Ramanujan series upside-down

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Ramanujan-type series for $1/\pi$

Let $(s)_n = s(s+1)\cdots(s+n-1)$. Ramanujan series are of the following form:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n}{(1)_n^3} z^n (a+bn) = \frac{1}{\pi},$$

where $s = 1/2,\ 1/4,\ 1/3$, or $1/6$ and z, a, b are algebraic numbers.

This kind of series were discovered by S. Ramanujan, who gave 17 examples in 1914. Two of them is

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (6n+1) = \frac{4}{\pi}, \quad \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (42n+5) = \frac{16}{\pi}.$$

The second one gives $\log 64 \simeq 1.8$ digits of π per term.

The form in which Ramanujan wrote his formulas

Nor the rising factorial, nor even the symbol of summation, were used by S. Ramanujan.

$$1+\left(\frac{1}{2}\right)^3\frac{6+1}{2^2}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^3\frac{12+1}{2^4}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3\frac{18+1}{2^6}+\cdots=\frac{4}{\pi},$$

and

$$5+\left(\frac{1}{2}\right)^3\frac{42+5}{2^6}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^3\frac{84+5}{2^{12}}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3\frac{126+5}{2^{18}}+\cdots=\frac{16}{\pi}.$$

The most impressive series discovered by Ramanujan are:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{882^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (21460n + 1123) = \frac{3528}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{1}{99^{4n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (26390n + 1103) = \frac{9801\sqrt{2}}{4\pi},$$

which give almost 6 and 8 digits per term respectively.

J. and P. Borwein were the first to prove the 17 Ramanujan series.

q-parametrization

The series

$$y_0 = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n}{(1)_n^3} z^n,$$

is the holomorphic solution of the differential equation $\mathcal{D}y = 0$,

$$\mathcal{D} = \theta^3 - z(\theta + \frac{1}{2})(\theta + s)(\theta + 1 - s), \qquad \theta = z\,\frac{d}{dz}.$$

As usual, we define $q = \exp(y_1/y_0)$. The mirror map $z(q)$ is a modular function, and

$$b = \frac{-\ln|q|}{\pi}\sqrt{1-z}, \quad a = \frac{1}{\pi y_0}\left(1 + \frac{\ln|q|}{y_0} q \frac{dy_0}{dq}\right),$$

are modular forms. Let $q = e^{2\pi i \tau}$. If τ is a quadratic irrational, then z, b, a are algebraic.

Dirichlet L series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}.$$

$$L_{-3}(s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \chi = (1,\ -1,\ 0),$$

$$L_{-4}(s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \chi = (1,\ 0,\ -1,\ 0),$$

$$L_{-7}(s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \chi = (1,\ 1,\ -1,\ 1,\ -1,\ -1,\ 0),$$

$$L_{5}(s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \chi = (1,\ -1,\ -1,\ 1,\ 0).$$

A new kind of series (1)

Case $s = \frac{1}{2}$:

(1) $$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(3n-1)}{n^3} \left(\frac{1}{2^2}\right)^n = \frac{\pi^2}{2},$$

(2) $$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(21n-8)}{n^3} \left(\frac{1}{2^6}\right)^n = \frac{\pi^2}{6},$$

(3) $$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(4n-1)}{n^3} (-1)^n = -16 L_{-4}(2),$$

(4) $$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(3n-1)}{n^3} \left(\frac{-1}{2^3}\right)^n = -2 L_{-4}(2),$$

Formulas (1) and (2) proved by D. Zeilberger (WZ-method).
Formulas (3) and (4) proved by G. (WZ-method).

A new kind of series (2)

Case $s = \frac{1}{3}$:

$$(5) \qquad \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n} \frac{(10n-3)}{n^3} \left(\frac{2}{27}\right)^{2n} = \frac{\pi^2}{2},$$

$$(6) \qquad \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n} \frac{(11n-3)}{n^3} \left(\frac{16}{27}\right)^{n} = 8\pi^2,$$

$$(7) \qquad \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n} \frac{(15n-4)}{n^3} \left(\frac{-1}{2^2}\right)^{n} = -27L_{-3}(2),$$

Conjectured by Z.W. Sun.

(6) proved by G. (WZ-method),
(7) proved by Hessami-Pilehroods (WZ-method).

