Universidad de Zaragoza

Ramanujan series. Generalizations and conjectures.

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Contents

We will discuss the following results:

1. Generalizations of some Ramanujan series for $1/\pi$ which can be automatically proved.
2. Hypergeometric identities which lead us to conjectures related to Ramanujan series.
3. A conjecture for the Ramanujan-Sato series.
4. A new kind of similar series for $1/\pi^2$.
5. Generalizations, hypergeometric identities and conjectures for the new kind of series for $1/\pi^2$.

Publications

- *Some binomial series obtained by the WZ-method.*
  Adv. in Appl. Math. ***29*** *(2002) 599-603.*
- *About a new kind of Ramanujan-type series.*
  Exp. Math. ***12*** *pp. 507-510, (2003).*
- *Generators of some Ramanujan's formulas.*
  The Ramanujan J. (2006) ***11*** *pp. 41-48.*
- *A new method to obtain series for* $1/\pi$ *and* $1/\pi^2$*.*
  Exp. Math. ***15*** *pp. 83-89, (2006).*
- *A class of conjectured series representations for* $1/\pi$*.*
  Exp. Math. ***15*** *pp. 409-414, (2006).*
- *Hypergeometric identities for 10 extended Ramanujan-type formulas.*
  The Ramanujan J. (to appear)

Rising factorials

The rising factorial or Pochhammer symbol is defined by

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \qquad (a)_x = \frac{\Gamma(a+x)}{\Gamma(a)}.$$

It generalizes the concept of factorial: $n! = (1)_n$.
We will need the following properties:

$$(0)_0 = 1, \qquad (0)_n = 0, \quad n = 1, 2, 3 \ldots$$

If we define $C_j(n) = \left(\frac{1}{j}\right)_n \left(\frac{2}{j}\right)_n \cdots \left(\frac{j-1}{j}\right)_n$, then

$$C_j(n) = \frac{1}{j^{jn}} \frac{(jn)!}{n!}, \quad j = 2, 3, \ldots$$

Ramanujan type series for $1/\pi$

They are of the form:

$$\sum_{n=0}^{\infty} z^n \frac{C(n)}{(1)_n^3}(an+b) = \frac{1}{\pi}, \qquad -1 \le z < 1,$$

where z, a and b are algebraic numbers and $C(n)$ is the product of 3 Pochhammer symbols obtained joining blocks:

$$(1/2)_n, \qquad (1/3)_n(2/3)_n, \qquad (1/4)_n(3/4)_n, \qquad (1/6)_n(5/6)_n.$$

Conversion to factorials:

$$(1/6)_n(5/6)_n = \frac{C_6(n)}{C_2(n)C_3(n)} = \frac{1}{2^{4n} \cdot 3^{3n}} \frac{(6n)!n!}{(2n)!(3n)!}.$$

An example and different kind of proofs

An example is

$$\sum_{n=0}^{\infty} \frac{1}{3^{4n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (10n+1) = \frac{9\sqrt{2}}{4\pi}.$$

It gives approximately $\log 81 \simeq 1.9$ digits of π per term.

1. All of them can be proved by finding some functions

$$z = z(q), \quad a = a(q), \quad b = b(q),$$

   which are related to elliptic modular functions and evaluating them at $q = e^{-\pi\sqrt{N}}$ for rational values of N.

2. We have proved 8 of them with the WZ-method; a method developed by Wilf and Zeilberger (Steele Prize).

The WZ-method

We say that $A(n,k)$ is hypergeometric or closed form if

$$\frac{A(n+1,k)}{A(n,k)} \quad \text{and} \quad \frac{A(n,k+1)}{A(n,k)}$$

are both rational functions.
We say that $F(n,k)$ and $G(n,k)$ is a WZ pair if F and G are closed forms which satisfy

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

If in addition we have $F(0,k) = 0$, then

$$\sum_{n=0}^{\infty} G(n,k) = \sum_{n=0}^{\infty} G(n,k+1).$$

Carlson's Theorem

If $f(z)$ is an entire function, $f(z) = 0$ for $z = 0, 1, 2, \cdots$ and $f(z) = O(e^{c|z|})$ for $c < \pi$ and $\Re(z) \geq 0$, then $f(z) = 0$.

