx)$. The function $g(x) = f_\lambda(x)\chi_E(x)$, ($E$ measurable $\subset (0,1)$) satisfies $|g(x)|^p = \lambda^p \chi_{\{|g(x)|>\lambda/2\}}(x)$ and then, in a similar way as in the previous proof, we get

$$\int_E |f_\lambda(x)|^p U(x)^p x dx \le C \int_E |f_\lambda(x)|^p V(x)^p x dx, \quad \forall E \subset (0,1), \quad \forall \lambda > 0.$$

Letting $\lambda \to 0$ and using the dominated convergence theorem, we obtain

$$\int_E \chi_{\{W(x)>0\}}(x)U(x)^p x dx \le C \int_E \chi_{\{W(x)>0\}}(x)V(x)^p x dx, \quad \forall E \subset (0,1)$$

which implies (1.7).

Proof of theorem 4.

$(ii) \Rightarrow (i)$. Observing that $\min\{\frac{1}{4}, \frac{\alpha+1}{2}\} = 1/4$ if $\alpha \ge -1/2$, it is a particular case of theorem 1. In fact, (ii) implies strong boundedness (that is, boundedness on L^p).

$(i) \Rightarrow (ii)$. By theorem 3, (i) implies (1.5)' and (1.6) and easy calculations show that the conditions

$$-1/4 \le \frac{1}{p} + \frac{a-1}{2} < 1/4;\ -1 < pb < p-1\ ;\ -1 < pb_k < p-1\ (1 \le k \le m) \tag{3.6}$$

follow from (1.5)' and (1.6). The strict inequality in $\frac{-1}{4} \le \frac{1}{p} + \frac{a-1}{2} < \frac{1}{4}$ cannot be deduced from conditions (1.5)' and (1.6).

In what follows, we shall use the following decomposition of the kernel

$$K_n^\alpha(x,y) = K_{n,13}^\alpha(x,y) + K_{n,24}^\alpha(x,y) + K_{n,5}^\alpha(x,y) + K_{n,6}^\alpha(x,y)$$

where

$$K_{n,13}^\alpha(x,y) = K_{n,1}^\alpha(x,y) + K_{n,3}^\alpha(x,y), \quad K_{n,24}^\alpha(x,y) = K_{n,2}^\alpha(x,y) + K_{n,4}^\alpha(x,y).$$

So, we can decompose

$$S_n^\alpha f(x) = S_{n,13}^\alpha f(x) + S_{n,24}^\alpha f(x) + S_{n,5}^\alpha f(x) + S_{n,6}^\alpha f(x)$$

where

$$S_{n,13}^\alpha f(x) = \int_0^1 \frac{M_n J_\alpha(M_n x) J_{\alpha+1}(M_n y) y}{y^2 - x^2} f(y) y dy$$

and

$$S_{n,24}^\alpha f(x) = \int_0^1 \frac{M_n J_\alpha(M_n y) J_{\alpha+1}(M_n x) x}{x^2 - y^2} f(y) y dy.$$

The plan is to prove that under conditions (3.6), $S_{n,24}^\alpha$, $S_{n,5}^\alpha$, $S_{n,6}^\alpha$ are bounded from $L^p((0,1); U(x)^p x dx)$ into $L_*^p((0,1); U(x)^p x dx)$ (in fact, $S_{n,24}^\alpha$ will be of strong type), while $S_{n,13}^\alpha$ is not bounded in the case $-\frac{1}{4} = \frac{1}{p} + \frac{a-1}{2}$.

Weak boundedness of $S_{n,5}^\alpha$. It is simply to observe that

$$|S_{n,5}^\alpha f(x)| \le C \left| \int_0^1 (xy)^\alpha f(y) y dy \right| \le C x^{-1/2} \int_0^1 y^{-1/2} |f(y)| y dy$$

and use Holder's inequality.

Weak boundedness of $S^\alpha_{n,6}$. It would be enough to prove (as in the last part of the proof of proposition 1) that $U(x)^p x^{1-p/2} \in A_p(0,1)$, but this is not true, since the condition $-\frac{1}{4} < \frac{1}{p} + \frac{a-1}{2}$ is needed as lemma 4 establishes.

