Arithmetical functions and zeros of zeta

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About this talk

This talk is based on my paper

Some sums over the non-trivial zeros
of the Riemann zeta function

The most interesting results that we will show are

- Some asymptotic behaviors involving the zeros of the Riemann zeta function related to the work by Crámer but more explicit.
- New representations of the main arithmetical functions in terms of the zeros of zeta.

Introduction 1

In 1737 Euler proved that for $s > 1$, the series and the product below are convergent and that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

In 1859 Riemann extends the function to the set $\mathcal{C}$

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

The function $\zeta(s)$ has trivial zeros at $s = -2n$. The other zeros $\rho = \beta + i\gamma$ are complex. Hasse in 1930 gives

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s},$$

which is convergent for all $s \neq 1$.

Introduction 2

Riemann also defines the function

$$\xi(s) = \Gamma\left(1+\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s), \qquad \xi(1-s) = \xi(s),$$

and proves that (rigorously proved by Hadamard):

$$\xi(s) = \xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right). \quad \text{Hence}$$

$$\zeta(s) = \frac{1}{2}\frac{\pi^{s/2}}{(s-1)\Gamma\left(1+\frac{s}{2}\right)}\prod_{\gamma>0}\left\{\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{\bar{\rho}}\right)\right\}.$$

Riemann relates $\pi(x)$ to the zeros of zeta. A simpler variant is

$$\psi(x) = \sum_{n\le x}\Lambda(n) = x - \log 2\pi - \frac{1}{2}\left(1-\frac{1}{x^2}\right) - \sum_{\rho}\frac{x^{\rho}}{\rho},$$

valid for $x \neq p^k$, where $\Lambda(n)$ is the Mangoldt function.

Introduction 3

The asymptotic approximation $\psi(x) \sim x$ is equivalent to the Prime Number Theorem conjectured by Gauss, namely

$$\pi(x) \sim \frac{x}{\log x}.$$

- From the functional equation of zeta, Riemann proved that all the non-trivial zeros of zeta are in the band $\beta \in [0, 1]$.
- In 1.896 Hadamard and de La Vallée Poussin achieved to prove the Prime Number Theorem by showing that $\zeta(s)$ has no zeros of the form $\rho = 1 + i\gamma$.
- The Riemann's famous conjecture stating that the real part of all the non-trivial zeros was equal to $1/2$ remains unproved.

In addition, Riemann gives the asymptotic behavior, as $T \to +\infty$:

$$\mathcal{N}(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi},$$

for the number of complex zeros with positive imaginary part less or equal than T. It was proved by von Mangoldt (1905):

$$\mathcal{N}(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\log T).$$

Backlund (1912) derived an exact formula for counting the zeros:

$$\mathcal{N}(T) = \frac{1}{\pi}\theta(T) + \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iT \right) + 1, \quad \text{where}$$

$$\theta(T) = \arg \Gamma \left(\frac{1}{4} + \frac{i}{2} T \right) - \frac{\log \pi}{2} T.$$

is the Riemann-Siegel theta function.

Notation

- $\rho = \beta + i\gamma$ for the non-trivial zeros.
- Following Riemann, we define $\tau = -i(\rho - \frac{1}{2})$. Hence $\rho = 1/2 + i\tau$ (Riemann used the notation α instead of τ).
- The Riemann Hypothesis is the statement that all the $\tau's$ are real.
- We use C for Euler's constant as Euler did.
- As usual in papers of Number Theory, log denotes the naperian logarithm.

In 1911 Landau proved the following formula

$$\Lambda(t) = \frac{-2\pi}{T}\sqrt{t} \sum_{0<\mathrm{Re}\,\tau \leq T} \cos(\tau \log t) + \mathcal{O}(\frac{\log T}{T}),$$

which implies

$$\Lambda(t) = -2\pi\sqrt{t} \lim_{T\to+\infty} \frac{1}{T} \sum_{0<\mathrm{Re}\,\tau \leq T} \cos(\tau \log t).$$

It has the surprising property that neglecting a finite number of zeros of zeta we still recover the Mangoldt function.

If we assume the Riemann Hypothesis then we can replace τ and $\mathrm{Re}\,\tau$ with γ.

