WZ-method proofs of some Ramanujan-type series for $1/\pi$ and new series for $1/\pi^2$

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The known proofs of the Ramanujan-type series for $1/\pi$ are based on the theory of modular functions. Instead, we use the Wilf-Zeilberger (WZ) method to find simpler proofs of some of the easiest ones. For example, we prove

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (42n+5) = \frac{16}{\pi},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, and $(a)_0 = 1$. With the same method we also prove three, somewhat similar, new series for $1/\pi^2$. An example is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} (20n^2+8n+1) = \frac{8}{\pi^2}.$$

Inspired by these new series, we discover others of the same form that we are not able to prove. Finally, by modifying the WZ pairs in our proofs and using a different strategy, we obtain new kinds of hypergeometric identities.

1. Ramanujan-type series for $1/\pi$.

2. The Wilf and Zeilberger (WZ) method. Package EKHAD.

3. Generalized series for $1/\pi$.

4. Generalized series for $1/\pi^2$.

5. Unproved series for $1/\pi^2$.

6. New kinds of hypergeometric identities.

Pochhammer Symbol:

$$a_n = a(a+1)(a+2)\cdots(a+n-1)$$

$$a_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$a_0 = 1, \quad \forall a$$

$$0_n = 0, \quad n = 1, 2, 3, \cdots$$

Ramanujan (1887-1920)
Ramanujan Series (1914). Are of the form:

$$\sum_{n=0}^{\infty} \frac{B(n)}{q^n}(an+b) = \frac{d\sqrt{k}}{\pi}$$

where $B(n) = n!^{-3}\ C(n)$ and $C(n)$ is one of the following products:

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n, \qquad \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n, \qquad \left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n.$$

The most impressive Ramanujan Series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{882^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{1_n^3} (21460n + 1123) = \frac{3528}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{99^{4n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{1_n^3} (26390n + 1103) = \frac{9801\sqrt{2}}{4\pi}.$$

8 digits/term.

And the most impressive Ramanujan Type Series is: (D. y G. Chudnovsky)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{53360^{3n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{1_n^3} \times (545140134n + 13591409) = \frac{426880\sqrt{10005}}{\pi}$$

Record computing π digits: (1991) 2.260.000.000, (1994) 4.0440.000.000

WZ Pairs (Wilf and Zeilberger).

A function $A(n,k)$ is hypergeometric if the quotions:

$$\frac{A(n+1,k)}{A(n,k)} \quad \text{and} \quad \frac{A(n,k+1)}{A(n,k)}$$

are both rational functions.

A couple of hypergeometric functions $F(n,k)$ and $G(n,k)$ is an WZ pair if

$$G(n,k+1) - G(n,k) = F(n+1,k) - F(n,k)$$

If F(n,k) has a companion then the package EKHAD (Maple) find it. (Remark: EKHAD cannot manage pochhammers symbols. So to use EKHAD first we need to convert them into factorials or binomials).

Strategy: Let $G(n,k)$ be an hypergeometric function. Suppose we want to know if:

$$\sum_{n=0}^{\infty} G(n,k) = CONSTANT$$

is true or false.

There is no problem. If it is true then EKHAD finds a rational function $C(n,k)$ such that:

$$F(n,k) = C(n,k)G(n,k),$$

$$F(0,k) = 0,$$

$$G(n,k+1) - G(n,k) = F(n+1,k) - F(n,k).$$

But this properties decide the question because:

$$\sum_{n=0}^{\infty} [G(n,k+1) - G(n,k)] =$$

$$\sum_{n=0}^{\infty} [F(n+1,k) - F(n,k)] = -F(0,k) = 0$$

$$\Longrightarrow \quad \sum_{n=0}^{\infty} G(n,k) = \sum_{n=0}^{\infty} G(n,k+1),$$

and applying Carlson Theorem (the conditions hold for the series we are going to consider) we conclude that

$$\sum_{n=0}^{\infty} G(n,k) = CONSTANT.$$

If we have

$$\sum_{n=0}^{\infty} G_1(n,k) = CONSTANT$$

and define:

$$G_2(n,k) = F_1(n+1, n+k) + G_1(n, n+k),$$

$$G_3(n,k) = F_2(n+1, n+k) + G_2(n, n+k),$$

$$etc.$$

Then, a Zeilberger's theorem, allows to get that

$$\sum_{n=0}^{\infty} G_1(n,k) = \sum_{n=0}^{\infty} G_2(n,k) = \cdots = CONSTANT.$$

So, we get a chain of series.