We will proceed as follows: we set the operator $S^\alpha_{n,6}$ in terms of the operator J and we only need to show

$$\|x^{-1/2}\chi_{(0,1/2)}(x)Jf(x)\|_{L^p_*((0,1);U(x)^p xdx)} \le C\,\|x^{-1/2}f(x)\|_{L^p((0,1);U(x)^p xdx)} \tag{3.7}$$

$$\|x^{-1/2}\chi_{(1/2,1)}(x)Jf(x)\|_{L^p_*((0,1);U(x)^p xdx)} \le C\,\|x^{-1/2}f(x)\|_{L^p((0,1);U(x)^p xdx)}. \tag{3.8}$$

The inequality (3.8) holds (even with strong norm) because the pair of weights

$$(U(x)^p x^{1-p/2}\chi_{(1/2,1)}(x), U(x)^p x^{1-p/2}) \in A^\delta_p(0,1)$$

since only the inequality $\frac{1}{p} + \frac{a-1}{2} < \frac{1}{4}$ is needed in $x = 0$.

For (3.7) notice that if $0 < x < 1/2$ then $|Jf(x)| \le C\,\int_0^1 |f(y)|dy$ and

$$\begin{aligned}
&\|x^{-1/2}\chi_{(0,1/2)}(x)Jf(x)\|_{L^p_*((0,1);U(x)^p xdx)} \le \\
&\le C\int_0^1 |f(y)|dy\,\|x^{-1/2}\chi_{(0,1/2)}(x)\|_{L^p_*((0,1);U(x)^p xdx)} \le \\
&\le C\int_0^1 |f(y)|dy\,\|x^{-1/2}\|_{L^p_*((0,1);U(x)^p xdx)} \le \\
&\le C\int_0^1 |f(y)|dy \le C\|x^{-1/2}U(x)f(x)\|_{L^p(xdx)}\|x^{-1/2}U(x)^{-1}f(x)\|_{L^{p'}(xdx)} \le \\
&\le C\|x^{-1/2}f(x)\|_{L^p((0,1);U(x)^p xdx)},
\end{aligned}$$

where we have used Holder's inequality and that

$$x^{-1/2} \in L^p_*((0,1);U(x)^p xdx) \cap L^{p'}((0,1);U(x)^{p'} xdx)$$

which is immediate from (3.6). The crucial fact is that $x^{-1/2} \in L^p_*$ even though we have the equality in $-\frac{1}{4} \le \frac{1}{p} + \frac{a-1}{2}$ (this is not true for strong L^p).

Strong boundedness of $S^\alpha_{n,24}$. Making the change of variables $t = y^2, z = x^2$ and taking the function $g(z) = M_n^{1/2} J_\alpha(M_n z^{1/2}) f(z^{1/2})$, the boundedness of $S^\alpha_{n,24}$ on $L^p((0,1);U(x)^p xdx)$ turns out to be equivalent to

$$\|M_n^{1/2} J_{\alpha+1}(M_n z^{1/2}) z^{1/2} Hg(z)\|_{L^p(U(z^{1/2})^p dz)} \le$$

$$\le C\|M_n^{-1/2} J_\alpha(M_n z^{1/2})^{-1} g(z)\|_{L^p(U(z^{1/2})^p dz)}.$$

Now, by using the estimate (3.1), we only need to see that

$$\|Hg(z)\|_{L^p(U(z^{1/2})^p z^{p/4} dz)} \le C\,\|g(z)\|_{L^p(U(z^{1/2})^p z^{p/4} dz)}$$

and this is equivalent to check that $(U(z^{1/2})^p z^{p/4}) \in A_p(0,1)$. Since $1 - x^{1/2} \approx 1 - x$, $|x^{1/2} - x_k| \approx |x - x_k^2|$, the A_p-condition appears as

$$U(x) = x^{pa/2+p/4}(1-x)^{pb} \prod_{k=1}^{m} |x - x_k^2|^{pb_k} \in A_p(0,1)$$

which is true from lemma 4 under the hypothesis (3.6).

The last step is to prove that, when $-\frac{1}{4} = \frac{1}{p} + \frac{a-1}{2}$ there exists no constant C such that

$$\|S_{n,13}^{\alpha} f(x)\|_{L^p_*((0,1);U(x)^p x dx)} \leq C \, \|f(x)U(x)\|_p \tag{3.9}$$

"uniformly" on n.