For the functions $G(n,k)$ that we will consider, the function

$$f(z) = \sum_{n=0}^{\infty} G(n,z) - \sum_{n=0}^{\infty} G(n,0),$$

satisfies the hypothesis of Carlson's theorem, and so

$$\sum_{n=0}^{\infty} G(n,k) = CONST.$$

Is there a method to determine F from G and G from F?

EKHAD

The answer is YES, H. Wilf and D. Zeilberger have discovered an algorithm that finds a rational function $C(n,k)$, called certificate, such that $F(n,k) = C(n,k)G(n,k)$.

In addition Zeilberger has written the Maple package EKHAD which implements the algorithm.

So, the proofs of the identities of the form

$$\sum_{n=0}^{\infty} G(n,k) = CONST.,$$

with $G(n,k)$ being a closed form, can be automatically carried over by a computer and are mathematically rigorous.

WZ method and 8 Ramanujan's series

$$\sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}(4n+1)=\frac{2}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{1}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}(6n+1)=\frac{4}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{3n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}(6n+1)=\frac{2\sqrt{2}}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}(42n+5)=\frac{16}{\pi},$$

WZ method and 8 ramanujan's series. Cont.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (20n+3) = \frac{8}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (8n+1) = \frac{2\sqrt{3}}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{48^{n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (28n+3) = \frac{16\sqrt{3}}{3\pi},$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} (154n+15) = \frac{32\sqrt{2}}{\pi}.$$

Can WZ prove all Ramanujan type series?

The most impressive Ramanujan type series with rational z are:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{882^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (21460n + 1123) = \frac{3528}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{1}{99^{4n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (26390n + 1103) = \frac{9801\sqrt{2}}{4\pi},$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{53360^{3n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \frac{545140134n + 13591409}{426880} = \frac{\sqrt{10005}}{\pi}.$$

Can they be proved by the WZ method?

Series for $1/\pi$. WZ1+Carlson's Th.

$$\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}(4n+1)=\frac{2}{\pi},\qquad \text{Ramanujan.}$$

Generalized series (proved by Zeilberger):

$$\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)_n^2\left(\frac{1}{2}-k\right)_n}{(1+k)_n(1)_n^2}(4n+1)\frac{\binom{2k}{k}}{2^{2k}}=\frac{2}{\pi}.$$

$$S(n,k)=\frac{n^2}{2n-2k-1}.$$

The value $2/\pi$ has been obtained for $k=1/2$, by observing that $n\neq 0 \Rightarrow (0)_n=0$ and $\binom{1}{1/2}=\frac{4}{\pi}$.

Program 1

We have written a program to look for functions $G(n,k)$ characterized by:

1. They have 3 rising factorials in the numerator and in the denominator.
2. The rational part is a polynomial of first degree in the symbols n and k.
3. One of the rising factorials in the numerator produces $(0)_n$ at $k = 1/2$.
4. The exclusive function of k produces the constant when it is evaluated at $k = 1/2$.

The program includes a function of EKHAD to certify if $G(n,k)$ is the second component of a WZ pair.

Series for $1/\pi$. WZ2+Carlson's Th.

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (6n+1) = \frac{4}{\pi}, \qquad \text{Ramanujan.}$$

Generalized series:

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n}{(1)_n^2 (1+k)_n} (6n+2k+1) \frac{\binom{2k}{k}}{2^{2k}} = \frac{4}{\pi}.$$

$$S(n,k) = \frac{16n^2}{2n-2k-1}.$$

The value $4/\pi$ has been obtained for $k=1/2$, by observing that $n \neq 0 \Rightarrow (0)_n = 0$ and $\binom{1}{1/2} = \frac{4}{\pi}$.