Generalized Series for $1/\pi$ (1)
With package EKHAD and using our WZ-strategy is a routine to prove that

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n}{1_n^2 (1+k)_n} \times (6n+2k+1) \frac{\binom{2k}{k}}{2^{2k}} = \frac{4}{\pi},$$

where the value $4/\pi$ has been determined by plugging $k = 1/2$.

The function : $\dfrac{\binom{2k}{k}}{2^{2k}}$, does not depend on n.

Remark: The companion of $G(n,k)$ is

$$F(n,k) = \frac{1}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n}{1_n^2 (1+k)_n} \times \frac{n^2}{2n-2k-1} \frac{\binom{2k}{k}}{2^{2k}}.$$

Generalized Series for $1/\pi$ (2-3-4)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}+k\right)_n}{1_n^2 (1+k)_n} \times (6n+2k+1) = \frac{2\sqrt{2}}{\pi} \frac{2^{3k}}{\binom{2k}{k}},$$

where the value $2\sqrt{2}/\pi$ has been determined by plugging $k=1/2$.

$$\frac{\binom{2k}{k}^2}{2^{4k}} \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^3}{1_n (1+k)_n^2} (6n+4k+1) = \frac{4}{\pi},$$

where the value $4/\pi$ has been determine by letting $k \to \infty$.

$$\frac{\binom{2k}{k}^2}{2^{4k}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)_n^2 \left(\frac{1}{2}+k\right)_n}{1_n (1+k)_n^2} (4n+2k+1) = \frac{2}{\pi},$$

where the value $2/\pi$ has been determined by letting $k \to \infty$.

Generalized Series for $1/\pi$ (5-6)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{1_n^2 (1+k)_n} \times (20n+2k+3) = \frac{8}{\pi} \frac{2^{2k}}{\binom{2k}{k}}.$$

$$\sum_{n=0}^{\infty} G_1(n,k) = \sum_{n=0}^{\infty} G_2(n,k)$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}+k\right)_n}{1_n^2 \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n} \times \frac{(2n+2k+1)(42n+2k+5)-30kn}{2n+k+1} = \frac{16}{\pi} \frac{2^{2k}}{\binom{2k}{k}}.$$

Generalized Series for $1/\pi$ (7-8)

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{1_n^2 (1+k)_n} (8n+2k+3) = \frac{2\sqrt{3}}{\pi} \frac{2^{4k}}{3^k \binom{2k}{k}}$$

$$\sum_{n=0}^{\infty} G_1(n,k) = \sum_{n=0}^{\infty} G_2(n,k)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n} 3^n} \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{4}+\frac{k}{2}\right)_n \left(\frac{3}{4}+\frac{k}{2}\right)_n \left(\frac{1}{2}+k\right)_n}{1_n^2 \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n}$$

$$\times \frac{(2n+2k+1)(28n+2k+3)-24kn}{2n+k+1} = \frac{16\sqrt{3}}{3\pi} \frac{2^{4k}}{3^k \binom{2k}{k}}$$

Generalized Series for $1/\pi$ (9)

$$\sum_{n=0}^{\infty}(-1)^n\frac{3^{3n}}{2^{9n}}A(n,k)R(n,k)=\frac{32\sqrt{2}}{\pi}\,\frac{2^{3k}}{\binom{2k}{k}}$$

$A(n,k)=$

$$\frac{\left(\frac{1}{2}-k\right)_n\left(\frac{1}{6}+\frac{k}{3}\right)_n\left(\frac{5}{6}+\frac{k}{3}\right)_n\left(\frac{1}{2}+k\right)_n\left(\frac{1}{2}+\frac{k}{3}\right)_n}{1_n^2\left(1+\frac{k}{2}\right)_n\left(\frac{1}{2}+\frac{k}{2}\right)_n\left(\frac{1}{2}\right)_n}$$