Formula with Mangoldt (1) $\Rightarrow$ (2)

Theorem

The following identity

$$(1) \qquad \sum_{\operatorname{Re}\tau>0} \frac{\sinh z\tau}{\sinh \pi\tau} - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{2\pi\sqrt{n}} \left(\frac{ie^{iz}}{e^{iz}+n} - \frac{ie^{-iz}}{e^{-iz}+n} \right) = f(z),$$

where

$$f(z) = \sin\frac{z}{2} - \frac{1}{8}\tan\frac{z}{4} - \frac{C+\log 8\pi}{4\pi}\tan\frac{z}{2} - \frac{1}{4\pi\cos\frac{z}{2}}\log\frac{1-\tan\frac{z}{4}}{1+\tan\frac{z}{4}},$$

holds for $|\operatorname{Re}(z)| < \pi$. *If in addition* $|\operatorname{Im}(z)| < \log 2$, *then we have*

$$(2) \qquad \sum_{\operatorname{Re}\tau>0} \frac{\sinh z\tau}{\sinh \pi\tau} + \frac{1}{\pi}\sum_{n=1}^{\infty} (-1)^n \frac{\zeta'(n+\frac{1}{2})}{\zeta(n+\frac{1}{2})} \sin nz = f(z).$$

Example from formula (2)

Example

Differentiating (2) with respect to z at $z = 0$, we get

$$\sum_{n=1}^{\infty} (-1)^n n \frac{\zeta'(n+\frac{1}{2})}{\zeta(n+\frac{1}{2})} + \sum_{\mathrm{Re}\,\tau>0} \frac{\pi\tau}{\sinh \pi\tau} = -\frac{3}{8}\log 2 - \frac{1}{8}\log \pi + \frac{15}{32}\pi - \frac{1}{8}C + \frac{1}{8}.$$

As $\gamma_1 \simeq 14.134725$, the sum over the zeros is of order 10^{-18}.

Asymptotic behavior

If we let $z = \pi - 1/T$ in formula (2), then as $T \to \infty$ we have

$$\sum_{\mathrm{Re}\,\tau>0} \exp\left(-\frac{\tau}{T}\right) = \frac{1}{2\pi} T \log T - \frac{C + \log 2 + \log \pi}{2\pi} T + \frac{7}{8} + \frac{1}{48\pi}\frac{\log T}{T} + A\frac{1}{T} - \frac{9}{64}\frac{1}{T^2} + \frac{7}{11520\pi}\frac{\log T}{T^3} + \mathcal{O}(\frac{1}{T^3}),$$

where A is the constant

$$A = \frac{1}{16} + \frac{4C - 1 + 16\log 2 + 4\log \pi}{96\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} n \frac{\zeta'(n+\frac{1}{2})}{\zeta(n+\frac{1}{2})} + \sum_{\mathrm{Re}\,\tau>0} \frac{\tau e^{-\pi\tau}}{\sinh \pi\tau} = -0.759578\ldots.$$

Other asymptotic behaviors

Differentiating with respect to T once and twice, we get

$$\sum_{\mathrm{Re}\,\tau>0} \tau \exp\left(-\frac{\tau}{T}\right) = \frac{1}{2\pi} T^2 \log T + \frac{1-C-\log 2-\log \pi}{2\pi} T^2 - \frac{1}{48\pi}\log T + \left(\frac{1}{48\pi} - A\right) + \frac{9}{32T} + \mathcal{O}(\frac{1}{T^2}),$$

and

$$\sum_{\mathrm{Re}\,\tau>0} \tau^2 \exp\left(-\frac{\tau}{T}\right) = \frac{1}{\pi} T^3 \log T + \frac{3-2C-2\log 2\pi}{2\pi} T^3 - \frac{1}{48\pi} T - \frac{9}{32} + \mathcal{O}(\frac{1}{T}).$$

as $T \to \infty$.

Corollary

For $t > 1$, we obtain the identity

$$
(3) \qquad \lim_{x \to \pi^-} \left(\frac{1}{4\pi} \frac{\Lambda(t)}{\sqrt{t}} \tan \frac{x}{2} + \sum_{\mathrm{Re}\, \tau > 0} \frac{\sinh x\tau}{\sinh \pi\tau} \cos(\tau \log t) \right) = \frac{1}{2} \left(\frac{t+1}{t} - \frac{t}{t^2 - 1} \right) \sqrt{t},
$$

If we assume the RH then we can replace τ and $\mathrm{Re}\, \tau$ with γ.

