A family of series for $1/\pi^2$ related to Calabi-Yau theory

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The Pochhammer symbol

The rising or sifting factorial (Pochhammer symbol) is defined by

$$(a)_x = \frac{\Gamma(a+x)}{\Gamma(a)}, \qquad (0)_0 = 1.$$

If x is a positive integer, it reduces to

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1),$$

For $a=1$, we have

$$(1)_n = n!.$$

Therefore, the rising factorial generalizes the ordinary factorial. In addition, for $j = 2, 3, 4, \dots,$ we have

$$\left(\frac{1}{j}\right)_n \cdots \left(\frac{j-1}{j}\right)_n = \frac{1}{j^{jn}} \frac{(jn)!}{n!},$$

In 1914 S. Ramanujan gave 17 fast hypergeometric series for $1/\pi$. Three examples are

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (42n+5) = \frac{16}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{1}{99^{4n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (26390n+1103) = \frac{9801\sqrt{2}}{4\pi},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} \frac{1}{2^{6n}} \left(\frac{\sqrt{5}-1}{2}\right)^{8n} \left[(42\sqrt{5}+30)n + (5\sqrt{5}-1)\right] = \frac{32}{\pi}.$$

J. and P. Borwein were the first to prove the 17 Ramanujan series by using elliptic integrals and modular equations.

They are of the form

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n}{(1)_n^3} z^n (a+bn) = \frac{1}{\pi},$$

where $s = 1/2$, $1/4$, $1/3$, or $1/6$ and z, a, b are algebraic numbers.

The ukranian brothers D. and G. Chudnovsky proved the formula

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{53360^{3n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \frac{545140134n + 13591409}{426880} = \frac{\sqrt{10005}}{\pi},$$

which is the fastest possible rational series due to

$$z(q) = \frac{12^3}{J(q)}, \quad J(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots,$$

$$z\left(e^{2\pi i \tau}\right) = \frac{-1}{53360^3}, \quad \text{for} \quad \tau = \frac{1+\sqrt{-163}}{2}.$$

Wadim Zudilin has used this series to prove that

$$\mu(\pi\sqrt{10005}) \leq 10.0213..., \quad \text{which beats} \quad \mu(\pi\sqrt{d}) \leq 10.8824...,$$

where μ denotes the measure of irrationality.

Modular parametrization and modular equations

To prove a Ramanujan series for $1/\pi$ we can follow these steps:

1. Step 1 Find some functions

$$z = z(q), \quad a = a(q), \quad b = b(q),$$

   related to elliptic modular functions.

2. Step 2 Evaluate the functions at $q = \pm e^{-\pi\sqrt{r}}$ for rational values of r, by means of a suitable modular equation.

For example, to prove the formulas corresponding to $q = e^{-\pi\sqrt{7}}$, we use equations that relate

$$f(q^7) \quad \text{and} \quad f(q),$$

in an algebraic way (modular equations of degree 7).

q-parametrization. Case $s = 1/2$

Let $q = e^{-\pi\sqrt{r}}$ (series of positive terms). One has:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} \left\{ \sqrt{r}(2\lambda(q)-1)n + [\alpha(q) - \sqrt{r}\lambda(q)] \right\} \Big[4\lambda(q)(1-\lambda(q)) \Big]^n = \frac{1}{\pi}.$$

For example, to obtain the series with $r = 7$ use the known identity

$$1 - \lambda(e^{-\pi\sqrt{r}}) = \lambda(e^{-\pi/\sqrt{r}}),$$

and substitute $q = e^{-\pi/\sqrt{r}}$ in the septic modular equation

$$\left\{ \lambda(q)\lambda(q^7) \right\}^{\frac{1}{8}} + \left\{ (1-\lambda(q))(1-\lambda(q^7)) \right\}^{\frac{1}{8}} = 1,$$

due to C. Guetzlaff in 1834 and rediscovered by Ramanujan.

Expansion in power series

Theorem

Let $0 < z < 1$ and $\sigma = 1$ (series of positive terms) or $\sigma = -1$ (alternating series). The expansion

$$\sum_{n=0}^{\infty} \sigma^n z^{n+x} \frac{(\frac{1}{2})_{n+x}(s)_{n+x}(1-s)_{n+x}}{(1)_{n+x}^3}[a+b(n+x)] = \frac{1}{\pi} + 0 \cdot x - \frac{k}{2}x^2 + \mathcal{O}(x^3),$$

determines the values of z, a, b as functions of k. In addition, if k is a rational number, then z, a, b are algebraic.

In a joint paper with Mathew Rogers we have also found the formula for the coefficient of x^3.

