Ramanujan-like series and String Theory

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Talk given in Ávila, Dublín, Zaragoza and Bilbao

Formulas proved by the WZ-method

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (820n^2+180n+13) = \frac{128}{\pi^2}, \qquad (2002).$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{1}{2^{4n}} (120n^2+34n+3) = \frac{32}{\pi^2}, \qquad (2002).$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{2n}} (20n^2+8n+1) = \frac{8}{\pi^2}, \qquad (2003).$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{4}\right)^{3n} (74n^2+27n+3) = \frac{48}{\pi^2}, \qquad (2010).$$

where $(s)_n = s(s+1)\cdots(s+n-1)$ is the Pochhammer symbol.

Formulas discovered by the PSLQ-algorithm

In 2003 I conjectured the following formulas:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (1640n^2+278n+15) = \frac{256\sqrt{3}}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{(-1)^n}{48^n} (252n^2+63n+5) = \frac{48}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{80^{3n}} (5418n^2+693n+29) = \frac{128\sqrt{5}}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{(1)_n^5} \frac{1}{7^{4n}} (1920n^2+304n+15) = \frac{56\sqrt{7}}{\pi^2},$$

which I discovered with the help of the PSLQ algorithm.

The PSLQ and WZ algorithms

The PSLQ algorithm is very good to discover formulas but it does not prove them. For example, looking for integer relations among the numbers t_0, t_1, t_2 and $1/\pi^2$, where

$$t_i = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{2n}} n^i, \quad \text{we get the vector} \quad (1, 8, 20, -8).$$

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With the WZ-method we can prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5 \left(\frac{1}{2}\right)_k^4}{(1)_n (1+k)_n^4 (1)_k^4} (20n^2 + 8n + 1 + 24kn + 8k^2 + 4k) = \frac{8}{\pi^2}$$

because it is an algorithm which can prove every identity of the form $\sum_{n=0}^{\infty} G(n,k) = \textit{constant}$, in case $G(n,k)$ is hypergeometric in its two variables.

- Although the WZ-proofs are beautiful and interesting they do not give us any insight of why there is is a family of similar formulas for the constant $1/\pi^2$.
- The theory we are going to explain now is a trial to solve this problem but it has the following important deficiencies:
  - We can only solve the equations by numerical approximations and not in an exact way.
  - We are unable to prove the main conjecture.
- For the moment the unique existing proofs, and only for some particular formulas, are by the WZ-method.

The theory we are going to explain is mainly based on the papers

- J. Guillera, A matrix form of Ramanujan-type series for $1/\pi$. *Contemp. Math.* 517 (2010), 189–206,
- G. Almkvist and J. Guillera, Ramanujan-like series for $1/\pi^2$ and String Theory,

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which are mainly inspired in:

- J. Guillera, A new method to obtain series for $1/\pi$ and $1/\pi^2$, *Exp. Math.* 15 (2006), 83-89,
- Y. Yang and W. Zudilin, On Sp_4 modularity of Picard–Fuchs differential equations for Calabi–Yau threefolds. *Contemp. Math.* 517 (2010), 381–413.

Ramanujan-like series for $1/\pi^2$

Let $s_0 = 1/2$, $s_3 = 1 - s_1$, $s_4 = 1 - s_2$ and

$$(s_1, s_2) = (1/2, 1/2), (1/2, 1/3), (1/2, 1/4), (1/2, 1/6), (1/3, 1/3), (1/3, 1/4), (1/3, 1/6), (1/4, 1/4), (1/4, 1/6), (1/6, 1/6), (1/5, 2/5), (1/8, 3/8), (1/10, 3/10), (1/12, 5/12).$$

We will call Ramanujan-like series for $1/\pi^2$ to the series which are of the form

$$\sum_{n=0}^{\infty} z^n \left[\prod_{i=0}^{4} \frac{(s_i)_n}{(1)_n} \right] (a + bn + cn^2) = \frac{1}{\pi^2}, \quad 0 < z < 1, \quad \text{or of the form}$$

$$\sum_{n=0}^{\infty} (-1)^n z^n \left[\prod_{i=0}^{4} \frac{(s_i)_n}{(1)_n} \right] (a + bn + cn^2) = \frac{1}{\pi^2}, \quad 0 < z \leq 1,$$

where z, a, b and c are algebraic numbers.

