A method for proving Ramanujan's series for $1/\pi$

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Definition of modular forms and functions

The Hecke congruence group is

$$\Gamma_0(\ell) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| \, a, b, c, d \in \mathbb{Z}, \; ad - bc = 1, \; c \equiv 0 \mod \ell \right\}.$$

A function $f(\tau)$ is a modular form of weight n and level ℓ for $\Gamma_0(\ell)$ if f is holomorphic in the upper semiplane and in the cusp, and

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(\tau), \quad \tau \in \mathbb{C}.$$

If $n = 0$ then $f(\tau)$ is a modular function, and has a Fourier expansion

$$f(\tau) = \sum_{k=0}^{\infty} h_k e^{2\pi i \tau k} = \sum_{k=0}^{\infty} h_k q^k = f(q), \quad q = e^{2\pi i \tau} \text{ (nome)}.$$

If $\ell = 1$ we see that the congruence group is just the modular group.

Ramanujan's series for $1/\pi$

In a famous paper of 1914 Ramanujan gave a list of 17 extraordinary formulas for the number $1/\pi$, which are of the following form:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{s}\right)_n \left(1-\frac{1}{s}\right)_n}{(1)_n^3} (a+bn)\, z^n = \frac{1}{\pi}, \qquad \ell = 4\sin^2\frac{\pi}{s},$$

where $s \in \{2,3,4,6\}$, and z, b, a are algebraic. Two examples are

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \frac{26390n+1103}{99^{4n+2}} = \frac{\sqrt{2}}{4\pi},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (-1)^n \frac{21460n+1123}{882^{2n+1}} = \frac{4}{\pi}.$$

Notation $(c)_n = c(c+1)(c+2)\cdots(c+n-1)$.

The tools of our method (A variant of Wan's method)

Notation:

$$F_s(\alpha) = {}_2F_1\left(\begin{matrix} \frac{1}{s}, 1-\frac{1}{s} \\ 1 \end{matrix} \,\middle|\, \alpha\right), \quad G_s(\alpha) = \alpha \frac{dF_s(\alpha)}{d\alpha},$$

(1) The following version of the Legendre's relation:

$$\alpha F_s(\alpha) G_s(\beta) + \beta F_s(\beta) G_s(\alpha) = \frac{1}{\pi} \sin \frac{\pi}{s} = \frac{\sqrt{\ell}}{2\pi}, \qquad \beta = 1 - \alpha.$$

(2) The Clausen's identity:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{s}\right)_n \left(1-\frac{1}{s}\right)_n}{(1)_n^3} z^n = F_s(\alpha) F_s(\alpha), \qquad z = 4\alpha(1-\alpha).$$

The main tool

(3) A transformation of Modular origin of level ℓ and degree $1/d$:

$$F_s(\alpha) = m(\alpha, \beta) F_s(\beta), \quad A(\alpha, \beta) = 0,$$

where $A(\alpha, \beta) = 0$ is the modular equation and $m(\alpha, \beta)$ is the multiplier. Differentiating with respect to x:

$$G_s(\alpha) = \alpha \, \frac{m'}{\alpha'} F_s(\beta) + \alpha \, \frac{m}{\beta} \, \frac{\beta'}{\alpha'} \, G_s(\beta).$$

If the transformation is of degree $1/d$, then

$$m^2 = \frac{1}{d} \, \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \, \frac{\alpha'}{\beta'}.$$

Modular origin: $\beta = f(q), \quad \alpha = f(q^d)$.

Explicit formulas (1)

Theorem (General explicit formulas for z, b and a)

$$z_0 = 4\alpha_0\beta_0 = 4\alpha_0(1-\alpha_0),$$

$$b = \frac{1}{\sqrt{\ell}}\left(m_0 d + \frac{1}{m_0}\right)(1-2\alpha_0), \qquad a = -2\alpha_0\beta_0 \frac{m_0'}{\alpha_0'} \frac{d}{\sqrt{\ell}}.$$

Proof: Applying the operator

$$a + bz\frac{d}{dz}\bigg|_{z_0} = a + b\frac{z}{z'}\frac{d}{d\alpha}\bigg|_{\alpha_0},$$

to both sides of the Clausen's identity, we get

$$aF(\alpha_0)F(\alpha_0) + (1-Ci)\frac{b}{2}\frac{1-\alpha_0}{1-2\alpha_0}F(\alpha_0)G(\alpha_0) + (1+Ci)\frac{b}{2}\frac{1-\alpha_0}{1-2\alpha_0}G(\alpha_0)F(\alpha_0).$$