Proof of formula (3)

Let $z = x - i \log t$ in (1) and take real parts. Observe that

$$\mathrm{Re} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{2\pi\sqrt{n}} \frac{ie^{ix}t}{e^{ix}t+n} = -\sin x \sum_{n=1}^{\infty} \frac{\Lambda(n)}{2\pi\sqrt{n}} \frac{tn}{(e^{ix}t+n)(e^{-ix}t+n)}.$$

We have to prove that for $t > 1$, the limit of

$$-\sin x \sum_{n=1}^{\infty} \frac{\Lambda(n)}{2\pi\sqrt{n}} \frac{tn}{(e^{ix}t+n)(e^{-ix}t+n)} + \frac{\Lambda(t)}{2\pi\sqrt{t}} \frac{\sin x}{2(1+\cos x)},$$

as $x \to \pi^-$, is equal to 0. But it is evident if t is not the power p^k of a prime p. When $t = p^k$ it comes observing that the only term that contributes to the sum is $n = p^k$.

Representation of the Mangoldt function

Corollary

From (3), as $x \to \pi^-$ we get

$$(4)\quad \Lambda(t) = -4\pi\sqrt{t}\cot\frac{x}{2}\sum_{\mathrm{Re}\,\tau>0}\frac{\sinh x\tau}{\sinh \pi\tau}\cos(\tau\log t) + \mathcal{O}\left(\cot\frac{x}{2}\right).$$

and

$$\begin{aligned}\Lambda(t) = & -4\pi\sqrt{t}\cot\frac{x}{2}\sum_{\mathrm{Re}\,\tau>0}\frac{\sinh x\tau}{\sinh \pi\tau}\cos(\tau\log t) \\ (5)\quad & + 2\pi\left(t+1-\frac{t^2}{t^2+1}\right)\cot\frac{x}{2} + o\left(\cot\frac{x}{2}\right).\end{aligned}$$

If we assume the Riemann Hypothesis then we can replace τ and $\mathrm{Re}\,\tau$ with γ.

Comparing representations

Landau's formula ($T \to \infty$):

$$\Lambda(t) = \frac{-2\pi}{T}\sqrt{t} \sum_{0<\gamma\leq T} \cos(\gamma \log t) + \mathcal{O}(\frac{\log T}{T}).$$

G1 formula ($x \to \pi^-$):

$$\Lambda(t) = -4\pi\sqrt{t} \cot\frac{x}{2} \sum_{\gamma>0} \frac{\sinh x\gamma}{\sinh \pi\gamma} \cos(\gamma \log t) + \mathcal{O}\left(\cot\frac{x}{2}\right).$$

G2 formula ($x \to \pi^-$):

$$\Lambda(t) = -\, 4\pi\sqrt{t} \cot\frac{x}{2} \sum_{\gamma>0} \frac{\sinh x\gamma}{\sinh \pi\gamma} \cos(\gamma \log t) + 2\pi\left(t + 1 - \frac{t^2}{t^2+1}\right)\cot\frac{x}{2} + o\left(\cot\frac{x}{2}\right).$$

Sage code for the Mangoldt function (G1)

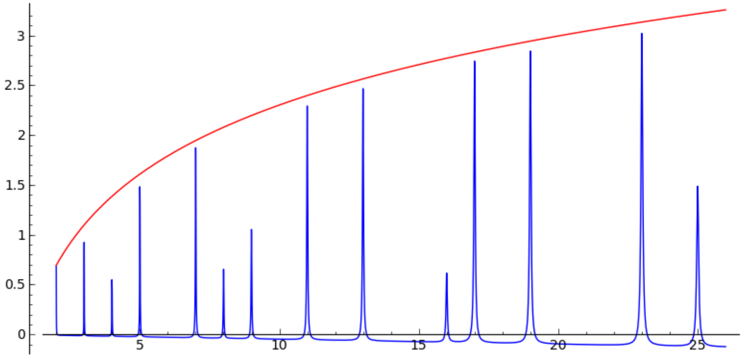
$$(x = 3.14,\ \gamma_k \text{ for } k = 1 \text{ to } k = 10000)$$