Expansions related to Ramanujan-like series for $1/\pi^2$

I propose a conjecture which motivate the study, that I am going to make, of the following expansions as $x \to 0$:

$$\sum_{n=0}^{\infty} z^{n+x} \left[\prod_{i=0}^{4} \frac{(s_i)_{n+x}}{(1)_{n+x}} \right] (a + b(n+x) + c(n+x)^2) \quad \text{or}$$

$$\sum_{n=0}^{\infty} (-1)^n z^{n+x} \left[\prod_{i=0}^{4} \frac{(s_i)_{n+x}}{(1)_{n+x}} \right] (a + b(n+x) + c(n+x)^2)$$

$$= \frac{1}{\pi^2} \left(1 - \frac{k}{2} \pi^2 x^2 + \frac{j}{24} \pi^4 x^4 \right) + O(x^5),$$

where now we use the generalized definition $(s)_x = \Gamma(s+x)/\Gamma(s)$.

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But k and j are not independent!.

Expansion in matrix form

Using the property $(s)_{n+x} = (s+x)_n (s)_x$, we have

$$\sum_{n=0}^{\infty} z^n \left[\prod_{i=0}^{4} \frac{(s_i + x)_n}{(1+x)_n} \right] \left(a + b(n+x) + c(n+x)^2\right) \quad \text{or}$$

$$\sum_{n=0}^{\infty} (-1)^n z^n \left[\prod_{i=0}^{4} \frac{(s_i + x)_n}{(1+x)_n} \right] \left(a + b(n+x) + c(n+x)^2\right)$$

$$= z^{-x} \left[\prod_{i=0}^{4} \frac{(1)_x}{(s_i)_x} \right] L_x + O(x^5), \qquad L_x = \frac{1}{\pi^2} - \frac{k}{2} x^2 + \frac{j}{24} \pi^2 x^4.$$

IDEA: We will replace the variable x with a fix nilpotent matrix X of order five.

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IDEA: We will replace the variable x with a fix nilpotent matrix X of order five. In this way we truncate the series in a natural way and we get rid of the derivatives with respect to x.

The matrices A, B, C and M

We define the matrices (series of positive terms)

$$A=\sum_{n=0}^{\infty} z^n \left[\prod_{i=0}^{4} \frac{(s_i+X)_n}{(1+X)_n}\right], \quad B=\sum_{n=0}^{\infty} z^n \left[\prod_{i=0}^{4} \frac{(s_i+X)_n}{(1+X)_n}\right](nI+X),$$

$$C=\sum_{n=0}^{\infty} z^n \left[\prod_{i=0}^{4} \frac{(s_i+X)_n}{(1+X)_n}\right](nI+X)^2, \quad M=z^{-X}\left[\prod_{i=0}^{4} \frac{(1)_X}{(s_i)_X}\right] L_X.$$

Then $aA+bB+cC=M$, $A=a_0 I+a_1 X+a_2 X^2+a_3 X^3+a_4 X^4$, etc.

All components a_0, a_1, a_2, etc are power series of z with rational coefficients. Let $\theta = z\, d/dz$. The easy relations $B = XA+\theta A$ and $C = XB+\theta B$, imply that (i=1,2,3,4)

$$b_0=\theta a_0, \quad c_0=\theta b_0, \quad b_i=a_{i-1}+\theta a_i, \quad c_i=b_{i-1}+\theta b_i.$$