The WZ-method

Let $G(n,k)$ be hypergeometric in its two symbols. The proof of

$$\sum_{n=0}^{\infty} G(n,k) = g(k) = \text{Constant},$$

can be automatically (EKHAD) carried over by a computer.

H. Wilf and D. Zeilberger have discovered an algorithm that finds a rational function $C(n,k)$ called certificate, such that

$$F(n,k) = C(n,k)G(n,k), \qquad F(0,k) = 0,$$
$$G(n,k+1) - G(n,k) = F(n+1,k) - F(n,k) \qquad (\textit{WZ–pair}).$$

Observe that if we sum for $n \geq 0$ the right side telescopes. Hence $g(k) = g(k+1)$. If in addition $g(k) = \mathcal{O}(e^{c|\mathrm{Im}(k)|})$, with $c < 2\pi$, then by Carlson's theorem, we have $g(k) = \mathrm{Constant}$.

Chains of WZ pairs

Let $F(n,k)$ and $G(n,k)$ be the two hypergeometric functions of a WZ-pair, and suppose that in addition $F(0,k)=0$. If we define

$$F_{s,t}(n,k)=F(sn,k+tn), \qquad s\in\mathbb{Z}-\{0\}, \qquad t\in\mathbb{Z},$$

then $F_{s,t}(n,k)$ and $G_{s,t}(n,k)$ are also the functions of WZ-pairs satisfying $F_{s,t}(0,k)=0$ and in addition, we have

$$\sum_{n=0}^{\infty} G_{s,t}(n,k)=\sum_{n=0}^{\infty} G(n,k)=\text{Constant}.$$

So we have a chain of formulas with the same sum.

Let K be the operator $KG(n,k) = G(n,k+1)$. The output of

```
Zeilberger(G(n,k),k,n,K)[1];
Zeilberger(G(n,k),k,n,K)[2];
```

is of the form

$$P_0(k) + P_1(k)K + \cdots P_m(k)K^m, \quad F(n,k),$$

where the functions $P_j(k)$ are polynomials and the function $F(n,k)$ is the companion. This means that $G(n,k)$ satisfies the recurrence

$$(P_0(k)+P_1(k)K+\cdots+P_m(k)K^m)\,G(n,k) = F(n+1,k)-F(n,k).$$

If we sum for $n \geq 0$, and call $g(k)$ the sum of $G(n,k)$, we see that

$$P_0(k)g(k) + P_1(k)g(k+1) + \cdots + P_m(k)g(k+m) = -F(0,k).$$

Problem 1

Prove the Ramanujan formulas:

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (8n+1) = \frac{2\sqrt{3}}{\pi},$$

and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \frac{(-1)^n}{48^n} (28n+3) = \frac{16\sqrt{3}}{3\pi}.$$

How to discover a proof of Problem 1

Let

$$H(n,k) = \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^2 (1+k)_n},$$

and

$$g(k) = \sum_{n=0}^{\infty} H(n,k).$$

Then, writing (in a Maple session):

```
degree(Zeilberger(H(n,k),k,n,K)[1],K);
```

we see that the degree d of the operator is $d = 2 < 3$ (candidate).

Proof of the first formula of Problem 1

Then, if we write

```
coK2:=coeff(Zeilberger(H(n,k)*(n+b*k+c),k,n,K)[1],K,2);
coes:=coeffs(coK2,k); solve({coes},{b,c});
```

we get the solution $b = 1/4$, $c = 1/8$. Then, writing

```
Zeilberger(H(n,k)*(8*n+2*k+1),k,n,K)[1];
```

we get the output $3(1+2k)K - 4(1+k)$. Hence

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^2(1+k)_n}(8n+2k+1) = \frac{2\sqrt{3}}{\pi} \left(\frac{4}{3}\right)^k \frac{(1)_k}{\left(\frac{1}{2}\right)_k},$$

where we get the constant taking $k = 1/2$. Finally, take $k = 0$.

Proof of the second formula of Problem 1

The transformation $F(n,k) \to F(n,k+n)$, leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n}3^n} \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{4}+\frac{k}{2}\right)_n \left(\frac{3}{4}+\frac{k}{2}\right)_n \left(\frac{1}{2}+k\right)_n}{(1)_n^2 \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n}$$
$$\times \frac{(28n+3)(2n+1)+4k(9n+k+2)}{2n+k+1} = \frac{16\sqrt{3}}{3\pi} \cdot \left(\frac{4}{3}\right)^k \frac{(1)_k}{\left(\frac{1}{2}\right)_k}.$$

Finally, take $k=0$.