Here is the Sage code:

```
var('t')
z=sage.databases.odlyzko.zeta_zeros()
p=pi.n(digits=15); x=3.14; ran=range(0,10000)
v(t)=sum([sinh(x*z[j])/sinh(p*z[j])*cos(log(t)*z[j])
          for j in ran])
r(t)=-4*p*sqrt(t)*v(t)*cot(x/2)
plot(r(t),t,2,26)+plot(log(t),t,2,26, color='red')
```

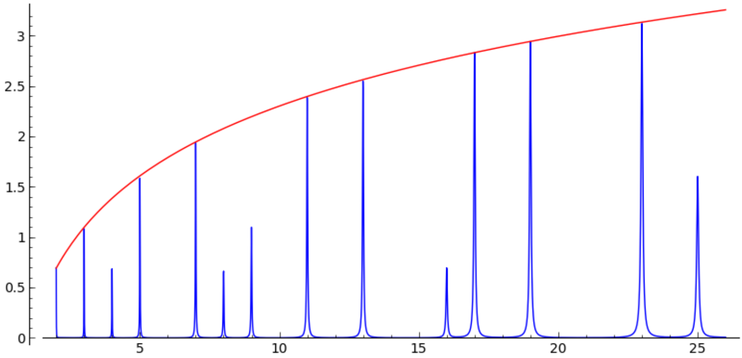
Mangoldt from G1

$$(x = 3.14,\ \gamma_k \text{ for } k = 1 \text{ to } k = 10000)$$



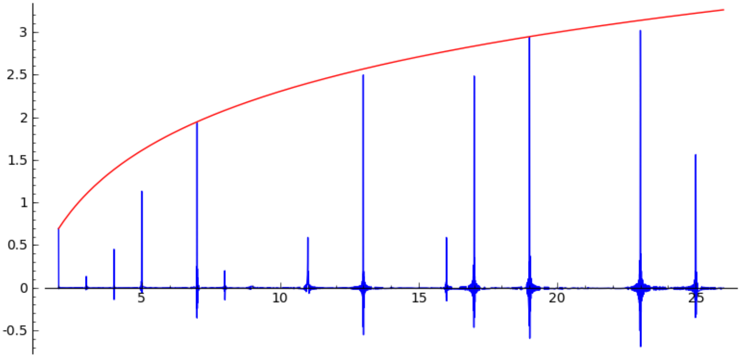
Mangoldt from G2

$(x = 3.14,\ \gamma_k \text{ for } k = 1 \text{ to } k = 10000)$



Mangoldt from Landau's formula

$$(T = 10000, \gamma_k \text{ for } k = 1 \text{ to } 10141)$$



Why the graphic with Landau's formula is so bad?

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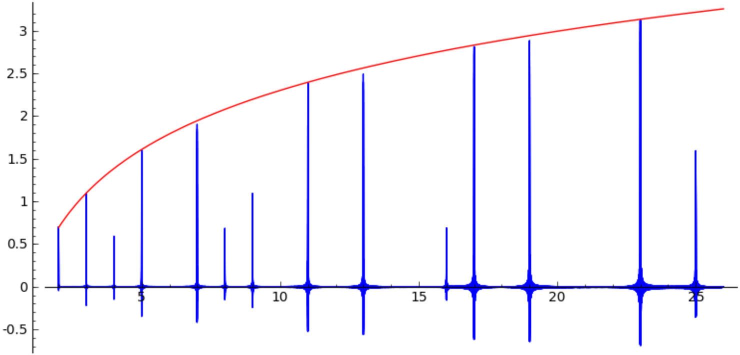


Corollary

The following formula holds

$$\lim_{x \to \pi^-} \sum_{\gamma > 0} \frac{\sinh x\gamma}{\sinh \pi\gamma} \left(\frac{\log 2}{\sqrt{2}} \cos(\gamma \log t) - \frac{\Lambda(t)}{\sqrt{t}} \cos(\gamma \log 2) \right)$$

$$= \frac{\log 2}{\sqrt{2}} \left(\frac{\sqrt{t}}{2} - \frac{1}{2(t^2-1)\sqrt{t}} \right) - \frac{\Lambda(t)}{\sqrt{t}} \frac{5\sqrt{2}}{12}. \tag{6}$$

Corollary

The following formula:

$$\lim_{z \to 1^-} \sum_{j=0}^{\infty} \cos(\gamma_{j+1} \log t)\, z^j = \frac{\sqrt{t}}{2} - \frac{1}{2(t^2-1)\sqrt{t}}, \tag{7}$$

holds whenever t is not a prime nor a power of prime. Otherwise the limit is infinite.