The matrix M

It is the product of the matrices

$$L_X = \frac{1}{\pi^2}\left(I - \frac{k}{2}\pi^2 X^2 + \frac{j}{24}\pi^4 X^4\right),$$

$$z^{-X} = I - (\ln z)X + \frac{1}{2}(\ln^2 z)X^2 - \frac{1}{6}(\ln^3 z)X^3 + \frac{1}{24}(\ln^4 z)X^4,$$

$$\prod_{i=0}^{4}\frac{(1)_X}{(s_i)_X} = \rho_0^{-X}\left(1 - \frac{\rho_1}{2}\pi^2 X^2 + \rho_2\zeta(3)X^3 - \frac{\rho_1^2 - 4\rho_3}{8}\pi^4 X^4\right).$$

Here

$$\rho_0 = \frac{1}{4}\exp\{4\gamma + \psi(s_1) + \psi(s_2) + \psi(1-s_1) + \psi(1-s_2)\},$$

$$\rho_1 = \frac{5}{3} + \cot^2(\pi s_1) + \cot^2(\pi s_2), \qquad \rho_3 = \frac{1}{\sin^2(\pi s_1)\sin^2(\pi s_2)},$$

$$\rho_2 = \frac{2}{\zeta(3)}\left\{\zeta(3,1/2) + \zeta(3,s_1) + \zeta(3,s_2) + \zeta(3,1-s_1) + \zeta(3,1-s_2)\right\}.$$

Components of the matrix M

We get the following results:

$$m_0 = \frac{1}{\pi^2},$$
$$m_1 = \frac{1}{\pi^2}\left\{-\ln(\rho_0 z)\right\},$$
$$m_2 = \frac{1}{\pi^2}\left\{\frac{\ln^2(\rho_0 z)}{2} - \frac{\pi^2}{2}(k+\rho_1)\right\},$$
$$m_3 = \frac{1}{\pi^2}\left\{-\frac{\ln^3(\rho_0 z)}{6} + \frac{\pi^2}{2}(k+\rho_1)\ln(\rho_0 z) + \rho_2\zeta(3)\right\},$$

and

$$2m_0 m_4 - 2m_1 m_3 + m_2^2 = \frac{j}{12} + \frac{k^2}{4} + \rho_1 k + \rho_3.$$

Picard-Fuchs differential equations

The matrix $G = z^X A$ is a solution of the differential equation

$$\theta^5 G = z(\theta + 1/2)(\theta + s_1)(\theta + s_2)(\theta + 1 - s_1)(\theta + 1 - s_2)G.$$

We prove the case $(s_1, s_2) = (1/2, 1/2)$. Writing

$$A = \sum_{n=0}^{\infty} E_n z^n, \quad \text{where} \quad E_n = \frac{\left(\frac{1}{2}I + X\right)_n^5}{(I + X)_n^5},$$

we have $\quad E_{n+1}\Big[(n+1)I + X\Big]^5 = E_n \left[\left(n + \frac{1}{2}\right) I + X\right]^5.$

If we substitute $G = z^X A$ in the differential equation, we obtain

$$z^X \sum_{n=0}^{\infty} E_n \Big(nI + X\Big)^5 z^n - z^X \sum_{n=0}^{\infty} E_n \left[\left(n + \frac{1}{2}\right) I + X\right]^5 z^{n+1} = 0.$$

Fundamental solutions

The fundamental solutions of the differential equation

$$\theta^5 g = z(\theta + 1/2)(\theta + s_1)(\theta + s_2)(\theta + 1 - s_1)(\theta + 1 - s_2)g$$

are the components of the matrix $G = z^X A$.

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$$g_0 = a_0, \quad g_1 = a_0 \ln z + a_1, \quad g_2 = a_0 \frac{\ln^2 z}{2} + a_1 \ln z + a_2,$$

$$g_3 = a_0 \frac{\ln^3 z}{6} + a_1 \frac{\ln^2 z}{2} + a_2 \ln z + a_3,$$

$$g_4 = a_0 \frac{\ln^4 z}{24} + a_1 \frac{\ln^3 z}{6} + a_2 \frac{\ln^2 z}{2} + a_3 \ln z + a_4.$$

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Applying θ once and twice we obtain $\theta G = z^X B$ and $\theta^2 G = z^X C$.

