Ramanujan-Orr type series

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The Pochhammer symbol

The Pochhammer symbol is defined by

$$(a)_x = \frac{\Gamma(a+x)}{\Gamma(a)}, \qquad (0)_0 = 1.$$

If x is a positive integer (rising factorial), it reduces to

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1),$$

For $a = 1$, we have

$$(1)_n = n!.$$

Therefore, the rising factorial generalizes the ordinary factorial.

Ramanujan series for $1/\pi$

In 1914 S. Ramanujan gave 17 fast hypergeometric series for $1/\pi$. Three examples are

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (42n+5) = \frac{16}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{1}{99^{4n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (26390n+1103) = \frac{9801\sqrt{2}}{4\pi},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} \frac{1}{2^{6n}} \left(\frac{\sqrt{5}-1}{2}\right)^{8n} \left[(42\sqrt{5}+30)n + (5\sqrt{5}-1)\right] = \frac{32}{\pi}.$$

J. and P. Borwein were the first to prove the 17 Ramanujan series by using elliptic integrals and modular equations.

The ukranian brothers D. and G. Chudnovsky proved the formula

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{53360^{3n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \frac{545140134n + 13591409}{426880} = \frac{\sqrt{10005}}{\pi},$$

which is the fastest possible rational series due to

$$z(q) = \frac{12^3}{J(q)}, \quad J(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots,$$

$$z\left(e^{2\pi i\tau}\right) = \frac{-1}{53360^3}, \quad \text{for} \quad \tau = \frac{1+\sqrt{-163}}{2}.$$

The convergence of this series is so fast that Wadim Zudilin could get with it the following irrationality measure:

$$\mu(\pi\sqrt{10005}) \le 10.021363...$$

The PSLQ algorithm

Let $(x_1, \ldots x_n)$ be a vector of real numbers and write all the numbers the x_j with a precision of d decimal digits.

The PSLQ algorithm finds a non-zero vector $(a_1, \ldots, a_n)$ of integers, such that:

$$a_1 x_1 + \cdots + a_n x_n = 0, \quad \text{(with a precision of d digits)},$$

Example: Let

$$f(j) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} \left(\frac{192}{2401}\right)^n \frac{n^j}{2n+1}.$$

With the PSLQ algorithm we guess that

$$15 f(0) + 216 f(1) + 376 f(2) = \frac{98\sqrt{21}}{9\pi}.$$

Elliptic integrals

Let $K(x)$ and $E(x)$ be the elliptic integrals

$$K(x)=\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-x^2\sin^2 t}},\quad E(x)=\int_0^{\frac{\pi}{2}}\sqrt{1-x^2\sin^2 t}\,dt.$$

Legendre's identity

$$-K(\sqrt{r_0})K(\sqrt{1-r_0})+K(\sqrt{r_0})E(\sqrt{1-r_0})+E(\sqrt{r_0})K(\sqrt{1-r_0})=\frac{\pi}{2},$$

Derivatives

$$\left(x\frac{d}{dx}\right)K=-K+\frac{E}{1-x^2},\quad \left(x\frac{d}{dx}\right)E=-K+E.$$

Modular transformation

$$\frac{K(\sqrt{1-\beta})}{K(\sqrt{\beta})}=n\frac{K(\sqrt{1-\alpha})}{K(\sqrt{\alpha})}\rightarrow\left(\frac{K(\sqrt{\alpha})}{K(\sqrt{\beta})}=m(\alpha,\beta),\quad P(\alpha,\beta)=0\right).$$

A variant of a method of Wan

Suppose we have a factorization of the following type:

$$G(x) = K(a(x))K(b(x)),$$

where $a(x)$ and $b(x)$ are algebraic functions. If x_0 is such that

$$a(x_0) = \sqrt{r_0}, \quad b(x_0) = \sqrt{1-r_0},$$

and D is a differential operator satisfying the property

$$D \Big|_{x=x_0} K(a(x))K(b(x)) =$$
$$-K(\sqrt{r_0})K(\sqrt{1-r_0}) + K(\sqrt{r_0})E(\sqrt{1-r_0}) + E(\sqrt{r_0})K(\sqrt{1-r_0}),$$

then, taking into account the Legendre's identity, we obtain

$$D \Big|_{x=x_0} G = \frac{\pi}{2}.$$

The following factorization is known:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} \left(\frac{-4x^2(x-1)^2}{(2x-1)^2}\right)^n = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^2} x^n \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^2} \left(\frac{x}{2x-1}\right)^n .$$