The proof is by contradiction. Assume that (3.9) holds, take $0 < d < 1$ and $f(x) = \chi_{(d,1)}(x) \operatorname{sgn}(J_{\alpha+1}(M_n x)) \, |g(x)| \, x^{-1/2}$, where $g(x)$ is a suitable function that we will choose below. It is clear that for $x \in (0,d)$,

$$|S_{n,13}^{\alpha} f(x)| \geq M_n^{1/2} |J_\alpha(M_n x)| \int_d^1 M_n^{1/2} |J_{\alpha+1}(M_n y)| y^{-1/2} |g(y)| dy.$$

Then, (3.9) implies

$$\left(\int_d^1 M_n^{1/2} |J_{\alpha+1}(M_n y)| y^{-1/2} |g(y)| dy\right) \|M_n^{1/2} J_\alpha(M_n x) \chi_{(0,d)}(x)\|_{L^p_*(U(x)^p x dx)} \leq$$

$$\leq C \, \|\chi_{(d,1)}(x) g(x) x^{-1/2} U(x)\|_p$$

with C independent of d, n and g. As $\lim M_n = +\infty$ we can apply lemma 5 and its weak version (as it was done in the proof of theorem 3) to the two factors in the left hand side of the inequality, and we obtain

$$\left(\int_d^1 y^{-1} \, |g(y)| dy\right) \|x^{-1/2} \chi_{(0,d)}(x)\|_{L^p_*(U(x)^p x dx)} \leq C \|x^{-1/2} \chi_{(d,1)}(x) g(x) U(x)\|_p. \tag{3.10}$$

Now, let us take $0 < d < r < x_1$ (therefore $U(x) \approx C x^a$ on $(0,r)$) and $g = \chi_{(d,r)}$. Then, it is easy to prove that

$$\int_d^1 y^{-1} |g(y)| dy = \log \frac{r}{d}, \quad \|x^{-1/2} \chi_{(d,1)}(x) g(x) U(x)\|_p \approx C (\log \frac{r}{d})^{1/p},$$

and also that if $-\frac{1}{4} = \frac{1}{p} + \frac{a-1}{2}$ then

$$\|x^{-1/2} \chi_{(0,d)}(x)\|_{L^p_*(U(x)^p x dx)} \approx C \|x^{-1/2} \chi_{(0,d)}(x)\|_{L^p_*(x^{ap+1} dx)} \approx C.$$

Thus, (3.10) implies the existence of a constant C such that

$$\log \frac{r}{d} \leq C \, (\log \frac{r}{d})^{1/p}$$

which is false.

This concludes the proof of theorem 4.

Proof of theorem 5.

Let $\alpha \geq -1/2$ and $p = 4$. According to the previous proof, we just need to show the $(4,4)$-restricted weak type of $S^{\alpha}_{n,13}$. Making the change of variables $t = y^2, z = x^2$, this boundedness is equivalent to

$$|\{z \in (0,1) : |M_n^{1/2} J_\alpha(M_n z^{1/2}) H(M_n^{1/2} J_{\alpha+1}(M_n t^{1/2}) t^{1/2} f(t^{1/2}))(z)| > \lambda\}| \leq$$

$$\leq C\,\lambda^{-4} \int_0^1 |f(x)|^p x dx$$

where f is a characteristic function. Now, denoting $g(t) = f(t^{1/2})$ and using that $|M_n^{1/2} J_\alpha(M_n z^{1/2})| \leq C\, z^{-1/4}$ (see (3.1)), it will be enough to prove that

$$|A_{\lambda,E}| \leq C\,\lambda^{-4}|E|, \quad \forall\lambda > 0, \quad \forall E \subseteq (0,1) \tag{3.11}$$

where

$$A_{\lambda,E} = \{z \in (0,1) : z^{-1/4}|H(M_n^{1/2} J_{\alpha+1}(M_n t^{1/2}) t^{1/2} \chi_E(t))(z)| > \lambda\}.$$

Observe that

$$|A_{\lambda,E} \cap (1/4,1)| \leq C\,\lambda^{-4} \int_0^1 |H(M_n^{1/2} J_{\alpha+1}(M_n t^{1/2}) t^{1/2} \chi_E(t))(z)|^4 dz \leq$$

$$\leq C\,\lambda^{-4} \int_0^1 |M_n^{1/2} J_{\alpha+1}(M_n t^{1/2}) t^{1/2} \chi_E(t)|^4 dt \leq$$

$$\leq C\,\lambda^{-4} \int_0^1 t^{1/4} |\chi_E(t)|^4 \leq C\,\lambda^{-4}|E|$$

due to the boundedness of the Hilbert transform on $L^4((0,1); dx)$ and to the estimate (3.1).