Proof (Continuation)

Then, using the substitutions

$$F_s(\alpha_0) = mF_s(\beta_0), \quad G_s(\alpha_0) = \alpha_0 \frac{m_0'}{\alpha_0'} F_s(\beta_0) + \frac{\alpha_0}{d\, m_0\, \beta_0} G_s(\beta_0),$$

and equating the coefficients of

$$F(\alpha_0)F(\beta_0), \quad F(\alpha_0)G(\beta_0), \quad F(\beta_0)G(\alpha_0),$$

to 0, α_0 and β_0 respectively, we get the system:

$$am_0 + (1+Ci)b\frac{1-\alpha_0}{1-2\alpha_0}\alpha_0\frac{m_0'}{\alpha_0'} = 0,$$

$$\frac{\sqrt{\ell}}{2}(1-Ci)\frac{b}{1-2\alpha_0} = \frac{1}{m_0}, \quad \frac{\sqrt{\ell}}{2}(1+Ci)\frac{b}{1-2\alpha_0} = m_0 d.$$

Explicit formulas (2)

We also deduce that

$$C = \frac{1 - m_0^2 d}{1 + m_0^2 d}.$$

In addition, if

$$m_0 = \frac{1}{\sqrt{d}} \quad \text{or} \quad m_0 = \frac{\sqrt{4d - \ell}}{2d} + \frac{\sqrt{\ell}}{2d}\, i,$$

then we have

$$b = (1 - 2\alpha_0) \sqrt{\frac{4d}{\ell}} \quad \text{or} \quad b = (1 - 2\alpha_0)\sqrt{\frac{4d}{\ell} - 1},$$

respectively. In all the cases that we have checked, we observe that the second situation corresponds to $z_0 < 0$.

The modular variable q (the nome)

Let $q = e^{-\pi\sqrt{r}}$ and $q = -e^{-\pi\sqrt{r}}$, the modular variable corresponding to the cases $z > 0$ and $z < 0$, respectively. From the known formula

$$r = \frac{b^2}{1-z} = \frac{b^2}{(1-2\alpha)^2},$$

we deduce that

$$r = \frac{4d}{\ell}, \quad r = \frac{4d}{\ell} - 1,$$

for the cases $z > 0$ and $z < 0$, respectively. Hence,

$$q = e^{-\pi\sqrt{\frac{4d}{\ell}}}, \qquad q = -e^{-\pi\sqrt{\frac{4d}{\ell}-1}},$$

for $z > 0$ and $z < 0$ respectively.

First example

We reprove the following two series for $1/\pi$ due to Ramanujan

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \frac{1}{3^{4n}} (10n+1) = \frac{9\sqrt{2}}{4\pi}.$$

and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \left(\frac{-1}{48}\right)^n (28n+3) = \frac{16\sqrt{3}}{3\pi}.$$

They are $(P, 2, 5)$ and $(A, 2, 5)$ respectively.

Proof

We use the Cooper-Zudilin's transformation of level $\ell = 2$ and degree $1/d$ with $d = 5$: $F_4(\alpha) = m\, F_4(\beta)$, where

$$\alpha = \frac{64x^5(1+x)}{(1+4x^2)(1-2x-4x^2)^2}, \quad \beta = \frac{64x(1+x)^5}{(1+4x^2)(1+22x-4x^2)^2},$$

$$m^2 = \frac{1-2x-4x^2}{1+22x-4x^2}.$$

One solution of $\beta = 1-\alpha$ is

$$\alpha_0 = \frac{1}{2} - \frac{2\sqrt{5}}{9}, \quad m_0 = \frac{1}{\sqrt{5}}, \quad \frac{m_0'}{\alpha_0'} = -\frac{72}{5}.$$

Another solution of $\beta = 1-\alpha$ is

$$\alpha_0 = \frac{1}{2} - \frac{7\sqrt{3}}{24}, \quad m_0 = \frac{(3+i)\sqrt{2}}{10}, \quad \frac{m_0'}{\alpha_0'} = \frac{18\sqrt{6}}{5}.$$

Degree d and degree N

The method of Chan, Chan and Liu uses a modular equation of degree N. For series of positive terms

$$q = e^{-\pi\sqrt{\frac{4d}{\ell}}}, \qquad q = e^{-\pi\sqrt{\frac{4N}{\ell}}}.$$

Hence $d = N$. For alternating series

$$q = -e^{-\pi\sqrt{\frac{4d}{\ell}-1}},$$

$$q = -e^{-\pi\sqrt{N}} \quad \text{for} \quad \ell = 1, 2, 4; \qquad q = -e^{-\pi\sqrt{\frac{N}{3}}} \quad \text{for} \quad \ell = 3.$$

Hence, we have

$$N = \frac{4d}{\ell} - 1 \quad \text{for} \quad \ell = 1, 2, 4; \qquad N = 4d - 3 \quad \text{for} \quad \ell = 3.$$

Wan also uses N (and not d).