Applying two quadratic transformations, we get

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} y^n = f(x) f\left(\frac{x}{2x-1}\right),$$

where $y = \dfrac{-4x^2(x-1)^2}{(2x-1)^2}$ and $f(x) = \dfrac{2}{\pi} \dfrac{K\left(\sqrt{\frac{2\sqrt{x}}{1+\sqrt{x}}}\right)}{\sqrt{1+\sqrt{x}}}.$

Proof of a Ramanujan-Orr series for $1/\pi$. Part 2

The arguments of the two elliptic integrals are complementary at

$$x_0 = \frac{1}{49} + \frac{4}{49}\sqrt{3}\, i, \quad y_0 = \frac{192}{2401}, \quad r_0 = \frac{1}{2} + \frac{\sqrt{3}}{6}\, i.$$

This suggest that there exist a formula with $y_0 = 192/2401$. Using the PSLQ algorithm, we get

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} \left(\frac{192}{2401}\right)^n \times (70668n^3 - 9216n^2 + 1428n + 90) = \frac{294\sqrt{21}}{\pi}.$$

Hence the differential operator that we need to apply is

$$D = 90 + 1428\left(y\frac{d}{dy}\right) - 9216\left(y\frac{d}{dy}\right)^2 + 70668\left(y\frac{d}{dy}\right)^3 \Bigg|_{y=y_0}.$$

The WZ-method

Let $G(n,k)$ be hypergeometric in its two symbols. The proof of

$$\sum_{n=0}^{\infty} G(n,k) = g(k) = \text{Constant},$$

can be automatically carried over by a computer.

H. Wilf and D. Zeilberger have discovered an algorithm that finds a rational function $C(n,k)$ called certificate, such that

$$F(n,k) = C(n,k)G(n,k), \qquad F(0,k) = 0,$$
$$G(n,k+1) - G(n,k) = F(n+1,k) - F(n,k) \qquad (\textit{WZ--pair}).$$

Observe that if we sum for $n \geq 0$ the right side telescopes. Hence $g(k) = g(k+1)$. If in addition $g(k) = \mathcal{O}(e^{c|\mathrm{Im}(k)|})$, with $c < 2\pi$, then by Carlson's theorem, we have $g(k) = \mathrm{Constant}$.

Chains of WZ pairs

Let $F(n,k)$ and $G(n,k)$ be the two hypergeometric functions of a WZ-pair, and suppose that in addition $F(0,k)=0$. If we define

$$F_{s,t}(n,k) = F(sn, k+tn), \qquad s \in \mathbb{Z} - \{0\}, \qquad t \in \mathbb{Z},$$

then $F_{s,t}(n,k)$ and $G_{s,t}(n,k)$ are also the functions of WZ-pairs satisfying $F_{s,t}(0,k)=0$ and in addition, we have

$$\sum_{n=0}^{\infty} G_{s,t}(n,k) = \sum_{n=0}^{\infty} G(n,k) = \text{Constant}.$$

So we have a chain of formulas with the same sum.

Zeilberger's algorithm

Let K be the operator $KG(n,k) = G(n,k+1)$. The output of

```
with(SumTools[Hypergeometric]);
Zeilberger(G(n,k),k,n,K)[1];
Zeilberger(G(n,k),k,n,K)[2];
```

is an operator $O(K)$, and the companion of $G(n,k)$, that is

$$O(K) := P_0(k) + P_1(k)K + \cdots P_m(k)K^m, \qquad F(n,k),$$

where $P_j(k)$ are polynomials of k. The above output means that

$$O(K)\, G(n,k) = F(n+1,k) - F(n,k).$$

If the operator is just $K-1$, then (F,G) is a WZ-pair because

$$(K-1)G(n,k) = F(n+1,k) - F(n,k).$$

Problem 1

Prove that

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (8n+1) = \frac{2\sqrt{3}}{\pi}.$$

This formula is due to Ramanujan.

Proof of Problem 1

Writing (in a Maple session):

```
d:=degree(Zeilberger(H(n,k),k,n,K)[1],K);
```

where $$H(n,k) = \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^2 (1+k)_n},$$

we have the output $d = 2 < 3$ (candidate). Then, writing

```
Zeilberger(H(n,k)*(8*n+2*k+1),k,n,K)[1];
```

we have the output $3(1+2k)K - 4(1+k)$. Hence

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^2 (1+k)_n} (8n+2k+1) \left(\frac{3}{4}\right)^k \frac{\left(\frac{1}{2}\right)_k}{(1)_k} = \frac{2\sqrt{3}}{\pi},$$

where we can get the constant taking $k = 1/2$. Finally, take $k = 0$.

