Self replication and Borwein-type algorithms

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Borwein-type algorithms for Pi

The Borweins' brothers found rapid algorithms for Pi: quadratic, cubic, etc. (Their proofs are based on modular functions and forms).

Borwein's quadratic algorithm (1985) :

$$d_0 = \frac{1}{\sqrt{2}}, \quad r_0 = \frac{1}{2}, \quad d_{n+1} = \frac{1-\sqrt{1-d_n^2}}{1+\sqrt{1-d_n^2}},$$

$$r_{n+1} = r_n(1+d_{n+1})^2 - 2^{n+1} d_{n+1}, \quad r_n \to \frac{1}{\pi}.$$

Our method is very general and simple. We need two ingredients:

- A rapid self-replicating transformation (quadratic, cubic, etc.)
- A suitable evaluation to fix the initial values and the limit.

We will refer to it as self-replication method.

First ingredient

Quadratic (Landen 1719-1790), $t \sim \frac{1}{4}x^2$:

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} x^{2k} = (1+t) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} t^{2k}, \quad t = \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}.$$

Cubic, $t \sim \frac{1}{9}x^3$:

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(1)_k^2} x^{3k} = (1+2t) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(1)_k^2} t^{3k}, \quad t = \frac{1-\sqrt[3]{1-x^3}}{1+2\sqrt[3]{1-x^3}}.$$

Quartic, $t \sim \frac{1}{8}x^4$:

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} x^{4k} = (1+t)^2 \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} t^{4k}, \quad t = \frac{1-\sqrt[4]{1-x^4}}{1+\sqrt[4]{1-x^4}}.$$

Second ingredient

Proof by the Wilf-Zeilberger's (WZ)-algorithm:

$$\sum_{k=0}^{\infty} \frac{(s)_k (1-s)_k}{(1)_k^2} \frac{1}{2^k} = \frac{\sqrt{\pi}}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)},$$

$$\sum_{k=0}^{\infty} \frac{(s)_k (1-s)_k}{(1)_k^2} \frac{k}{2^k} = \frac{s\sqrt{\pi}}{\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}.$$

Hence

$$\sum_{k=0}^{\infty} \frac{(s)_k (1-s)_k}{(1)_k^2} \frac{1}{2^k} \sum_{k=0}^{\infty} \frac{(s)_k (1-s)_k}{(1)_k^2} \frac{k}{2^k} = \frac{\sin \pi s}{\pi}.$$

For $s = 1/2,\ 1/3, 1/4, 1/6$ there exist algebraic tranformations.

Proof of Borweins' quadratic algorithm for π (1)

Applying the following operator:

$$a+\frac{b}{2}\,\vartheta_x = a+b\,\frac{1+t}{1-t}\,\vartheta_t, \qquad \vartheta_x = x\,\frac{d}{dx}, \quad \vartheta_t = t\,\frac{d}{dt}$$

to the Landen's transformation, we get

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2}(a+bk)x^{2k} = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2}(\alpha+\beta k)t^{2k},$$

$$\alpha = a(1+t)+b\frac{t(1+t)}{1-t}, \quad \beta = 2b\frac{(1+t)^2}{1-t}.$$

Multipying side to side by the Landen's transformation, we arrive at

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2}x^{2k}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2}(a+bk)x^{2k}=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2}t^{2k}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2}(\mu+\nu k)t^{2k}.$$

Proof of Borweins' quadratic algorithm for π (2)

Let

$$A_n = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} d_n^{2k} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} (a_n + b_n k) d_n^{2k}.$$

We obtain the following recurrences:

$$d_{n+1} = \frac{1 - \sqrt{1 - d_n^2}}{1 + \sqrt{1 - d_n^2}}, \qquad b_{n+1} = 2 b_n \frac{(1 + d_{n+1})^3}{1 - d_{n+1}},$$

$$a_{n+1} = a_n (1 + d_{n+1})^2 + \frac{b_{n+1} d_{n+1}}{2(1 + d_{n+1})}.$$

As A_n is a constant sequence, it implies that $\lim A_n = A_0$, and as $d_n^{2k} \to 0$ and $b_n d_n^{2k} \to 0$ when $n \to \infty$ and $k \neq 0$, we deduce that $\lim A_n = \lim a_n$. Hence $\lim a_n = A_0$.