$R(n,k)=$

$$\frac{(2n+2k+1)(6n+2k+3)}{(2n+1)(6n+3k+3)}(154n+6k+15)+\frac{kn}{3}\,\frac{448k-1216n+608}{(2n+1)(2n+k+1)}$$

By taking $k=0$, we get the following series

$$\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\,\frac{\left(\frac{1}{2}\right)_n^3}{1_n^3}\,(6n+1)=\frac{4}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n^3}{1_n^3} (42n+5) = \frac{16}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{\left(\frac{1}{2}\right)_n^3}{1_n^3} (6n+1) = \frac{2\sqrt{2}}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{1_n^3} (20n+3) = \frac{8}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{1_n^3} (8n+1) = \frac{2\sqrt{3}}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{48^n} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{1_n^3} (28n+3) = \frac{16\sqrt{3}}{3\pi}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{1_n^3} (154n+15) = \frac{32\sqrt{2}}{\pi}$$

Generalized Series for $1/\pi^2$ (1-2)

$$\frac{\binom{2k}{k}^4}{2^{8k}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{1_n(1+k)_n^4}$$
$$\times (20n^2 + 8n + 1 + 24kn + 8k^2 + 4k) = \frac{8}{\pi^2},$$

where the value $8/\pi^2$ has been determined letting $k \to \infty$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n}{1_n^3(1+k)_n^2}$$
$$\times (20n^2 + 12kn + 8n + 2k + 1) = \frac{8}{\pi^2} \frac{2^{4k}}{\binom{2k}{k}^2},$$

where the value $8/\pi^2$ has been determined by plugging $k = 1/2$.

Generalized Series for $1/\pi^2$ (3-4)

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{1_n^3 (1+k)_n^2}$$
$$\times (120n^2 + 84kn + 34n + 10k + 3) = \frac{32}{\pi^2} \frac{2^{4k}}{\binom{2k}{k}^2}$$

$$\sum_{n=0}^{\infty} G_1(n,k) = \sum_{n=0}^{\infty} G_2(n,k)$$

$$\frac{\binom{2k}{k}^2}{2^{4k}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n^3}{1_n^3 \left(1+\frac{k}{2}\right)_n^2 \left(\frac{1}{2}+\frac{k}{2}\right)_n^2}$$
$$\times \left[820n^2 + 180n + 13 + \frac{k\, P(n,k)}{(2n+k+1)^2}\right] = \frac{128}{\pi^2}$$

$$\begin{aligned} P(n,k) = {} & 1312n^3 + 1340n^2k + 336nk^2 \\ & + 1456n^2 + 828nk + 40k^2 \\ & + 472n + 79k + 36 \end{aligned}$$

Plugging $k = 0$ we get series for $1/\pi^2$ similar to R.T.S. for $1/\pi$. Explicitly

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{1_n^5}(20n^2 + 8n + 1) = \frac{8}{\pi^2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{1_n^5}(120n^2+34n+3) = \frac{32}{\pi^2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \frac{\left(\frac{1}{2}\right)_n^5}{1_n^5}(820n^2 + 180n + 13) = \frac{128}{\pi^2}$$

The last one gives 3 digits/term.
These series are of the type

$$\sum_{n=0}^{\infty} \frac{B(n)}{q^n}(an^2 + bn + c) = \frac{d\sqrt{k}}{\pi^2},$$

where $B(n) = n!^{-5}\, C(n)$ and $C(n)$ is the product of 5 Pochhammer symbols.