Problem 2

Prove the Ramanujan series for $1/\pi$:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \frac{(-1)^n}{48^n}(28n+3) = \frac{16\sqrt{3}}{3\pi},$$

and the Orr-type series for $1/\pi$:

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n\left(\frac{1}{6}\right)_n\left(\frac{5}{6}\right)_n}{(1)_n^3 \left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n} \frac{720n^3+804n^2+236n+15}{(n+\frac{1}{3})(n+\frac{2}{3})} = \frac{128\sqrt{3}}{\pi},$$

and

$$\sum_{n=0}^{\infty} \frac{5^{5n}}{2^{6n}3^{5n}} \frac{\left(\frac{1}{10}\right)_n\left(\frac{3}{10}\right)_n\left(\frac{7}{10}\right)_n\left(\frac{9}{10}\right)_n}{(1)_n^3\left(\frac{1}{2}\right)_n} \frac{2924n^2+1668n+105}{n+\frac{1}{2}} = \frac{432\sqrt{3}}{\pi}.$$

Proof of Problem 2

We already know the identity:

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^2 (1+k)_n} (8n+2k+1) \left(\frac{3}{4}\right)^k \frac{\left(\frac{1}{2}\right)_k}{(1)_k} = \frac{2\sqrt{3}}{\pi}$$

$F(n,k) \to F(n,k+n)$, leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n} 3^n} \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{4}+\frac{k}{2}\right)_n \left(\frac{3}{4}+\frac{k}{2}\right)_n \left(\frac{1}{2}+k\right)_n}{(1)_n^2 \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n} \cdot \left(\frac{3}{4}\right)^k \frac{\left(\frac{1}{2}\right)_k}{(1)_k} \times \frac{(28n+3)(2n+1)+4k(9n+k+2)}{2n+k+1} = \frac{16\sqrt{3}}{3\pi}.$$

To prove the second and third identities, we apply respectively:

$$F(n,k) \to F(n,k+2n), \quad F(n,k) \to F(2n,k-3n).$$

Problem 3

Use the WZ-method to prove the following formula:

$$\sum_{n=0}^{\infty} \left(\frac{-1}{16} \right)^n \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3} (51n + 7) = \frac{12\sqrt{3}}{\pi},$$

due to Chan, Liaw and Tan, who proved it in 2003 using modular equations.

Proof of Problem 3

If we write in a Maple session:

```
degree(Zeilberger(H(n,k),k,n,K)[1],K);
```

where

$$H(n,k) = \left(\frac{-1}{16}\right)^n \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}+2k\right)_n \left(\frac{1}{3}+k\right)_n \left(\frac{2}{3}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n \left(1+\frac{k}{2}\right)_n (1+k)_n (1)_n} \cdot \frac{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{(1)_k^2},$$

the output is $d := 3 < 4$ (candidate). We conjecture

$$\sum_{n=0}^{\infty} H(n,k) \frac{(51n+7)(2n+1) + k(pn+qk+r)}{2n+k+1} = \frac{12\sqrt{3}}{\pi}.$$

Letting for example $k=-1/3$, $k=-2/3$ and $k=-4/3$, we get

$$p = 114, \quad q = 36, \quad r = 37.$$

Problem 4

Prove that

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} \left(\frac{192}{2401}\right)^n \times (70668n^3 - 9216n^2 + 1428n + 90) = \frac{294\sqrt{21}}{\pi},$$

is equivalent to

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} \left(\frac{192}{2401}\right)^n \frac{376n^2 + 216n + 15}{2n+1} = \frac{98\sqrt{21}}{9\pi}.$$

Proof of Problem 4

We can prove automatically the identity

$$\sum_{n=0}^{\infty} \frac{\left(\frac{k}{2}\right)_n \left(\frac{1-k}{2}\right)_n \left(\frac{1+k}{2}\right)_n \left(1-\frac{k}{2}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} z^n \left[-\frac{8(1-z)}{z} n^3 + 12n^2 + \frac{k(k^2-1)(k-2) + 4(1+2k-2k^2)n + 8(1+k-k^2)n^2}{2n+1} \right] = 0,$$

using Zeilberger's algorithm:

$$G(n,k) = F(n+1,k) - F(n,k), \quad F(0,k) = 0,$$

where $G(n,k)$ is the function inside the sum and $F(n,k)$ is the companion. The identity is due to telescoping cancellation when we sum for $n \geq 0$.