Proof of Borweins' quadratic algorithm for π (3)

With the substitutions $c_n = b_n/(1-d_n^2)$ and $r_n = a_n + c_0\, 2^{n-1}\, d_n^2$, we get the iterations

$$d_{n+1} = \frac{1-\sqrt{1-d_n^2}}{1+\sqrt{1-d_n^2}}, \quad r_{n+1} = r_n(1+d_{n+1})^2 - c_0\, 2^n d_{n+1},$$

with $\lim r_n = \lim a_n = A_0$, where

$$A_0 = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} d_0^{2k} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} (a_0 + b_0 k) d_0^{2k}.$$

For $d_0 = \frac{1}{\sqrt{2}}$, $b_0 = 1$, and $a_0 = 0$, we have $A_0 = 1/\pi$. Then

$$c_0 = \frac{b_0}{1-d_0^2} = 2, \quad r_0 = a_0 + c_0 \cdot 2^{-1} d_0^2 = \frac{1}{2}.$$

Hidden modularity

With the λ-function:

$$\lambda(q) = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}} \right)^8,$$

and the elliptic α-function by Borwein & Borwein, we have that

$$d_n^2 = \lambda\left(e^{-\pi \cdot 2^n \sqrt{N_0}}\right), \quad r_n = \alpha\left(e^{-\pi \cdot 2^n \sqrt{N_0}}\right), \qquad N_0 = 1.$$

The Borweins' proved their algorithms introducing the function $\alpha(q)$, and using Ramanujan's modular equations for $\lambda(q)$.

Modular equations:

$$\left[\lambda(q)\lambda(q^7)\right]^{1/8} + \left[(1-\lambda(q))(1-\lambda(q^7)\right]^{1/8} = 1,$$

which is of degree 7.

Borwein' quartic algorithm

From the quartic transformation

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} x^{4k} = (1+t)^2 \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} t^{4k}, \quad t = \frac{1-\sqrt[4]{1-x^4}}{1+\sqrt[4]{1-x^4}},$$

letting

$$A_n = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} d_n^{4k} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} (a_n + b_n k) d_n^{4k},$$

and defining $c_n = b_n/(1-d_n^4)$, $r_n = a_n + 2c_0 \cdot 4^{n-1}\, d_n^4$, we have

$$d_{n+1} = \frac{1-\sqrt[4]{1-d_n^4}}{1+\sqrt[4]{1-d_n^4}}, \; r_{n+1} = r_n(1+d_{n+1})^4 - 2c_0 4^n d_{n+1}(1+d_{n+1}+d_{n+1}^2),$$

with $\lim r_n = \lim a_n = A_0$.

Borwein' cubic algorithm

From the cubic transformation

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(1)_k^2} x^{3k} = (1+2t) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(1)_k^2} t^{3k}, \quad t = \frac{1-\sqrt[3]{1-x^3}}{1+2\sqrt[3]{1-x^3}},$$

letting

$$A_n = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(1)_k^2} d_n^{3k} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{(1)_k^2} (a_n + b_n k) d_n^{3k},$$

and defining $c_n = b_n/(1-d_n^3)$, $\quad r_n = a_n + c_0 3^{n-1} d_n^3$, we have

$$d_{n+1} = \frac{1-\sqrt[3]{1-d_n^3}}{1+2\sqrt[3]{1-d_n^3}}, \quad r_{n+1} = r_n(1+2d_{n+1})^2 - c_0 3^n d_{n+1}(1+d_{n+1}),$$

with $\lim r_n = \lim a_n = A_0$.

Generalization of Borweins' quadratic algorithm

Let $$A_n = \left(\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} d_n^{2k} \right)^w \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} (a_n + c_n(1-d_n^2)k) d_n^{2k}.$$

Quadratic algorithm (2017)

$$d_0 = \frac{1}{\sqrt{2}}, \quad c_0 = 2, \quad a_0 = 0,$$

$$d_{n+1} = \frac{1-\sqrt{1-d_n^2}}{1+\sqrt{1-d_n^2}}, \quad c_{n+1} = 2c_n(1+d_{n+1})^{w-1},$$

$$a_{n+1} = a_n(1+d_{n+1})^{w+1} + \frac{1}{2} c_{n+1} d_{n+1} (1-d_{n+1}),$$

$$a_n(w) \to \frac{1}{\Gamma\left(\frac{3}{4}\right)^{2w-2} \pi^{\frac{3}{2}-\frac{w}{2}}}.$$

Modular forms

The modular group

$$\Gamma = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}.$$

Congruence group $\Gamma_0(m)$ is the subgroup of Γ with $c \equiv 0 \mod m$.