Problem 2

Use the WZ-method to prove the following formula:

$$\sum_{n=0}^{\infty} \left(\frac{-1}{16}\right)^n \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3}(51n+7) = \frac{12\sqrt{3}}{\pi},$$

due to Chan, Liaw and Tan, who proved it in 2003 using modular equations.

How to discover a WZ proof of Problem 2. Part 1

We consider the following expression:

$$H(n,k)=\left(\frac{-1}{16}\right)^n \frac{\left(\frac{1}{2}+j_1\,k\right)_n \left(\frac{1}{2}+j_2\,k\right)_n \left(\frac{1}{3}+j_3\,k\right)_n \left(\frac{2}{3}+j_3\,k\right)_n}{\left(1+j_4\,\frac{k}{2}\right)_n \left(\frac{1}{2}+j_4\,\frac{k}{2}\right)_n (1+j_5\,k)_n (1)_n},$$

For most of the values of j_1, j_2, j_3, j_4 and j_5, we see (Maple):

```
with(SumTools[Hypergeometric]);
degree(Zeilberger(H(n,k),k,n,K)[1],K);
```

is equal to 4, but for $j_1=0$, $j_2=2$, $j_3=j_4=j_5=1$, we see that

```
degree(Zeilberger(H(n,k),k,n,K)[1],K);
```

is equal to 3. Hence, this is a candidate.

How to discover a WZ proof of Problem 2. Part 2

Let

$$H(n,k) = \left(\frac{-1}{16}\right)^n \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}+2k\right)_n \left(\frac{1}{3}+k\right)_n \left(\frac{2}{3}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n \left(1+\frac{k}{2}\right)_n (1+k)_n (1)_n},$$

We conjecture a formula of the type

$$\sum_{n=0}^{\infty} H(n,k) \frac{(51n+7)(2n+1)+k(pn+qk+r)}{2n+k+1} = \frac{12\sqrt{3}}{\pi} f(k).$$

$$\text{Trying with:} \quad f(k) = \frac{(1)_k^2}{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}.$$

Then, letting for example $k=-1/3$, $k=-2/3$ and $k=-4/3$, we get a system of equations. Solving it, we get

$$p = 114, \quad q = 36, \quad r = 37.$$

A family of series for $1/\pi^2$

Let $s_0 = 1/2$, $s_3 = 1 - s_1$, $s_4 = 1 - s_2$ and

$$
\begin{aligned}
(s_1, s_2) = & (1/2, 1/2), (1/2, 1/3), (1/2, 1/4), (1/2, 1/6), (1/3, 1/3), \\
& (1/3, 1/4), (1/3, 1/6), (1/4, 1/4), (1/4, 1/6), (1/6, 1/6), \\
& (1/5, 2/5), (1/8, 3/8), (1/10, 3/10), (1/12, 5/12).
\end{aligned}
$$

We will call Ramanujan-like series for $1/\pi^2$ to the series which are of the form

$$
\sum_{n=0}^{\infty} z^n \left[\prod_{i=0}^{4} \frac{(s_i)_n}{(1)_n} \right] (a + bn + cn^2) = \frac{1}{\pi^2},
$$

where z, a, b and c are algebraic numbers.

The PSLQ algorithm

Let $(x_1, \dots x_n)$ be a vector of real numbers and write all the numbers the x_j with a precision of d decimal digits.

The PSLQ algorithm finds a vector $(a_1, \dots, a_n)$ of integers (with $a_j \neq 0$ for some j), such that:

$$a_1 x_1 + \cdots + a_n x_n = 0, \quad \text{(with a precision of d digits)},$$

and which has the smallest possible norm.

The PSLQ algorithm discovers identities but do not prove them.

Example: Let

$$f(j) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} \frac{(-1)^n}{2^{kn}} n^j,$$

and look for integer relations among $f(0)$, $f(1)$, $f(2)$ and $1/\pi^2$.

With PSLQ I discovered the formulas

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{2n}} (20n^2+8n+1) = \frac{8}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{1}{2^{4n}} (120n^2+34n+3) = \frac{32}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (820n^2+180n+13) = \frac{128}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{4}\right)^{3n} (74n^2+27n+3) = \frac{48}{\pi^2}.$$

I proved the three first formulas by the WZ-method in 2002 and 2003 and the last one in 2010.