Let z, a, b be algebraic, and $s \in \{1/2, 1/4, 1/3, 1/6\}$, then

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n}{(1)_n^3} z^n (a+bn) = \frac{1}{\pi}, \quad \text{Ramanujan series.}$$

Let $s_0 = 1/2$, $s_3 = 1 - s_1$, $s_4 = 1 - s_2$,

$$\begin{aligned}(s_1, s_2) =& (1/2, 1/2), (1/2, 1/3), (1/2, 1/4), (1/2, 1/6), (1/3, 1/3), \\ & (1/3, 1/4), (1/3, 1/6), (1/4, 1/4), (1/4, 1/6), (1/6, 1/6), \\ & (1/5, 2/5), (1/8, 3/8), (1/10, 3/10), (1/12, 5/12),\end{aligned}$$

and z, a, b and c algebraic numbers, then

$$\sum_{n=0}^{\infty} z^n \left[\prod_{i=0}^{4} \frac{(s_i)_n}{(1)_n} \right] (a + bn + cn^2) = \frac{1}{\pi^2}, \quad \text{Ramanujan-like series.}$$

Formulas in the new family

With PSLQ we discovered the formulas

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{2n}} (20n^2+8n+1) = \frac{8}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{1}{2^{4n}} (120n^2+34n+3) = \frac{32}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (820n^2+180n+13) = \frac{128}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{4}\right)^{3n} (74n^2+27n+3) = \frac{48}{\pi^2}.$$

I proved the three first formulas by the WZ-method in 2002 and 2003 and the last one in 2010.

Problem 5

Prove the following series for $1/\pi^2$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} (20n^2 + 8n + 1) = \frac{8}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} (120n^2 + 34n + 3) = \frac{32}{\pi^2},$$

and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (820n^2 + 180n + 13) = \frac{128}{\pi^2}.$$

Proof of Problem 5

We have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n(1+k)_n^4}(20n^2+8n+1+24kn+8k^2+4k) = \frac{8}{\pi^2}\frac{(1)_k^4}{\left(\frac{1}{2}\right)_k^4},$$

and

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3\left(\frac{1}{4}-\frac{k}{2}\right)_n\left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^3(1+k)_n^2}(120n^2+84kn+34n+10k+3) = \frac{32}{\pi^2}\frac{(1)_k^2}{\left(\frac{1}{2}\right)_k^2}.$$

Then taking $k=0$, we get the first and second formula.

If we make the transformation $F(n,k) \to F(n,k+n)$ in either of them, and then let $k=0$ we obtain the third one.

Problem 6

Prove the identity

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{4}\right)^{3n} (74n^2 + 27n + 3) = \frac{48}{\pi^2}.$$

Proof of Problem 6

We have

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}+\frac{k}{3}\right)_n \left(\frac{2}{3}+\frac{k}{3}\right)_n \left(1+\frac{k}{3}\right)_n}{(1)_n^3 (1+k)_n^3} \left(\frac{3}{4}\right)^{3n} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2}$$
$$\times \frac{(74n^2+27n+3)n + k(108n^2+42kn+24n+5k+1)}{3n+k} = \frac{16}{\pi^2}.$$

Below we show how we have determined the constant:

$$\lim_{k\to\infty} k \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (42n+5) \frac{1}{64^n} = \frac{1}{\pi} \cdot \frac{16}{\pi}.$$

To complete the proof, take $k=0$.

By the PSLQ algorithm we discovered the formulas

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{(-1)^n}{48^n} (252n^2 + 63n + 5) = \frac{48}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{(1)_n^5} \frac{1}{7^{4n}} (1920n^2 + 304n + 15) = \frac{56\sqrt{7}}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (1640n^2 + 278n + 15) = \frac{256\sqrt{3}}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{80^{3n}} (5418n^2 + 693n + 29) = \frac{128\sqrt{5}}{\pi^2}.$$

They remain unproved.