Parameterization of α and β

For levels $\ell = 2, 3, 4$:

$$\alpha_2(q) = \frac{64}{64 + \eta^{24}(q)\eta^{-24}(q)}, \qquad \alpha_3(q) = \frac{27}{27 + \eta^{12}(q)\eta^{-12}(q)},$$

$$\alpha_4(q) = \frac{16}{16 + \eta^{8}(q)\eta^{-8}(q)}, \quad \text{where} \quad \eta(q) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n),$$

is the Dedekind eta function, and

$$J = \frac{432}{\alpha_1(1-\alpha_1)} = \frac{64(1+3\alpha_2)^3}{\alpha_2(1-\alpha_2)^2} = \frac{27(1+8\alpha_3)^3}{\alpha_3(1-\alpha_3)^3}$$
$$= \frac{16(1+14\alpha_4+\alpha_4^2)^3}{\alpha_4(1-\alpha_4)^4} \quad \text{is the modular invariant.}$$

Trasnsformation of degree d: $\beta(q) = \alpha(q^d)$.

Table for level $\ell = 4$ (The classical theory)

d	a	b	$z<0$	d	a	b	$z>0$
3	$\frac{1}{2}$	2	-1	3	$\frac{1}{4}$	$\frac{6}{4}$	$\frac{1}{4}$
5	$\frac{1}{2\sqrt{2}}$	$\frac{6}{2\sqrt{2}}$	$-\frac{1}{8}$	7	$\frac{5}{16}$	$\frac{42}{16}$	$\frac{1}{64}$

Table: Rational Ramanujan-type series for $1/\pi$: $N = d-1$, $N = d$.

Modular equations by Ramanujan ($\ell = 4$):

$$d = 2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29, 31,\ 37, 47, 71.$$

degree 11: $u^{12} = \alpha\beta, \quad v^{12} = (1-\alpha)(1-\beta), \quad P(u,v) = 0,$

$$P(u,v) = u^3 + v^3 + 2\sqrt[3]{2}\,u\,v - 1.$$

Table for level $\ell = 2$

d	a	b	$z < 0$	d	a	b	$z > 0$
3	$\frac{3}{8}$	$\frac{20}{8}$	$-\frac{1}{4}$	2	$\frac{2}{9}$	$\frac{14}{9}$	$\frac{32}{81}$
4	$\frac{8}{9\sqrt{7}}$	$\frac{65}{9\sqrt{7}}$	$-\frac{16^2}{63^2}$	3	$\frac{1}{2\sqrt{3}}$	$\frac{8}{2\sqrt{3}}$	$\frac{1}{9}$
5	$\frac{3\sqrt{3}}{16}$	$\frac{28\sqrt{3}}{16}$	$-\frac{1}{48}$	5	$\frac{4}{9\sqrt{2}}$	$\frac{40}{9\sqrt{2}}$	$\frac{1}{81}$
7	$\frac{23}{72}$	$\frac{260}{72}$	$-\frac{1}{18^2}$	9	$\frac{27}{49\sqrt{3}}$	$\frac{360}{49\sqrt{3}}$	$\frac{1}{7^4}$
13	$\frac{41\sqrt{5}}{288}$	$\frac{644\sqrt{5}}{288}$	$-\frac{1}{5\cdot 72^2}$	11	$\frac{19}{18\sqrt{11}}$	$\frac{280}{18\sqrt{11}}$	$\frac{1}{99^2}$
19	$\frac{1123}{3528}$	$\frac{21460}{3528}$	$-\frac{1}{882^2}$	29	$\frac{4412}{9801\sqrt{2}}$	$\frac{105560}{9801\sqrt{2}}$	$\frac{1}{99^4}$

Table: Rational Ramanujan-type series for $1/\pi$: $N = 2d - 1$, $N = d$.