Inspired by them we are going to look for more series like these. We consider the following products of Pochhammer symbols:

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{5}\right)_n \left(\frac{2}{5}\right)_n \left(\frac{3}{5}\right)_n \left(\frac{4}{5}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{12}\right)_n \left(\frac{5}{12}\right)_n \left(\frac{7}{12}\right)_n \left(\frac{11}{12}\right)_n,$$

$$\left(\frac{1}{2}\right)_n \left(\frac{1}{10}\right)_n \left(\frac{3}{10}\right)_n \left(\frac{7}{10}\right)_n \left(\frac{9}{10}\right)_n.$$

$$\text{Writing :} \quad F_j = \sum_{n=0}^{\infty} \frac{B(n)}{q^n} n^j, \quad G = \frac{\sqrt{k}}{\pi^2},$$

we look for integers a, b, c and d such that (Integer relations algorithms, PSLQ)

$$aF_0 + bF_1 + cF_2 + dG = 0, \quad d \neq 0.$$

We find four (unproved) new series for $1/\pi^2$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^5 2^{10n}} \times (1640n^2 + 278n + 15) = \frac{256\sqrt{3}}{3\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{n!^5 48^{n}} \times (252n^2 + 63n + 5) = \frac{48}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^5 80^{3n}} \times (5418n^2 + 693n + 29) = \frac{128\sqrt{5}}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{n!^5 7^{4n}} \times (1920n^2 + 304n + 15) = \frac{56\sqrt{7}}{\pi^2}.$$

Similar series for $1/\pi^3$. Boris Gourevitch (Integer relations):

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n^7}{1_n^7} (168n^3 + 76n^2 + 14n + 1) = \frac{32}{\pi^3}$$

We consider interesting to compare the following series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3 80^{3n}} \times (5418n + 263) = \frac{640\sqrt{15}}{3\pi},$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^5 80^{3n}} \times (5418n^2 + 693n + 29) = \frac{128\sqrt{5}}{\pi^2},$$

Coincidence?

Series for $1/\pi$	Series for $1/\pi^2$
Modular functions	A new theory?
WZ-method (all?)	WZ-method (all?)

$$F_a(n,k) = F(n+a,k), \quad G_a(n,k) = G(n+a,k),$$

$$G_a(n,k+1) - G_a(n,k) = F_a(n+1,k) - F_a(n,k)$$

and summing from $n = 0$ to ∞ we get

$$\sum_{n=0}^{\infty} [G_a(n,k+1) - G_a(n,k)]$$
$$= \sum_{n=0}^{\infty} [F_a(n+1,k) - F_a(n,k)] = -F_a(0,k)$$

which implies that

$$\sum_{n=0}^{\infty} G_a(n,0) = \sum_{n=0}^{\infty} G_a(n,1) + F_a(0,0)$$
$$= \sum_{n=0}^{\infty} G_a(n,2) + F_a(0,1) + F_a(0,0)$$
$$= \sum_{n=0}^{\infty} G_a(n,3) + \sum_{k=0}^{2} F_a(0,k)$$

and continuing the recursion we arrive to

$$\sum_{n=0}^{\infty} G_a(n,0) = \lim_{k\to\infty} \sum_{n=0}^{\infty} G_a(n,k) + \sum_{k=0}^{\infty} F_a(0,k).$$

A kind of hypergeometric identities

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(a+\frac{1}{2}\right)_n^3}{(a+1)_n^3} \times [6(n+a)+1)] = 8a \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(a+1)_n^2}$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(a+\frac{1}{2}\right)_n^3}{(a+1)_n^3} \times [42(n+a)+5)] = 32a \sum_{n=0}^{\infty} \frac{\left(a+\frac{1}{2}\right)_n^2}{(2a+1)_n^2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{\left(a+\frac{1}{2}\right)_n^3}{(a+1)_n^3}$$

$$\times\, [6(n+a)+1)] = 4a \sum_{n=0}^{\infty} \frac{\left(\frac{a}{2}+\frac{1}{4}\right)_n \left(\frac{a}{2}+\frac{3}{4}\right)_n}{(a+1)_n^2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(a+\frac{1}{2}\right)_n \left(a+\frac{1}{4}\right)_n \left(a+\frac{3}{4}\right)_n}{(a+1)_n^3}$$

$$\times[20(n+a)+3)] = 16a \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(a+\frac{1}{2}\right)_n}{(a+1)_n(2a+1)_n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{\left(a+\frac{1}{2}\right)_n \left(a+\frac{1}{6}\right)_n \left(a+\frac{5}{6}\right)_n}{(a+1)_n^3}$$