Let $q = e^{2\pi i \tau}$. We can view f as a function of τ.

A function $f(\tau)$ is a modular form of weight n and level m for Γ_0 if $f(\tau)$ is holomorphic in the upper semiplane and as $z \to i\infty$, and

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(\tau).$$

A modular function is a modular form of weight $n = 0$.

Arithmetic sequences

We will refer to C_n as an arithmetic sequence if the function

$$f(x) = \sum_{n=0}^{\infty} C_n x^n,$$

has a modular origin. Let $C_n = c(n)$

1−) For primes p, the $c(n)$ satisfy
Lucas congruences: $c(n) \equiv c(n_0)c(n_1)\cdots c(n_r) \pmod{p}$,
where $n = n_0 + n_1 p + \cdots + n_r p^r$.

2−) The $c(n)$ satisfy or are conjectured to satisfy
$p^{\ell r}$-congruences: $c(mp^r) \equiv c(mp^{r-1}) \pmod{p^{\ell r}}$,
with $\ell \in \{1, 2, 3\}$. If $\ell > 1$ (supercongruences).

Self replication (1)

Example 1: The solution of the functional equation

$$\frac{1}{(1+4z)^2} f\left(\frac{z}{(1+4z)^3}\right) = \frac{1}{(1+2z)^2} f\left(\frac{z^2}{(1+2z)^3}\right),$$

is a modular form

$$f(x) = \sum_{n=0}^{\infty} C_n x^n,$$

of weight 2 and level $m = 7$, when a related modular parametrization $x = x(\tau)$ is properly chosen. The numbers C_n are given by

$$C_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n} \binom{2k}{n} = \sum_{k=0}^{n} (-1)^{n-k} \binom{3n+1}{n-k} \binom{n+k}{n}^3.$$

Self replication (2)

In "Ramanujan-type series for $1/\pi$: the art of translation" (J. G. & W.Zudilin), we proved that

$$\sum_{n=0}^{\infty} C_n(4+21n) \times \frac{1}{5^{3n+3}} = \frac{1}{8\pi}.$$

Hence, we can get a quadratic algorithm for π.

Example 2:

$$\frac{1}{1+8z} f\left(\frac{z}{(1+4z)(1+8z)^2}\right) = \frac{1}{1+2z} f\left(\frac{z^2}{(1+4z)(1+2z)^2}\right).$$

The solution is:

$$f(x) = \sum_{n=0}^{\infty} \left\{ \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \right\} x^n.$$

Self-replication (3)

Example 3:

$$\frac{1}{(1+4z)^2} f\left(z\left(\frac{1-z}{1+4z}\right)^5\right) = f\left(z^5\left(\frac{1-z}{1+4z}\right)\right).$$

The solution is

$$f(x) = \sum_{n=0}^{\infty} \binom{2n}{n}^3 x^n = {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1\end{matrix} \,\middle|\, 64x\right).$$

To get the initial values consider

$$\sum_{n=0}^{\infty} \binom{2n}{n}^3 \left(\frac{1}{4}+n\right)\left(\frac{-1}{64}\right)^n = \frac{1}{2\pi},$$

due to Bauer.

Quintic algorithm for $1/(2\pi)$

Initial values: $a_0 = \frac{1}{4}, \quad b_0 = 1, \quad x_0 = -\frac{1}{64}.$

Iterations:

$$\begin{aligned} z_n &= \text{the real root of } z(1-z)^5 = x_n(1+4z)^5, \\ a_{n+1} &= a_n(1+4z_n)^2 + 8b_n \frac{z_n(1-z_n)(1+4z_n)}{1-22z_n-4z_n^2}, \\ b_{n+1} &= 5b_n(1+4z_n)^2 \frac{1+2z_n-4z_n^2}{1-22z_n-4z_n^2}, \\ x_{n+1} &= z_n^5 \frac{1-z_n}{1+4z_n}. \end{aligned}$$

Convergence:

$$\left| a_2 - \frac{1}{2\pi} \right| \approx 8.25 \times 10^{-47}, \qquad \left| a_3 - \frac{1}{2\pi} \right| \approx 4.57 \times 10^{-239}.$$

References

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Thank you for your attention