Problem 1

In 2002 I solved the following problem: Prove the formulas

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5}(20n^2+8n+1) = \frac{8}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5}(120n^2+34n+3) = \frac{32}{\pi^2},$$

and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{10n}}(820n^2+180n+13) = \frac{128}{\pi^2}.$$

Proof of Problem 1

We have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n(1+k)_n^4}(20n^2+8n+1+24kn+8k^2+4k) = \frac{8}{\pi^2} \frac{(1)_k^4}{\left(\frac{1}{2}\right)_k^4},$$

and

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^3(1+k)_n^2}(120n^2+84kn+34n+10k+3) = \frac{32}{\pi^2} \frac{(1)_k^2}{\left(\frac{1}{2}\right)_k^2}.$$

Then taking $k=0$, we get the first and second formula.

If we make the transformation $F(n,k) \to F(n,k+n)$ in either of them, and then let $k=0$ we obtain the third one.

Problem 2

In 2010 I solved the following problem: Prove the identity

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{4}\right)^{3n} (74n^2 + 27n + 3) = \frac{48}{\pi^2}.$$

Proof of Problem 2

We have

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}+\frac{k}{3}\right)_n \left(\frac{2}{3}+\frac{k}{3}\right)_n \left(1+\frac{k}{3}\right)_n}{(1)_n^3 (1+k)_n^3} \left(\frac{3}{4}\right)^{3n}$$

$$\times \frac{(74n^2+27n+3)n + k(108n^2+42kn+24n+5k+1)}{n+\frac{k}{3}}$$

$$= \frac{48}{\pi^2} \frac{(1)_k^2}{\left(\frac{1}{2}\right)_k^2}, \quad \text{(we get the constant taking the limit as } k \to \infty).$$

Then, taking $k=0$.

Conjectured formulas

By the PSLQ algorithm we discovered the formulas

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{(-1)^n}{48^n} (252n^2 + 63n + 5) = \frac{48}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{(1)_n^5} \frac{1}{7^{4n}} (1920n^2 + 304n + 15) = \frac{56\sqrt{7}}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (1640n^2 + 278n + 15) = \frac{256\sqrt{3}}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{80^{3n}} (5418n^2 + 693n + 29) = \frac{128\sqrt{5}}{\pi^2}.$$

They remain unproved.

Expansion in power series

Conjecture

Let $0 < z < 1$ *and* $\sigma = 1$ *(series of positive terms) or* $\sigma = -1$ *(alternating series). The expansion*

$$\sum_{n=0}^{\infty} \sigma^n z^{n+x} \left[\prod_{i=0}^{4} \frac{(s_i)_{n+x}}{(1)_{n+x}} \right] \left[a + b(n+x) + c(n+x)^2 \right]$$
$$= \frac{1}{\pi^2} + 0 \cdot x - \frac{k}{2} x^2 + 0 \cdot x^3 + \frac{j}{24} \pi^2 x^4 + \mathcal{O}(x^5),$$

determines the values of j, z, a, b, c *as functions of* k. *In addition, if* k *is a rational number such that* j *is rational too, then* z, a, b, c *are algebraic.*

More conjectured formulas

In 2010 we discovered three more series

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} (-1)^n \left(\frac{3}{4}\right)^{6n} (1936n^2 + 549n + 45) = \frac{384}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \left(\frac{3}{5}\right)^{6n} (532n^2 + 126n + 9) = \frac{375}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{\phi}\right)^{3n} \left[\left(32 - \frac{216}{\phi}\right) n^2 + \left(18 - \frac{162}{\phi}\right) n + \left(3 - \frac{30}{\phi}\right) \right] = \frac{3}{\pi^2},$$

where ϕ is the fifth power of the golden ratio. This formula is the unique irrational example that I have found for $1/\pi^2$.

The second formula is joint with G. Almkvist.

Calabi-Yau onefolds are elliptic curves. The periods

$$f(z) = \int_1^\infty \frac{dz}{y},$$

of the family $y^2 = x(x-1)(x-z)$ satisfy

$$(1-z)\theta^2 f - z\,\theta f - \frac{z}{4} f = 0, \quad \theta = z\frac{d}{dz}.$$

C-Y twofolds are either 2-dimensional complex tori or $K3$ surfaces.

An example of Calabi-Yau threfolds is the quintic

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - z^{-1} x_1 x_2 x_3 x_4 x_5 = 0.$$

The periods satisfy the folowing differential equation

$$\left\{ \theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4) \right\} y = 0.$$

Fifth order differential equation

The function

$$a_0 = \sum_{n=0}^{\infty} z^n \left[\prod_{i=0}^{4} \frac{(s_i)_n}{(1)_n} \right],$$

is a solution of the differential equation

$$\theta^5 g - z(\theta + 1/2)(\theta + s_1)(\theta + s_2)(\theta + 1 - s_1)(\theta + 1 - s_2) g = 0.$$