$$\times[154(n+a)+15)] = 128a \sum_{n=0}^{\infty} \frac{\left(\frac{a}{2}+\frac{1}{4}\right)_n \left(\frac{a}{2}+\frac{3}{4}\right)_n}{(a+1)_n(2a+1)_n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(a+\frac{1}{2}\right)_n^5}{(a+1)_n^5} [20(n+a)^2+8(n+a)+1)]$$

$$= 8a \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^4}{(a+1)_n^4} (4n+2a+1)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \frac{\left(a+\frac{1}{2}\right)_n^5}{(a+1)_n^5} [820(n+a)^2+180(n+a)+13]$$

$$= 128a \sum_{n=0}^{\infty} \frac{\left(a+\frac{1}{2}\right)_n^4}{(2a+1)_n^4} (4n+6a+1)$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(a+\frac{1}{2}\right)_n^3 \left(a+\frac{1}{4}\right)_n \left(a+\frac{3}{4}\right)_n}{(a+1)_n^5}$$

$$\times [120(n+a)^2+34(n+a)+3)]$$

$$= 32a \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \left(a+\frac{1}{2}\right)_n^2}{(a+1)_n^2 (2a+1)_n^2} (4n+4a+1)$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(a+\frac{1}{2}\right)_n^3}{(a+1)_n^3}[6(n+a)+1]$$

$$= \frac{4}{\pi}\ \frac{4^a}{\cos^2 \pi a}\ \frac{1_a^3}{\left(\frac{1}{2}\right)_a^3} + \frac{16a^2}{2a-1} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(a+\frac{1}{2}\right)_n}{(a+1)_n \left(\frac{3}{2}-a\right)_n}$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(a+\frac{1}{2}\right)_n^3}{(a+1)_n^3}[42(n+a)+5]$$

$$= \frac{16}{\pi}\ \frac{64^a}{\cos^2 \pi a}\ \frac{1_a^3}{\left(\frac{1}{2}\right)_a^3} + \frac{128a^2}{2a-1} \sum_{n=0}^{\infty} \frac{\left(a+\frac{1}{2}\right)_n^2}{(2a+1)_n \left(\frac{3}{2}-a\right)_n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{\left(a+\frac{1}{2}\right)_n^3}{(a+1)_n^3}[6(n+a)+1]$$

$$= \frac{2\sqrt{2}\ 8^a}{\pi \cos \pi a}\ \frac{1_a^3}{\left(\frac{1}{2}\right)_a^3} + \frac{16a^2}{2a-1} \sum_{n=0}^{\infty} \frac{1}{2^n}\ \frac{\left(a+\frac{1}{2}\right)_n^2}{(a+1)_n \left(\frac{3}{2}-a\right)_n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(a+\frac{1}{2}\right)_n \left(a+\frac{1}{4}\right)_n \left(a+\frac{3}{4}\right)_n}{(a+1)_n^3} [20(n+a)+3]$$

$$= \frac{8}{\pi} \frac{4^a}{\cos \pi a} \frac{1_a^3}{\left(\frac{1}{2}\right)_a \left(\frac{1}{4}\right)_a \left(\frac{3}{4}\right)_a}$$

$$+ \frac{48a^2}{2a-1} \sum_{n=0}^{\infty} \frac{1}{4^n} \frac{\left(a+\frac{1}{2}\right)_n \left(2a+\frac{1}{2}\right)_n}{(a+1)_n \left(\frac{3}{2}-a\right)_n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{\left(a+\frac{1}{2}\right)_n \left(a+\frac{1}{6}\right)_n \left(a+\frac{5}{6}\right)_n}{(a+1)_n^3}$$

$$\times [154(n+a)+15]$$

$$= \frac{32\sqrt{2}}{\pi} \frac{512^a}{27^a \cos \pi a} \frac{1_a^3}{\left(\frac{1}{2}\right)_a \left(\frac{1}{6}\right)_a \left(\frac{5}{6}\right)_a}$$

$$+ \frac{512a^2}{2a-1} \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\left(a+\frac{1}{2}\right)_n \left(3a+\frac{1}{2}\right)_n}{(2a+1)_n \left(\frac{3}{2}-a\right)_n}$$