Case $s = \frac{1}{4}$:

$$(8) \qquad \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n} \frac{(5n-1)}{n^3} \left(\frac{-9}{16}\right)^n = -\frac{45}{2} L_{-3}(2),$$

$$(9) \qquad \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n} \frac{(35n-8)}{n^3} \left(\frac{3}{4}\right)^{4n} = 12\pi^2.$$

Conjectured by Z.W. Sun.

Experiments...(1)

The series

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n} \frac{(15n-4)}{n^3} \left(\frac{-1}{2^2}\right)^n = -27L_{-3}(2),$$

is the upside-down of $\displaystyle\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3}(15n+4)(-4)^n,$

which is divergent but can be interpreted as

$$\frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \frac{\left(\frac{1}{2}\right)_s \left(\frac{1}{3}\right)_s \left(\frac{2}{3}\right)_s}{(1)_s^2}(15s+4)\Gamma(-s)2^{2s}\, ds$$

$$= 4 \, {}_4\mathrm{F}_3\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{19}{15} \\ 1, 1, \frac{4}{15} \end{matrix} \right| -4 \right) \stackrel{?}{=} \frac{3\sqrt{3}}{\pi}.$$

Experiments...(2)

We consider the Ramanujan series

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (42n+5) \left(\frac{1}{2^6}\right)^n = \frac{16}{\pi}$$

The upside-down is $\displaystyle\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(-42n+5)}{n^3} 64^n$, which is divergent,

but can be interpreted as $\displaystyle -1184 \, {}_5F_4\left(\begin{matrix} 1, 1, 1, 1, \frac{79}{42} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{37}{42} \end{matrix} \,\middle|\, 64 \right)$,

and experimentally, we guess that

$${}_5F_4\left(\begin{matrix} 1, 1, 1, 1, \frac{79}{42} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{37}{42} \end{matrix} \,\middle|\, 64 \right) \stackrel{?}{=} -\frac{2}{37} L_{-4}(2) - \frac{1}{296} \pi^2 i.$$

Upside-down

Theorem (Rogers-G., (2012))

Let $s \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$. *If*

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n}{(1)_n^3} (a+bn) z^n = \frac{1}{\pi},$$

where and z, b, a *are algebraic numbers, then*

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n} \frac{(a-bn)}{n^3} z^{-n},$$

reduces to two Epstein zeta values and some elementary constants in the following two cases:

$$1) \quad 2\,\mathrm{Re}(\tau) \in \mathbb{Z}, \qquad 2) \quad \frac{2\,\mathrm{Re}(\tau)}{|\tau|^2} \in \mathbb{Z}.$$

Epstein zeta function

We use the notation by Glasser and Zucker:

$$S(A, B, C; t) := \sum_{(n,m)\neq(0,0)} \frac{1}{(An^2 + Bnm + Cm^2)^t}.$$

We will take the values at $t = 2$ calculated by Glasser and Zucker.

$$S(1, 0, 8; 2) = \frac{7\pi^2}{48} L_{-8}(2) + \frac{\pi^2}{8\sqrt{2}} L_{-4}(2),$$
$$S(3, 4, 4; 2) = \frac{7\pi^2}{48} L_{-8}(2) - \frac{\pi^2}{8\sqrt{2}} L_{-4}(2).$$

The series

$$y=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n(s)_n(1-s)_n}{(1)_n^3}z^n, \quad y=\sum_{n=1}^{\infty}\frac{(1)_n^3}{\left(\frac{1}{2}\right)_n(s)_n(1-s)_n}\frac{z^{-n}}{n^3},$$

satisfy the differential equations $\mathcal{D}y=0$ and $\mathcal{D}y=1$, where

$$\mathcal{D}=\theta^3-z(\theta+\frac{1}{2})(\theta+s)(\theta+1-s), \qquad \theta=z\,\frac{d}{dz}.$$

The operator $a+b\theta$ introduces the factors $a+bn$ and $a-bn$.

Step 1

Lemma (Rogers-G.)

Let

$$H_x(z) = \frac{\cos \pi x (\cos^2 \pi x - \cos^2 \pi s)}{e^{i\pi x} \sin^2 \pi s} Y_x(z), \quad \textit{where}$$

$$Y_x(z) = \sum_{n \in \mathbb{Z}} \frac{(s)_{n+x} \left(\frac{1}{2}\right)_{n+x} (1-s)_{n+x}}{(1)_{n+x}^3} z^{n+x}. \quad \textit{Then}$$

1-) $H_{x+1}(z) = H_x(z)$.
2-) We can write $H_x(z)$ *in hypergeometric form (analytic).*
3-) $|H_x(z)| = \mathcal{O}\left(|\operatorname{Im}(x)|^{-3/2} e^{4\pi |\operatorname{Im}(x)|}\right)$ *for* $|\operatorname{Im}(x)|$ *suff. large.*
4-) $H_0(z) = y_0(z)$.
5-) $\mathcal{D} H_x(z) = 0$, *where* $\mathcal{D}$ *is the operator which annihilates* y_0.
6-) $H_x(z) = y_0(z)[-u + (u+1)\cos 2\pi x - iv \sin 2\pi x]$,
where u and v depend only on z.