The five fundamental solutions are

$$g_0 = a_0, \quad g_1 = a_0 \ln z + a_1, \quad g_2 = a_0 \frac{\ln^2 z}{2} + a_1 \ln z + a_2,$$

$$g_3 = a_0 \frac{\ln^3 z}{6} + a_1 \frac{\ln^2 z}{2} + a_2 \ln z + a_3,$$

$$g_4 = a_0 \frac{\ln^4 z}{24} + a_1 \frac{\ln^3 z}{6} + a_2 \frac{\ln^2 z}{2} + a_3 \ln z + a_4.$$

The pullback

The five solutions g_0, g_1, g_2, g_3, g_4 can be recovered from the four fundamental solutions y_0, y_1, y_2, y_3 of a Calabi-Yau diff. eq.

$$\theta^4 y = e_3(z)\theta^3 y + e_2(z)\theta^2 y + e_1(z)\theta y + e_0(z) \quad \text{by means of :}$$

$$g_0 = \begin{vmatrix} y_0 & y_1 \\ \theta y_0 & \theta y_1 \end{vmatrix}, \quad g_1 = \begin{vmatrix} y_0 & y_2 \\ \theta y_0 & \theta y_2 \end{vmatrix}, \quad g_3 = \frac{1}{2}\begin{vmatrix} y_1 & y_3 \\ \theta y_1 & \theta y_3 \end{vmatrix},$$

$$g_4 = \frac{1}{2}\begin{vmatrix} y_2 & y_3 \\ \theta y_2 & \theta y_3 \end{vmatrix}, \quad g_2 = \begin{vmatrix} y_0 & y_3 \\ \theta y_0 & \theta y_3 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ \theta y_1 & \theta y_2 \end{vmatrix}.$$

The following relations hold:

$$2g_0g_4 - 2g_1g_3 + g_2^2 = 0,$$

$$2(\theta g_0)(\theta g_4) - 2(\theta g_1)(\theta g_3) + (\theta g_2)^2 = 0,$$

$$2(\theta^2 g_0)(\theta^2 g_4) - 2(\theta^2 g_1)(\theta^2 g_3) + (\theta^2 g_2)^2 = \exp^2\left(\int \frac{e_3(z)}{2z} dz\right).$$

The mirror map and the Yukawa coupling

Let $q = \exp(y_1/y_0)$. We can invert it to get

$$z = z(q),$$

which we call the "mirror map". The Yukawa coupling is defined by

$$K(q) := \theta_q^2\left(\frac{y_2}{y_0}\right), \qquad \theta_q = q\frac{d}{dq}.$$

The following equivalence is known: $K(q) = \theta_q^3 \Phi$, where

$$\Phi(q) := \frac{1}{2}\left(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\right) = \frac{1}{6}\ln^3 q - T(q)$$

is known in String Theory as the Gromov-Witten potential.

Other important related functions

Definition. $\tau(q)=c(q)\int \frac{e_3(z)}{2z}dz=c(q)\sqrt{1-z(q)}$ (hyperg. cases).

Theorem. For series of positive terms, we have

$$k(q)=\frac{2}{\pi^2}\frac{\Phi(q)-2h}{\ln q}-\left(\frac{5}{3}+\cot^2(\pi s_1)+\cot^2(\pi s_2)\right)$$

$$\tau(q)=\frac{1}{2}(q\ln q)\frac{dk}{dq}, \quad \text{where} \quad h=\sum_{i=0}^{4}\zeta(3,s_i).$$

In addition, we have

$$\tau^2=\frac{j}{12}+\frac{k^2}{4}+\frac{5k}{3}+1+(\cot^2\pi s_1)(\cot^2\pi s_2)+(1+k)(\cot^2\pi s_1+\cot^2\pi s_2).$$

Conjecture: z, a, b, c are algebraic $\iff$ k and τ^2 are rational.

Case $s_1 = 1/2$, $s_2 = 1/2$

The 5^{th} order diff. op. is $\theta^5 - 32z(\theta+1)^5$, and its pullback is

$$\theta^4 - 16z(128\theta^4 + 256\theta^3 + 304\theta^2 + 176\theta + 39) + 2^{20}z^2(\theta+1)^4.$$

We get

$$J(q) := \frac{2^{10}}{z(q)} = \frac{1}{q} + 320 + 68000q + 12646400q^2 + 2251836880q^3 + 396687233024q^4 + 69974001492480q^5 + \cdots$$

and the Yukawa coupling

$$K(q) := 1 - 160q - 55520q^2 - 14571520q^3 - 3492443360q^4 - 800369820160q^5 - 178601623193600q^6 - \cdots .$$

Thank you