Theorem

The following identity

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{2\pi\sqrt{n}} \frac{ie^{iz}}{e^{iz}+n} = \sum_{\gamma} \frac{1}{\zeta'(\frac{1}{2}+i\gamma)} \frac{e^{-z\gamma}}{\sinh \pi\gamma} + \frac{i}{2\pi\zeta(\frac{1}{2})} - \sum_{n=1}^{\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!\zeta(2n+1)} ie^{-\frac{4n+1}{2}iz} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{\zeta(\frac{1}{2}-n)} ie^{-\frac{n}{2}iz}, \tag{8}$$

holds for $|\mathrm{Re}(z)| < \pi$ *and* $\mathrm{Im}(z) < 0$ *assuming the RH and that all the zeros of zeta are simple.*

Corollary

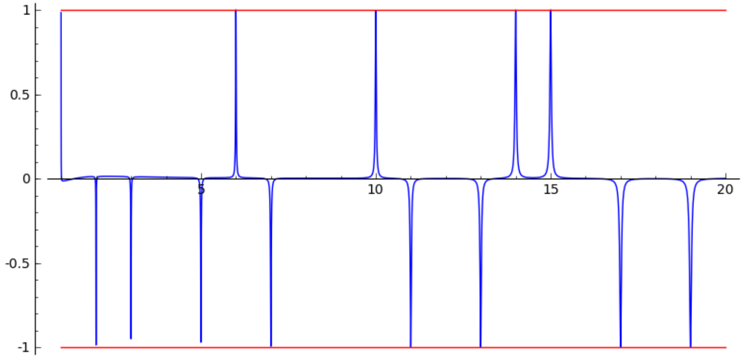
For the Moebius function $\mu(t)$, we get the following representation as $x \to \pi^-$:

$$\mu(t) = 4\pi\sqrt{t}\cot\frac{x}{2}\sum_{\gamma>0}\left[\mathrm{Re}\left(\frac{1}{\zeta'(\frac{1}{2}+i\gamma)}\right)\frac{\sinh x\gamma}{\sinh \pi\gamma}\cos(\gamma\log t) - \mathrm{Im}\left(\frac{1}{\zeta'(\frac{1}{2}+i\gamma)}\right)\frac{\cosh x\gamma}{\sinh \pi\gamma}\sin(\gamma\log t)\right] + 4\pi\cot\frac{x}{2}\sum_{n=1}^{\infty}\frac{(-1)^n(2\pi)^{2n}}{(2n)!\zeta(2n+1)}t^{-2n} + o\left(\cot\frac{x}{2}\right), \tag{9}$$

assuming the RH and that all the zeros of zeta are simple.

Observe that neglecting a finite number of zeros of zeta, we still recover the Moebius function.

$$(x = 3.14,\ \gamma_k \text{ for } k = 1 \text{ to } k = 10000)$$



Example

Taking $t = 1$ in formula (9), we get

$$\lim_{x \to \pi^-} \cot \frac{x}{2} \sum_{\gamma > 0} \frac{\sinh x\gamma}{\sinh \pi\gamma} \left(\frac{1}{\zeta'(\frac{1}{2} + i\gamma)} + \frac{1}{\zeta'(\frac{1}{2} - i\gamma)} \right) = \frac{1}{2\pi}.$$

Then, with the substitution $x = \pi + \log z$, we get

$$\lim_{z \to 1^-} (1 - z) \sum_{\gamma > 0} \left(\frac{1}{\zeta'(\frac{1}{2} + i\gamma)} + \frac{1}{\zeta'(\frac{1}{2} - i\gamma)} \right) z^{\gamma} = \frac{1}{\pi}.$$

Theorem

If we assume the Riemann Hypothesis and that all the zeros of zeta are simple, then the identity