Pullback

It is known that there exists functions y_0, y_1, y_2, y_3 satisfying a Calabi-Yau diff. equation $\theta^4 y = e_3(z)\theta^3 y + \cdots$, such that

$$g_0 = \begin{vmatrix} y_0 & y_1 \\ \theta y_0 & \theta y_1 \end{vmatrix}, \quad g_1 = \begin{vmatrix} y_0 & y_2 \\ \theta y_0 & \theta y_2 \end{vmatrix}, \quad g_3 = \frac{1}{2}\begin{vmatrix} y_1 & y_3 \\ \theta y_1 & \theta y_3 \end{vmatrix},$$

$$g_4 = \frac{1}{2}\begin{vmatrix} y_2 & y_3 \\ \theta y_2 & \theta y_3 \end{vmatrix}, \quad g_2 = \begin{vmatrix} y_0 & y_3 \\ \theta y_0 & \theta y_3 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ \theta y_1 & \theta y_2 \end{vmatrix}.$$

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The following relations hold:

$$\begin{aligned} 2g_0g_4 - 2g_1g_3 + g_2^2 &= 0, \\ 2(\theta g_0)(\theta g_4) - 2(\theta g_1)(\theta g_3) + (\theta g_2)^2 &= 0, \\ 2(\theta^2 g_0)(\theta^2 g_4) - 2(\theta^2 g_1)(\theta^2 g_3) + (\theta^2 g_2)^2 &= f^2, \end{aligned}$$

$$\text{where} \quad f = \exp\left(\int \frac{e_3(z)}{2z} dz\right).$$

Proof of the third identity

$$2(\theta^2 g_0)(\theta^2 g_4) - 2(\theta^2 g_1)(\theta^2 g_3) + (\theta^2 g_2)^2 = f^2,$$

$$f = \begin{vmatrix} y_0 & y_3 \\ \theta^3 y_0 & \theta^3 y_3 \end{vmatrix} - \begin{vmatrix} y_1 & y_2 \\ \theta^3 y_1 & \theta^3 y_2 \end{vmatrix} = \begin{vmatrix} \theta y_1 & \theta y_2 \\ \theta^2 y_1 & \theta^2 y_2 \end{vmatrix} - \begin{vmatrix} \theta y_0 & \theta y_3 \\ \theta^2 y_0 & \theta^2 y_3 \end{vmatrix}$$

Then we obtain

$$2\theta f = \begin{vmatrix} y_0 & y_3 \\ \theta^4 y_0 & \theta^4 y_3 \end{vmatrix} - \begin{vmatrix} y_1 & y_2 \\ \theta^4 y_1 & \theta^4 y_2 \end{vmatrix}.$$

But $\theta^4 y = e_3(z)\theta^3 y + e_2(z)\theta^2 y + e_1(z)\theta y + e_0(z)y$. Hence

$$2\theta f = e_3(z) f, \quad \text{and} \quad \ln f = \int \frac{e_3(z)}{2z} dz.$$

As $e_3(z) = z/(1-z)$, we obtain $f = 1/\sqrt{1-z}$ (hyperg. cases).

Relations among the components of A, B and C

The following non-trivial relations hold:

$$2a_0a_4 - 2a_1a_3 + a_2^2 = 0,$$

$$2b_0b_4 - 2b_1b_3 + b_2^2 = 0,$$

$$2c_0c_4 - 2c_1c_3 + c_2^2 = f^2.$$

From them we get some more important relations

$$a_0b_4 + a_4b_0 - a_1b_3 - a_3b_1 + a_2b_2 = 0,$$
$$a_0c_4 + a_4c_0 - a_1c_3 - a_3c_1 + a_2c_2 = 0,$$
$$b_0c_4 + b_4c_0 - b_1c_3 - b_3c_1 + b_2c_2 = 0.$$