Modular equations of level 2

degree 7 (Bendt, Bhargava, Garvan):

$$u^8 = \alpha\beta, \quad v^8 = (1-\alpha)(1-\beta), \quad P(u,v) = 0,$$

$$P(u,v) = u^4 + v^4 + 20u^2v^2 + 8\sqrt{2}(u^3v + uv^3) - 1.$$

degree 11 (Bendt, Bhargava, Garvan):

$$u^{12} = \alpha\beta, \quad v^{12} = (1-\alpha)(1-\beta), \quad P(u,v) = 0,$$

$$P(u,v) = u^6 + v^6 + 68u^3v^3 + 16(u^5v + uv^5) + 48(u^4v^2 + u^2v^4) - 1.$$

Table for level $\ell = 3$

d	a	b	$z < 0$	d	a	b	$z > 0$
3	$\frac{\sqrt{3}}{4}$	$\frac{5\sqrt{3}}{4}$	$-\frac{9}{16}$	2	$\frac{1}{3\sqrt{3}}$	$\frac{6}{3\sqrt{3}}$	$\frac{1}{2}$
5	$\frac{7}{12\sqrt{3}}$	$\frac{51}{12\sqrt{3}}$	$-\frac{1}{16}$	4	$\frac{8}{27}$	$\frac{60}{27}$	$\frac{2}{27}$
7	$\frac{\sqrt{15}}{12}$	$\frac{9\sqrt{15}}{12}$	$-\frac{1}{80}$	5	$\frac{8}{15\sqrt{3}}$	$\frac{66}{15\sqrt{3}}$	$\frac{4}{125}$
11	$\frac{106}{192\sqrt{3}}$	$\frac{1230}{192\sqrt{3}}$	$-\frac{1}{2^{10}}$				
13	$\frac{26\sqrt{7}}{216}$	$\frac{330\sqrt{7}}{216}$	$-\frac{1}{3024}$				
23	$\frac{827}{1500\sqrt{3}}$	$\frac{14151}{1500\sqrt{3}}$	$-\frac{1}{500^2}$				

Table: Rational Ramanujan-type series for $1/\pi$: $N = 4d - 3$, $N = d$.

Modular equations of level 3

degree 11 (Berndt, Bhargava, Garvan):

$$u^{12} = \alpha\beta, \;\; v^{12} = (1-\alpha)(1-\beta), \;\; P(u,v) = 0,$$

$$P(u,v) = u^4 + v^4 + 6u^2v^2 + 3\sqrt{3}(uv^3 + u^3v) - 1.$$

degree 23 (Chan, Liaw):

$$u^{12} = \alpha\beta, \;\; v^{12} = (1-\alpha)(1-\beta), \;\; P(u,v) = 0,$$

$$\begin{aligned} P(u,v) = &(u^8+v^8) - 12\sqrt{3}(u^7v + uv^7) - 87(u^6v^2 + u^2v^6) \\ &- 84\sqrt{3}(u^5v^3 + u^3v^5) - 160(u^4v^4) - 2(u^4+v^4) \\ &- 15\sqrt{3}(u^3v + uv^3) - 48(u^2v^2) + 1. \end{aligned}$$

Proof of the formula $(A, 3, 23)$ (First part)

If we let $\beta = 1 - \alpha$, then we see that $u^{12} = v^{12}$. If we choose

$$v = \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\, i \right) u,$$

One solution of $P(u, v(u)) = 0$ is

$$u_0 = \frac{\sqrt{15} + \sqrt{5}}{20} + \frac{\sqrt{15} - \sqrt{5}}{20}\, i, \quad v_0 = \frac{\sqrt{15} + \sqrt{5}}{20} - \frac{\sqrt{15} - \sqrt{5}}{20}\, i.$$

Then, from $\alpha_0(1 - \alpha_0) = u_0^{12} = -10^{-6}, \quad \beta_0 = 1 - \alpha_0$, we get

$$\alpha_0 = \frac{1}{2} - \frac{53}{1000}\sqrt{89}, \quad \beta_0 = \frac{1}{2} + \frac{53}{1000}\sqrt{89}.$$

Proof of the formula $(A, 3, 23)$ (Second part)

From $P(u, v) = 0$, we get v_0', and v_0''. From

$$u^{12} = \alpha\beta, \quad u^{12} - v^{12} + 1 = \alpha + \beta,$$

we get α_0', β_0', α_0'', β_0''. From the formula for the multiplier we obtain

$$m_0 = \sqrt{\frac{1}{23}\frac{\alpha_0'}{\beta_0'}} = \frac{\sqrt{89}}{46} + \frac{\sqrt{3}}{46}i = \frac{\sqrt{4d-\ell}}{2d} + \frac{\sqrt{\ell}}{2d}\, i,$$

$$\frac{m_0'}{\alpha_0'} = \frac{m_0}{2\alpha_0'}\left(\frac{\beta_0'}{\beta_0} - \frac{\beta_0'}{1-\beta_0} - \frac{\alpha_0'}{\alpha_0} + \frac{\alpha_0'}{1-\alpha_0} + \frac{\alpha_0''}{\alpha_0'} - \frac{\beta_0''}{\beta_0'}\right) = \frac{827000}{69}.$$