(10)

$$i\frac{\zeta(\frac{1}{2})}{2\pi\zeta(\frac{3}{2})}e^{iz} - e^{iz}\sum_{n=1}^{\infty}\frac{\varphi(n)}{2\pi n^{3/2}}\frac{ie^{iz}}{e^{iz}+n} = \sum_{\gamma}\frac{\zeta(-\frac{1}{2}+i\gamma)}{\zeta'(\frac{1}{2}+i\gamma)}\frac{e^{-z\gamma}}{\sinh \pi\gamma}$$

$$+ i\frac{\zeta(-\frac{1}{2})}{2\pi\zeta(\frac{1}{2})} + i\frac{3}{\pi^2}e^{\frac{3}{2}iz} - \frac{1}{2\pi^2}\sum_{n=1}^{\infty}(2n+1)\frac{\zeta(2n+2)}{\zeta(2n+1)}ie^{-\frac{4n+1}{2}iz}$$

$$+ \frac{1}{4\pi^3}\sum_{n=1}^{\infty}(-1)^n\frac{\zeta(-\frac{1}{2}-n)}{\zeta(\frac{1}{2}-n)}ie^{-\frac{n}{2}iz},$$

holds for $|\mathrm{Re}(z)| < \pi$ *and* $\mathrm{Im}(z) < 0$.

Representation of the Euler-Phi function

Corollary

As $x \to \pi^-$, for the Euler-Phi function $\varphi(t)$ we get

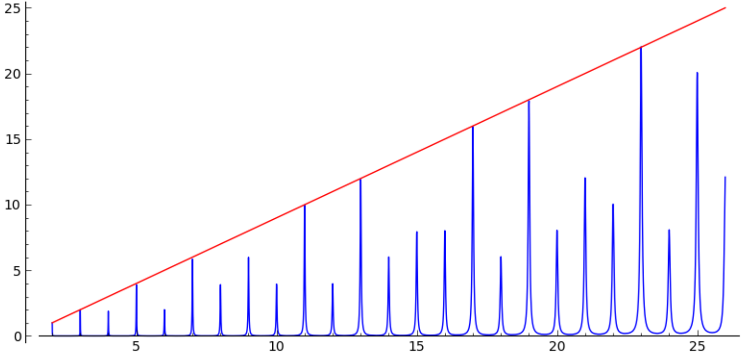
$$\varphi(t) = 4\pi\sqrt{t}\cot\frac{x}{2}\sum_{\gamma>0}\left[\operatorname{Re}\left(\frac{\zeta(-\frac{1}{2}+i\gamma)}{\zeta'(\frac{1}{2}+i\gamma)}\right)\frac{\sinh x\gamma}{\sinh \pi\gamma}\cos(\gamma\log t) - \operatorname{Im}\left(\frac{\zeta(-\frac{1}{2}+i\gamma)}{\zeta'(\frac{1}{2}+i\gamma)}\right)\frac{\cosh x\gamma}{\sinh \pi\gamma}\sin(\gamma\log t)\right] + \frac{12}{\pi}t^2\cot\frac{x}{2}$$

$$- \frac{2}{\pi}\cot\frac{x}{2}\sum_{n=1}^{\infty}(2n+1)\frac{\zeta(2n+2)}{\zeta(2n+1)}t^{-2n} + o\left(\cot\frac{x}{2}\right), \tag{11}$$

assuming the RH and that all the zeros of zeta are simple.

Observe that neglecting a finite number of zeros of zeta, we still recover the Euler-Phi function.

$(x = 3.14,\ \gamma_k \text{ for } k = 1 \text{ to } k = 10000)$



Proof of (1). Integral along $z = -1/2$

Let $I(z)$ be the analytic continuation of the integral $I(z)$ along the vertical axis $z = -1/2$, where

$$I(z) = \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{\zeta'(s+\frac{1}{2})}{\zeta(s+\frac{1}{2})} \frac{\pi}{\sin \pi s} z^s ds.$$