Another kind of hypergeometric identities

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(a+\frac{1}{2}\right)_n \left(a+\frac{1}{4}\right)_n \left(a+\frac{3}{4}\right)_n}{(a+1)_n^3} [8(n+a)+1]$$

$$= \frac{2\sqrt{3}}{\pi} \frac{9^a}{\cos 2\pi a} \frac{1_a^3}{\left(\frac{1}{2}\right)_a \left(\frac{1}{4}\right)_a \left(\frac{3}{4}\right)_a}$$

$$+ \frac{36a^2}{4a-1} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \frac{\left(\frac{1}{2}\right)_n \left(a+\frac{1}{2}\right)_n}{(a+1)_n \left(\frac{3}{2}-2a\right)_n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{48^n} \frac{\left(a+\frac{1}{2}\right)_n \left(a+\frac{1}{4}\right)_n \left(a+\frac{3}{4}\right)_n}{(a+1)_n^3} [28(n+a)+3]$$

$$= \frac{16\sqrt{3}}{3\pi} \frac{48^a}{\cos \pi a} \frac{1_a^3}{\left(\frac{1}{2}\right)_a \left(\frac{1}{4}\right)_a \left(\frac{3}{4}\right)_a}$$

$$+ \frac{96a^2}{2a-1} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \frac{\left(a+\frac{1}{2}\right)_n \left(2a+\frac{1}{2}\right)_n}{(2a+1)_n \left(\frac{3}{2}-a\right)_n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(a+\frac{1}{2}\right)_n^5}{(a+1)_n^5} [20(n+a)^2 + 8(n+a) + 1]$$

$$= \frac{2}{\pi} \frac{1}{\cos \pi a} \frac{1_a^2}{\left(\frac{1}{2}\right)_a^2} \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(a+\frac{1}{2}\right)_n^3}{(a+1)_n^3} [6(n+a)+1]$$

$$+ \frac{32a^3}{2a-1} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2 \left(a+\frac{1}{2}\right)_n}{(a+1)_n^2 \left(\frac{3}{2}-a\right)_n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \frac{\left(a+\frac{1}{2}\right)_n^5}{(a+1)_n^5} [820(n+a)^2 + 180(n+a) + 13]$$

$$= \frac{8}{\pi} \frac{16^a}{\cos \pi a} \frac{1_a^2}{\left(\frac{1}{2}\right)_a^2} \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(a+\frac{1}{2}\right)_n^3}{(a+1)_n^3} [42(n+a)+5]$$

$$+ \frac{2048a^3}{2a-1} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}+a\right)_n^3}{(2a+1)_n^2 \left(\frac{3}{2}-a\right)_n}$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(a+\frac{1}{2}\right)_n^3 \left(a+\frac{1}{4}\right)_n \left(a+\frac{3}{4}\right)_n}{(a+1)_n^5}$$

$$\times \left[120(n+a)^2 + 34(n+a) + 3\right]$$

$$= \frac{2}{\pi} \frac{1}{4^a \ \cos 2\pi a} \frac{1_a^2}{\left(\frac{1}{4}\right)_a \left(\frac{3}{4}\right)_a}$$

$$\times \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(a+\frac{1}{2}\right)_n^3}{(a+1)_n^3} [42(n+a)+5]$$

$$+ \frac{512a^3}{4a-1} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(a+1)_n^2 \left(\frac{3}{2}-2a\right)_n}$$

Appendix. Carlson Theorem.

$$f(z) \text{ analytic,}$$

$$f(z) = 0, \quad \text{for} \quad z = 0, 1, 2, \cdots$$

$$f(z) = O(e^{c|z|}) \quad \text{for} \quad c < \pi,\ \Re(z) \geq 0$$

$$\Rightarrow f(z) = 0$$

We apply Carlson Theorem to the functions

$$f(z) = \sum_{n=0}^{\infty} G(n, z) - CONSTANT$$