Main relations among determinants

$$M_3 = \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = f \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix},$$

$$M_2 = \begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = f \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix},$$

$$M_1 = \begin{vmatrix} a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = f \begin{vmatrix} a_0 & b_0 \\ a_3 & b_3 \end{vmatrix},$$

$$M_0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = f \begin{vmatrix} a_0 & b_0 \\ a_4 & b_4 \end{vmatrix}.$$

The system of equations

We have to solve the equation $aA + bB + cC = M$, which is the (overdetermined) system

$$\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}.$$

As we want the system to be compatible we first impose that

$$\begin{vmatrix} a_0 & b_0 & c_0 & m_0 \\ a_1 & b_1 & c_1 & m_1 \\ a_2 & b_2 & c_2 & m_2 \\ a_3 & b_3 & c_3 & m_3 \end{vmatrix} = m_0 M_0 - m_1 M_1 + m_2 M_2 - m_3 M_3 = 0.$$

First equation in q

Dividing by M_3 we have $m_3 = m_0 H_0 - m_1 H_1 + m_2 H_2$, where

$$H_2 = \frac{a_0 b_2 - a_2 b_0}{a_0 b_1 - a_1 b_0}, \quad H_1 = \frac{a_0 b_3 - a_3 b_0}{a_0 b_1 - a_1 b_0}, \quad H_0 = \frac{a_0 b_4 - a_4 b_0}{a_0 b_1 - a_1 b_0}$$

and we can prove the relation $H_2^2 = 2H_1$. We define

$$q = \rho_0 \, z \, e^{H_2(z)}.$$

Inverting it we get z as a power series of q. We also have

$$\ln(\rho_0 z) = \ln q - H_2,$$

$$\frac{1}{6} \ln^3 q - \frac{\pi^2}{2}(k + \rho_1) \ln q - \rho_2 \zeta(3) - T(q) = 0, \quad T = \frac{1}{6} H_2^3 - H_0.$$

Second equation in q

From the system $a_i a + b_i b + c_i c = m_i$, $i = 0, 1, 2, 3, 4$, we get

$$\tau^2 := 2m_0 m_4 - 2m_1 m_3 + m_2^2 = (2c_0 c_4 - 2c_1 c_3 + c_2^2)c^2 = f^2 c^2,$$
$$= \frac{j}{12} + \frac{k^2}{4} + \rho_1 k + \rho_3.$$

Solving c by Cramer's rule from the three first equations, we obtain

$$c = m_0 \frac{a_1 b_2 - a_2 b_1}{M_3} - m_1 \frac{a_0 b_2 - a_2 b_0}{M_3} + m_2 \frac{a_0 b_1 - a_1 b_0}{M_3}.$$

Hence $\tau = m_0 J - m_1 H_2 + m_2, \quad J = (a_1 b_2 - a_2 b_1)/(a_0 b_1 - a_1 b_0)$,

$$\frac{1}{2} \ln^2 q - \frac{\pi^2}{2}(2\tau + k + \rho_1) - U(q) = 0, \quad U = H_1 - J.$$

Fortunately we can prove that $U(q) = \theta_q T(q)$.

The main equations and how to solve them.

Let $t = \ln q$. In case of series of positive terms the equations are

$$\frac{1}{6}t^3 - \frac{\pi^2}{2}(k+\rho_1)t - \rho_2\zeta(3) - T(e^t) = 0,$$

$$\frac{1}{2}t^2 - \frac{\pi^2}{2}(2\tau + k + \rho_1) - U(e^t) = 0,$$

and

$$\tau^2 = \frac{j}{12} + \frac{k^2}{4} + \rho_1 k + \rho_3, \quad c = \frac{\tau}{f}.$$

For alternating series we replace $z(q)$ with $-z(-q)$, $T(q)$ with $T(-q)$ and $U(q)$ with $U(-q)$. I wrote a program with two parts

1. It obtains the function $T(q)$ from the matrices A and B and the function $z(q)$.
2. It solves the equations with the Maple function fsolve.