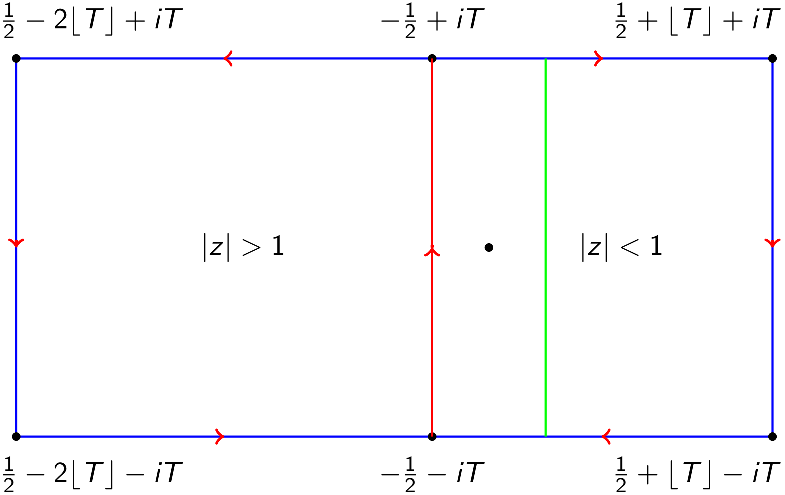
The poles of the integrand to the right of $z = -1/2$ are at

$$s = \frac{1}{2}, \quad s = n, \quad \rho - \frac{1}{2}, \qquad n \in \{0, 1, 2 \dots\}.$$

The poles of the integrand to the left of $z = -1/2$ are at

$$s = -2n - \frac{1}{2}, \quad s = -n, \qquad n \in \{1, 2, 3 \dots\}.$$

Proof of (1). Contour of integration



Proof of (1). Application of the residues theorem

For $|z| < 1$, we have

$$I = -\pi\sqrt{z} + \sum_{n=0}^{\infty} (-1)^n \frac{\zeta'(n+\frac{1}{2})}{\zeta(n+\frac{1}{2})} z^n + \pi \sum_{\rho} \frac{z^{\rho-\frac{1}{2}}}{\sin \pi(\rho - \frac{1}{2})}.$$

For $|z| > 1$, we have

$$I = \sum_{n=1}^{\infty} \frac{\zeta'(-2n)}{\zeta(-2n)} \sin(2\pi n) z^{-2n-\frac{1}{2}} + \sum_{n=1}^{\infty} (-1)^n \frac{\zeta'(\frac{1}{2}-n)}{\zeta(\frac{1}{2}-n)} z^{-n}.$$

To simplify it we use the functional equation:

$$\frac{\zeta'(1-s)}{\zeta(1-s)} = \log 2\pi - \psi(s) - \frac{\pi}{2} \tan \frac{\pi s}{2} - \frac{\zeta'(s)}{\zeta(s)}.$$

Proof of (1). The function $I(z)$ for $|z| < 1$ and $|z| > 1$

For $|z| < 1$, we have

$$I = -\pi\sqrt{z} + \sum_{n=0}^{\infty} (-1)^n \frac{\zeta'(n+\frac{1}{2})}{\zeta(n+\frac{1}{2})} z^n + \pi \sum_{\rho} \frac{z^{\rho-\frac{1}{2}}}{\sin \pi(\rho - \frac{1}{2})}.$$

For $|z| > 1$, we have

$$I = \frac{\pi}{\sqrt{z}(z^2-1)} + \frac{\log 2\pi}{z+1} + \frac{C + \log 4}{z+1} - \frac{\pi}{2z-2} + i \frac{\sqrt{z}}{z+1} \log \frac{\sqrt{z}+i}{\sqrt{z}-i} + \sum_{n=1}^{\infty} (-1)^n \frac{\zeta'(n+\frac{1}{2})}{\zeta(n+\frac{1}{2})} z^{-n}.$$

Analytic continuation to $\mathbb{C} - (-\infty, 0]$.

Proof of (1). Analytic continuation

The following identities are valid in $\mathbb{C} - (-\infty, 0]$:

$$I = -\pi\sqrt{z} + \frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} + \sum_{n=1}^{\infty} \frac{\Lambda(n) z}{\sqrt{n}(z+n)} + \pi \sum_{\rho} \frac{z^{\rho - \frac{1}{2}}}{\sin \pi(\rho - \frac{1}{2})}$$

and

$$I = \frac{\pi}{\sqrt{z}(z^2-1)} + \frac{C + \log 8\pi}{z+1} - \frac{\pi}{2z-2} + i \frac{\sqrt{z}}{z+1} \log \frac{\sqrt{z}+i}{\sqrt{z}-i} + \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}(1+zn)}.$$

Identifying both identities and replacing z with e^{iz} we arrive at (1).

Gracias

Thank you