Finally, we obtain the formula $(A, 3, 23)$ from the formulas

$$z = 4\alpha_0\beta_0, \qquad b = (1-2\alpha_0)\sqrt{\frac{4d}{\ell} - 1}, \qquad a = -2\alpha_0\beta_0 \frac{m_0'}{\alpha_0'}\frac{d}{\sqrt{\ell}}.$$

Table for level $\ell = 1$ (there is one more with $d = 41$)

d	a	b	$z<0$	d	a	b	$z>0$
2	$\frac{8}{5\sqrt{15}}$	$\frac{63}{5\sqrt{15}}$	$-\frac{4^3}{5^3}$	2	$\frac{3}{5\sqrt{5}}$	$\frac{28}{5\sqrt{5}}$	$\frac{3^3}{5^3}$
3	$\frac{15}{32\sqrt{2}}$	$\frac{154}{32\sqrt{2}}$	$-\frac{3^3}{8^3}$	3	$\frac{6}{5\sqrt{15}}$	$\frac{66}{5\sqrt{15}}$	$\frac{4}{5^3}$
5	$\frac{25}{32\sqrt{6}}$	$\frac{342}{32\sqrt{6}}$	$-\frac{1}{8^3}$	4	$\frac{20}{11\sqrt{33}}$	$\frac{252}{11\sqrt{33}}$	$\frac{2^3}{11^3}$
7	$\frac{279}{160\sqrt{30}}$	$\frac{4554}{160\sqrt{30}}$	$-\frac{9}{40^3}$	7	$\frac{144\sqrt{3}}{85\sqrt{85}}$	$\frac{2394\sqrt{3}}{85\sqrt{85}}$	$\frac{4^3}{85^3}$
11	$\frac{526\sqrt{15}}{80^2}$	$\frac{10836\sqrt{15}}{80^2}$	$-\frac{1}{80^3}$				
17	$\frac{10177\sqrt{330}}{3 \cdot 440^2}$	$\frac{261702\sqrt{330}}{3 \cdot 440^2}$	$-\frac{1}{440^3}$				

Table: Rational Ramanujan-type series for $1/\pi$: $N = 4d - 1$, $N = d$.

$(A, 1, 41)$: G. and D. Chudnovsky's series for $1/\pi$

It is the most efficient series for computing π:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} (-1)^n \frac{545140134n + 13591409}{53360^{3n+2}} = \frac{3}{2\sqrt{10005}} \frac{1}{\pi},$$

and gives 14 correct digits of π per term.

$$A, \quad \ell = 1, \quad d = 41, \quad q = -e^{-2\pi\sqrt{\frac{41}{1}-\frac{1}{4}}} = -e^{-\pi\sqrt{163}}, \quad N = 163.$$

It is known that $Q(\sqrt{-163})$ has unique factorization, and that 163 is the greatest integer satisfying that property.

$$z(q) = \frac{12^3}{J(q)}, \quad J(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots,$$

$$z\left(-e^{-\pi\sqrt{163}}\right) = \frac{-1}{53360^3}.$$

Other levels

Ramanujan-Sato series are of the form:

$$\sum_{n=0}^{\infty} A_n (a + bn) z^n = \frac{1}{\pi},$$

where the A_n are integers satisfying a certain type of recurrences of order 3, and z, b, a are algebraic numbers. Examples of A_n are

$$\ell = 5, \quad \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k},$$

$$\ell = 7, \quad \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n}{k} \binom{n+k}{k},$$

$$\ell = 10, \quad \sum_{k=0}^{n} \binom{n}{k}^4.$$

Some important related previous results

- J. Borwein and P. Borwein (First rigorous proofs).
- G. Chudnovsky and D. Chudnovsky (The fastest series).
- B. Berndt, S. Bhargava, and F. Garvan (Alternative theories).
- Baruah and Berndt (Proofs close to Ramanujan's ideas).
- H.H. Chan, S.H. Chan, Z. Liu (General method for all levels).
- S. Cooper and W. Zudilin (Hyperg. modular equations).
- J. Wan (Method for series that factorize into ${}_2F_1$ series).

Thank you