Expansion in power series

Conjecture

Let $0 < z < 1$ and $\sigma = 1$ (series of positive terms) or $\sigma = -1$ (alternating series). The expansion

$$\sum_{n=0}^{\infty} \sigma^n z^{n+x} \left[\prod_{i=0}^{4} \frac{(s_i)_{n+x}}{(1)_{n+x}} \right] \left[a + b(n+x) + c(n+x)^2 \right]$$
$$= \frac{1}{\pi^2} + 0 \cdot x - \frac{k}{2} x^2 + 0 \cdot x^3 + \frac{j}{24} \pi^2 x^4 + \mathcal{O}(x^5),$$

determines the values of j, z, a, b, c as functions of k. In addition, if k is a rational number such that j is rational too, then z, a, b, c are algebraic.

More conjectured formulas

In 2010 we discovered three more series

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} (-1)^n \left(\frac{3}{4}\right)^{6n} (1936n^2 + 549n + 45) = \frac{384}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \left(\frac{3}{5}\right)^{6n} (532n^2 + 126n + 9) = \frac{375}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{\phi}\right)^{3n} \left[(32 - \frac{216}{\phi}) n^2 + (18 - \frac{162}{\phi}) n + (3 - \frac{30}{\phi}) \right] = \frac{3}{\pi^2},$$

where ϕ is the fifth power of the golden ratio. This formula is the unique irrational example that I have found for $1/\pi^2$.

These three formulas correspond to $k = 5/3$, $8/3$, $8/3$ respectively. The second formula is joint with G. Almkvist.

Two expansions proved by the WZ-method

We have proved the following expansions:

$$\frac{1}{32}\sum_{n=0}^{\infty}\frac{1}{2^{4(n+x)}}\frac{\left(\frac{1}{2}\right)_{n+x}^{3}\left(\frac{1}{4}\right)_{n+x}\left(\frac{3}{4}\right)_{n+x}}{(1)_{n+x}^{5}}[120(n+x)^2+34(n+x)+3]$$
$$=\frac{1}{\pi^2}-x^2+\frac{10}{3}\pi^2x^4-224\zeta(3)x^5+O(x^6).$$

$$\frac{1}{48}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{3(n+x)}\frac{\left(\frac{1}{2}\right)_{n+x}^{3}\left(\frac{1}{3}\right)_{n+x}\left(\frac{2}{3}\right)_{n+x}}{(1)_{n+x}^{5}}[74(n+x)^2+27(n+x)+3]$$
$$=\frac{1}{\pi^2}-\frac{1}{3}x^2+\frac{2}{3}\pi^2x^4-\frac{112}{3}\zeta(3)x^5+O(x^6).$$

To prove formulas like these we need special WZ-pairs which have the following property: $F(n,k)$ has the factor n^5.

Formulas for $1/\pi^3$ and $1/\pi^4$

B. Gourevitch (2002) found with PSLQ the formula

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7} \frac{1}{2^{6n}} (168n^3 + 76n^2 + 14n + 1) = \frac{32}{\pi^3},$$

and Jim Cullen (2010) found with PSLQ the formula

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^9} \frac{1}{2^{12n}} \times$$

$$(43680n^4 + 20632n^3 + 4340n^2 + 466n + 21) = \frac{2^{12}}{\pi^4}.$$

These are unproved formulas.

Ramanujan-Orr type series upside-down

Using Wan's method we can prove the "divergent" formula

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} \left(\frac{-2^{14}}{7^4}\right)^n \frac{2210n^2+1273n+120}{7^2(2n+1)} \text{ “}=\text{” } \frac{\sqrt{7}}{\pi}.$$

Looking at it and making a numerical calculation, I have guessed the exact value of the upside-down of the above series:

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n (1)_n^3}{\left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n} \left(\frac{-7^4}{2^{14}}\right)^n \frac{2210n^2-1273n+120}{7^4(-2n+1)n^3} \stackrel{?}{=} \mathrm{L}_{-7}(2).$$

Inspired by works of Van Hamme and W. Zudilin, we observe that

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{\left(\frac{1}{2}\right)_n (1)_n^3} \left(\frac{-2^{14}}{7^4}\right)^n \frac{2210n^2+1273n+120}{2n+1}$$

$$\stackrel{?}{\equiv} 120p\left(\frac{-7}{p}\right) \pmod{p^3}. \quad \text{Curiously } \mathrm{L}_{-7}(2) := \sum_{n=1}^{\infty} \left(\frac{-7}{n}\right) n^{-2}.$$

Thank you