On the other hand, if $E \subseteq [1/2,1)$ and $z \in (0,1/4)$

$$|H(M_n^{1/2} J_{\alpha+1}(M_n t^{1/2}) t^{1/2} \chi_E(t))(z)| \leq C \int_0^1 t^{1/4} \chi_E(t) dt \leq C\,|E|.$$

Hence,

$$|A_{\lambda,E} \cap (0,1/4)| \leq |\{z \in (0,1/4) : z^{-1}\, C\,|E|^4 > \lambda^4\}| \leq C\,\lambda^{-4}|E|^4 \leq C\,\lambda^{-4}|E|$$

and (3.11) holds, if $E \subseteq [1/2,1)$.

If $E \subseteq (0,1/2)$, let $I_k = (\frac{1}{2^{k+1}}, \frac{1}{2^k})$, $k = 2,3,....$ We shall use the following notation:

$$I_{k1} = (0, \frac{1}{2^{k+2}}), \quad I_{k2} = (\frac{1}{2^{k+2}}, \frac{1}{2^{k-1}}), \quad I_{k3} = (\frac{1}{2^{k-1}}, 1/2), \quad k = 2,3,...$$

$$A^k_{\lambda,E} = A_{\lambda,E} \cap I_k, \quad A^{km}_{\lambda,E} = A^k_{\lambda/3,E_{km}}, \ m = 1,2,3$$

where $E_{km} = E \cap I_{km}$, $m = 1,2,3$.

Then, $A^k_{\lambda,E} \subseteq A^{k1}_{\lambda,E} \cup A^{k2}_{\lambda,E} \cup A^{k3}_{\lambda,E}$ and

$$|A_{\lambda,E} \cap (0,1/4)| \leq \sum_{k=2}^{\infty} |A^k_{\lambda,E}| \leq \sum_{k=2}^{\infty} \{|A^{k1}_{\lambda,E}| + |A^{k2}_{\lambda,E}| + |A^{k3}_{\lambda,E}|\}. \tag{3.12}$$

For $m = 3$, notice that if $z \in I_k$ and $t \in I_{k3}$ then $t - z \approx t$. Hence, using (3.1)

$$|H(M_n^{1/2} J_{\alpha+1}(M_n t^{1/2}) t^{1/2} \chi_{E_{k3}}(t))(z)| \leq C \int_0^1 t^{-3/4} \chi_E(t) dt \leq C\,|E|^{1/4}$$

(the last inequality can be deduced from duality between Lorentz spaces). Therefore,

$$|A^{k3}_{\lambda,E}| \leq |\{z \in I_k : z < C\,\lambda^{-4} |E|\}|. \tag{3.13}$$

The same arguments work in order to prove that $|A^{k1}_{\lambda,E}|$ has the estimate (3.13).

For $m = 2$, we have

$$\begin{aligned} |A^{k2}_{\lambda,E}| &\leq |\{z \in I_k : |H(M_n^{1/2} J_{\alpha+1}(M_n t^{1/2}) t^{1/2} \chi_{E_{k2}}(t))(z)| > C\,\lambda\, 2^{-\frac{k}{4}}\}| \\ &\leq C\,\lambda^{-4}\, 2^k \int_0^1 |H(M_n^{1/2} J_{\alpha+1}(M_n t^{1/2}) t^{1/2} \chi_{E_{k2}}(t))(z)|^4 dz \\ &\leq C\,\lambda^{-4}\, 2^k \int_0^1 |M_n^{1/2} J_{\alpha+1}(M_n t^{1/2}) t^{1/2} \chi_{E_{k2}}(t)|^4 dt \\ &\leq C\,\lambda^{-4}\, 2^k \int_0^1 t \chi_{E_{k2}}(t) \leq C\,\lambda^{-4} |E_{k2}|. \end{aligned} \tag{3.14}$$

Finally, putting together (3.12), (3.13) and (3.14) we get (3.11) and the proof for $p = 4$ is concluded.

The proof for $p = 4/3$ follows from standard arguments of duality between Lorentz spaces.

Remarks. The decomposition in four terms which we have used to study the weak boundedness, also works to study the strong boundedness, instead of using the decomposition in six terms. However, the results are somehow different. For instance, to obtain the uniform boundedness of the proposition 1, we must change the condition by

$$(U(x^{1/2})^p x^{-p/4}, V(x^{1/2})^p x^{-p/4}) \in A_p^\delta(0,1), \quad (U(x^{1/2})^p x^{p/4}, V(x^{1/2})^p x^{p/4}) \in A_p^\delta(0,1).$$

Something similar happens with proposition 2. If we apply these conditions to particular weights as in theorem 1, the inequalities (1.3), (1.4) are the same.

Analogous results can be obtained in a similar way for Fourier-Dini series.

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