Two new Ramanujan-like series

Let $(s_1, s_2) = (1/2, 1/3)$. For $k = 8/3$ we get $j = 112.0000000$, and we guess that $j = 112$.

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$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{15\sqrt{5}-33}{2}\right)^{3n} \Big[(1220/3 - 180\sqrt{5})n^2 + (303 - 135\sqrt{5})n + (56 - 25\sqrt{5}) \Big] = \frac{1}{\pi^2}.$$

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Let $(s_1, s_2) = (1/3, 1/6)$. For $k = 5/3$ we guess that $j = 85$ and we find

$$\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \left(\frac{3}{4}\right)^{6n} (1930n^2 + 549n + 45) = \frac{384}{\pi^2}.$$

I have discovered these two formulas in 2010.

Calabi-Yau differential equations

It is a 4^{th} order differential equation with rational coefficients

$$\theta^4 y = (e_3(z)\theta^3 + e_2(z)\theta^2 + e_1(z)\theta + e_0(z))y,$$

satisfying the following conditions:

1. It has a solution of the form

$$y_0 = \alpha_0, \quad y_1 = \alpha_0 \ln(z) + \alpha_1, \quad y_2 = \alpha_0 \frac{\ln^2(z)}{2} + \alpha_1 \ln(z) + \alpha_2,$$

$$y_3 = \alpha_0 \frac{\ln^3(z)}{6} + \alpha_1 \frac{\ln^2(z)}{2} + \alpha_2 \ln(z) + \alpha_3.$$

   where the functions α_0, α_1, α_2, α_3 are power series of z and $\alpha_0(0) = 1$, and $\alpha_1(0) = \alpha_2(0) = \alpha_3(0) = 0$.

2. The coefficients satisfy a relation which imply that

$$\begin{vmatrix} y_0 & y_3 \\ \theta y_0 & \theta y_3 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ \theta y_1 & \theta y_2 \end{vmatrix}.$$

The mirror map and the Yukawa coupling

Let $q = \exp(y_1/y_0)$. We can invert it to get $z = z(q)$ which is called the "mirror map". The Yukawa coupling is defined by

$$K(q) = \theta_q^2(\frac{y_2}{y_0}), \qquad \theta_q = q\frac{d}{dq}.$$

The following equivalence is known $K(q) = \theta_q^3\Phi$, where

$$\Phi = \frac{1}{2}(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0})$$

is known in String Theory as the Gromov-Witten potential. Wadim Zudilin suggested to me that the functions $z(q)$ and $T(q)$ that I was using were related to the mirror map and the Yukawa coupling of the Calabi-Yau pullback respectively.

Almkvist theorem

Gert Almkvist has proved that

$$H_2 = \frac{y_1}{y_0} - \ln \rho_0 z.$$

Comparing with

$$H_2 = \ln(q) - \ln \rho_0 z,$$

we see that

$$\ln q = \frac{y_1}{y_0}.$$

Hence the $z(q)$ which we have been using is precisely the mirror map. He also has proved that

$$\frac{1}{6} H_2^3 - H_0 = \frac{1}{6} \ln^3 q - \Phi(q).$$

Hence

$$T(q) = \frac{1}{6} \ln^3 q - \Phi(q).$$

One more Ramanujan-like series for $1/\pi^2$

Gert Almkvist modified the first part of my program to obtain $T(q)$ from the Calabi-Yau diff. equation. This new version of the program is so fast that we could try all values of k of the form $k = i/60$ for $i = 0, \cdots, 1200$ for the 14 hypergeometric cases. However we only found ($k = 8/3$, $j = 160$) the new formula

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \left(\frac{3}{5}\right)^{6n} (532n^2 + 126n + 9) = \frac{375}{\pi^2}.$$

It can be written in the nice form

$$\frac{1}{\pi^2} = 32 \sum_{n=0}^{\infty} \frac{(6n)!}{3n!^6} \frac{1}{10^{6n+3}} (532n^2 + 126n + 9),$$

where the summands contain no infinite decimal fractions.