Step 2

We have

$$Y_x(z) = y_x(z) + x^3 \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n} \frac{z^{-n}}{n^3} + O(x^4)$$

$$= \frac{(e^{i\pi x} \sin^2 \pi s) y_0(z)}{\cos \pi x (\cos^2 \pi x - \cos^2 \pi s)} [-u(z) + (u(z)+1) \cos 2\pi x - iv(z) \sin 2\pi x],$$

where

$$y_x(z) = \sum_{n=0}^{\infty} \frac{(s)_{n+x} \left(\frac{1}{2}\right)_{n+x} (1-s)_{n+x}}{(1)_{n+x}^3} z^{n+x}, \qquad \mathcal{D} y_x(z) = x^3 z^x.$$

We will introduce the factor $(a - bn)$ in the way

$$\left(a + bz \frac{d}{dz}\right) \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n} \frac{z^{-n}}{n^3}.$$

Step 3

We deduce that

$$\frac{1}{y_0(z)} \sum_{n=1}^{\infty} \frac{(1)_n^3}{(s)_n \left(\frac{1}{2}\right)_n (1-s)_n} \frac{z^{-n}}{n^3}$$
$$= i\pi^3 \csc^2(\pi s) - \frac{\pi^2}{3}\left(1 + 3\csc^2(\pi s)\right) \phi_1(q) - i\pi\phi_2(q) + \phi_3(q),$$

where

$$\left(q\frac{d}{dq}\right)^3 \phi_1(q) = 0, \quad \left(q\frac{d}{dq}\right)^3 \phi_2(q) = 0,$$
$$\left(q\frac{d}{dq}\right)^3 \phi_3(q) = \sqrt{1-z}\, y_0^2(z).$$

For $s = 1/2$ we know that $\sqrt{1-z}\, y_0^2(z) = \theta_3^8(q) - 2\theta_3^4(q)\theta_2^4(q)$.

Step 4

Lemma (Rogers-G.)

Suppose that $f(z)$ is a differentiable function, and let

$$\phi_f(q) = \frac{f(z)}{y_0(z)}.$$

Then

$$\left(a + bz\frac{d}{dz}\right) f(z) = \frac{1}{\pi}\left(\phi_f(q) - \ln|q|\, q\frac{d\phi_f(q)}{dq}\right).$$

We find that

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(a-bn)}{n^3} z^{-n} = -\frac{1}{15} F(q) + \frac{1}{60} F(q^4)$$
$$+ \frac{\log(q)^3}{6\pi} - \frac{\log(q)^2 \log|q|}{2\pi} + \frac{\log|q|^3}{3\pi}$$
$$- \frac{i}{2} \log(q)^2 + i \log(q) \log|q|$$
$$- \frac{5}{6}\pi \log(q) + \frac{5}{6}\pi \log|q| + \frac{i\pi^2}{2},$$

where

$$F(q) := -\frac{\log^3|q|}{3\pi} + \frac{120}{\pi}\zeta(3) + \frac{240}{\pi} \sum_{j=1}^{\infty} \mathrm{Li}_3(q^j) - \log|q^j| \, \mathrm{Li}_2(q^j).$$

Series for $s = 1/2,\ 1/3,\ 1/4$

We have

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(a-bn)}{n^3} z^{-n} = -\frac{1}{15} F(q) + \frac{1}{60} F(q^4) + (\cdot)\pi^2,$$

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{3}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n} \frac{(a-bn)}{n^3} z^{-n} = -\frac{1}{8} F(q) + \frac{1}{24} F(q^3) + (\cdot)\pi^2,$$