Special values of Φ and $\theta_q\Phi$: A proven example.

One of the formulas I proved by the WZ-method is

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{1}{2^{4n}} (120n^2 + 34n + 3) = \frac{32}{\pi^2}.$$

and later I also proved by the WZ-method that the corresponding value of k is $k = 2$. From the establish formula $\tau^2 = c^2/(1-z)$ we get $\tau = \sqrt{15}$. We also know that $\rho_1 = 8/3$ and $\rho_2 = 24$. Substituting all these values in our formulas, and using Almkvist theorem, we arrive to the following result:

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Theorem: Let $q = q(k)$ and $q_0 = q(2)$. Then $z(q_0) = 1/16$ and

$$\Phi(q_0) = \frac{14}{3}\pi^2 \ln q_0 + 24\zeta(3), \qquad (\theta_q \Phi)(q_0) = \left(\frac{7}{3} + \sqrt{15}\right)\pi^2.$$

When the theory works?

Let

$$g_0 = \sum_{n=0}^{\infty} E_n z^n.$$

The theory works if

1. g_0 is the solution of a fifth order differential equation.
2. The fifth order differential equation has a pullback to a forth order Calabi-Yau differential equation.
3. E_x has an expansion of the form

$$E_x = \rho_0^x \left(1 + \frac{\rho_1}{2}\pi^2 x^2 - \rho_2 \zeta(3) x^3 + \frac{3\rho_1^2 - 4\rho_3}{8}\pi^4 x^4 + O(x^5)\right),$$

where ρ_0, ρ_1, ρ_2 and ρ_3 are rational numbers.

A non-hypergeometric example

Let

$$E_n = \frac{\left(\frac{1}{2}\right)_n^2 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^4} \sum_{i=0}^{n} \frac{\left(\frac{1}{2}\right)_i^2 (-n)_i^2}{(1)_i^2 \left(\frac{1}{2}-n\right)_i^2}, \qquad g_0 = \sum_{n=0}^{\infty} E_n z^n.$$

The pullback is a differential equation of Calabi-Yau type:

$$\theta^4 = \frac{2z}{1-z}\theta^3 + \cdots .$$

We get the relation $c = \tau(1-z)$. In addition E_x has the expansion required with $\rho_0 = 2^4 \cdot 3^3$, $\rho_1 = 2$, $\rho_2 = 7$, $\rho_3 = 13/12$. For $k = -2/3$ we get $j = 10$ and we obtain

$$\sum_{n=0}^{\infty} E_n \left(\frac{27}{32}\right)^n (25n^2 - 15n - 6) = \frac{192}{\pi^2}.$$

A higher Ramanujan-like series

At the end of 2002 B. Gourevitch discovered (PSLQ) the following series for $1/\pi^3$:

$$\frac{1}{32}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7}\frac{1}{2^{6n}}(168n^3+7n^2+14n+1)=\frac{1}{\pi^3}.$$

The corresponding extended series has the expansion

$$\frac{1}{\pi^3}\left(1-2\frac{\pi^2x^2}{2!}+32\frac{\pi^4x^4}{4!}-4112\frac{\pi^6x^6}{6!}\right)+O(x^7).$$

The highest known Ramanujan-like series

At the end of 2010 Jim Cullen discovered (PSLQ) the following series for $1/\pi^4$:

$$\frac{1}{2048}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^7\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(1)_n^9}\frac{1}{2^{12n}}\times$$
$$(43680n^4+20632n^3+4340n^2+466n+21)=\frac{1}{\pi^4}.$$

The corresponding extended series has the expansion

$$\frac{1}{\pi^4}\left(1-2^2\frac{\pi^2x^2}{2!}+2^3\cdot 11\frac{\pi^4x^4}{4!}-2^5\cdot 227\frac{\pi^6x^6}{6!}+\right.$$
$$\left.2^8\cdot 97\cdot 139\frac{\pi^8x^8}{8!}\right)+O(x^9).$$