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n} \frac{(a-bn)}{n^3} z^{-n} = -\frac{1}{3} F(q) + \frac{1}{6} F\left(q^2\right) + (\cdot)\pi^2,$$

$$F(q) = \frac{120 y^3}{\pi^2} S(1, 2x, x^2 + y^2; 2) + \frac{60 i}{\pi^2} \sum_{\substack{n,k \\ n \neq 0}} \frac{(k+nx)\left((k+nx)^2 + 3n^2y^2\right)}{n^3\left((k+nx)^2 + n^2y^2\right)^2}, \qquad \tau = x + iy.$$

The function $F(q)$

Theorem

Let $\tau = x + iy$. *If* $2x \in \mathbb{Z}$ *and* $y > 0$, *then*

$$F(q) = \frac{120y^3}{\pi^2} S(1, 2x, x^2 + y^2; 2).$$

If $2x/(x^2 + y^2) \in \mathbb{Z}$ *and* $y > 0$, *then*

$$F(q) = \frac{120y^3}{\pi^2} S(1, 2x, x^2 + y^2; 2) + \frac{4i\pi^2}{3} x \left(\frac{x^2 + 3y^2}{(x^2 + y^2)^2} + x^2 + 3y^2 - 5 \right).$$

Deriving a rational formula

Set $q = -e^{-\pi\sqrt{15}/3}$. We have

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{3}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n} \frac{(15n-4)}{n^3} \frac{(-1)^{n+1}}{4^n} = 27\, L_{-3}(2)$$

$$= \frac{675\sqrt{5}}{8\pi^2} \left(S(1,1,4;2) - S(2,3,3;2) \right).$$

From tables by Glasser and Zucker, we see that

$$S(1,1,4;2) = \frac{\pi^2}{6} L_{-15}(2) + \frac{4\pi^2}{25\sqrt{5}} L_{-3}(2),$$

$$S(2,3,3;2) = \frac{\pi^2}{6} L_{-15}(2) - \frac{4\pi^2}{25\sqrt{5}} L_{-3}(2).$$

Irrational formulae

Set $q = e^{9\pi i/8} e^{-\pi\sqrt{15}/8}$. We have

$$\sum_{n=1}^{\infty} \frac{3(35+16\sqrt{5})n - 4(11+5\sqrt{5})}{n^3 \binom{2n}{n}^3} \left(\frac{\sqrt{5}-1}{2}\right)^{8n} = \frac{\pi^2}{30}.$$

Set $q = -e^{-2\pi\sqrt{3}/3}$. We get

$$\sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(6\sqrt{6}+30\sqrt{2})n - (3\sqrt{6}+7\sqrt{2})}{24n^3} \left(\frac{-1}{2(\sqrt{3}-1)^6}\right)^n$$

$$= 16\sqrt{2}L_{-8}(2) - 8\sqrt{6}L_{-24}(2).$$

More irrational formulae

Set $q=-e^{-\pi\sqrt{21}/3}$. We get

$$\sum_{n=1}^{\infty}\frac{(1)_n^3}{\left(\frac12\right)_n\left(\frac13\right)_n\left(\frac23\right)_n}\frac{(21+39\sqrt{7})n-(10+7\sqrt{7})}{54n^3}\times\left(\frac{-1}{26\sqrt{7}-68}\right)^n=-20L_{-4}(2)+\frac{35}{4}\sqrt{7}L_{-7}(2),$$

and

$$\sum_{n=1}^{\infty}\frac{(1)_n^3}{\left(\frac12\right)_n\left(\frac14\right)_n\left(\frac34\right)_n}\frac{(84+112\sqrt{3})n-(27+20\sqrt{3})}{72n^3}\times\left(\frac{-1}{(42-24\sqrt{3})^2}\right)^n=-\frac{160}{3}L_{-4}(2)+40\sqrt{3}L_{-3}(2).$$

Other values of the Epstein zeta function

Set $q = -e^{-\pi/3}$. We obtain

$$\frac{48}{\pi^2} S(1,0,36;2) = \frac{140}{27} L_{-4}(2) + \frac{13}{\sqrt{3}} L_{-3}(2) - \sum_{n=1}^{\infty} \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3} \frac{(a-bn)}{n^3} z^{-n},$$

where

$$z = -8\left(74977 + 40284r + 21644r^2 + 11629r^3\right),$$
$$a = \frac{1}{18}\left(1038 + 558r + 300r^2 + 161r^3\right),$$
$$b = \frac{1}{3}\left(387 + 208r + 112r^2 + 60r^3\right),$$

and $r = \sqrt[4]{12}$. It converges very rapidly because $z \simeq -2.4 \times 10^6$.

The end

Gracias
Thank you