Series for $1/\pi$. WZ3+Carlson's Th.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}(6n+1) = \frac{2\sqrt{2}}{\pi}, \qquad \text{Ramanujan.}$$

Generalized series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}+k\right)_n}{(1)_n^2 (1+k)_n} (6n+2k+1) \frac{\binom{2k}{k}}{2^{3k}} = \frac{2\sqrt{2}}{\pi}.$$

$$S(n,k) = \frac{16n^2}{2n-2k-1}.$$

The value $2\sqrt{2}/\pi$ has been obtained by taking $k=1/2$.

Series for $1/\pi$. WZ4+Carlson's Th.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (20n+3) = \frac{8}{\pi}, \qquad \text{Ramanujan.}$$

Generalized series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^2 (1+k)_n} (20n+2k+3) \frac{\binom{2k}{k}}{2^{2k}} = \frac{8}{\pi}.$$

$$S(n,k) = \frac{64n^2}{4n-2k-1}.$$

The value $8/\pi$ has been obtained by taking $k=1/2$.

Series for $1/\pi$. WZ5+Carlson's Th.

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (8n+1) = \frac{2\sqrt{3}}{\pi}, \qquad \text{Ramanujan.}$$

Generalized series:

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{(1)_n^2 (1+k)_n} (8n+2k+1) \frac{3^k \binom{2k}{k}}{2^{4k}} = \frac{2\sqrt{3}}{\pi}.$$

$$S(n,k) = \frac{128n^2}{2n-2k-1}.$$

The value $2\sqrt{3}/\pi$ has been obtained by taking $k=1/2$.

Chains of WZ pairs

Let $F(n,k)$ and $G(n,k)$ be the two components of a WZ pair such that $F(0,k)=0$. If we define

$$F_{s,t}(n,k) = F(sn, k+tn), \qquad s \in \mathbb{Z} - \{0\}, \qquad t \in \mathbb{Z},$$

then $F_{s,t}(n,k)$ and $G_{s,t}(n,k)$ are also the components of WZ pairs such that $F_{s,t}(0,k)=0$ and

$$\sum_{n=0}^{\infty} G_{s,t}(n,k) = \sum_{n=0}^{\infty} G(n,k) = CONST.$$

The proofs of the Ramanujan type series that we are going to see now are based on these kind of transformations.

Series for $1/\pi$. WZ6+Carlson's Th.

We make the transformation $F(n,k) = F_4(n, k+n)$.

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}+k\right)_n}{(1)_n^2 \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n}$$

$$\times \frac{(2n+2k+1)(42n+2k+5) - 30kn}{2n+k+1} \frac{\binom{2k}{k}}{2^{2k}} = \frac{16}{\pi}.$$

The value $16/\pi$ has been obtained by taking $k = 1/2$.
Setting $k = 0$, we get

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (42n+5) = \frac{16}{\pi}, \quad \text{Ramanujan.}$$

Series for $1/\pi$. WZ7+Carlson's Th.

We make the transformation $F(n,k) = F_5(n, k+n)$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n}3^n} \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{4}+\frac{k}{2}\right)_n \left(\frac{3}{4}+\frac{k}{2}\right)_n \left(\frac{1}{2}+k\right)_n}{(1)_n^2 \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n}$$

$$\times \frac{(2n+2k+1)(28n+2k+3)-24kn}{2n+k+1} \frac{3^k \binom{2k}{k}}{2^{4k}} = \frac{16\sqrt{3}}{3\pi}.$$

The value $(16/3)\sqrt{3}/\pi$ has been obtained by taking $k = 1/2$. Setting $k = 0$, we get

$$\sum_{n=0}^{\infty} \frac{(-1)}{2^{4n}3^n} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} (28n+3) = \frac{16\sqrt{3}}{3\pi}, \qquad \text{Ramanujan}$$

Series for $1/\pi$. WZ8+Carlson's Th.

We finally prove the following Ramanujan type series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} (154n+15) = \frac{32\sqrt{2}}{\pi}.$$

Proof: We make the transformation $F(n,k) = F_3(2n, k-n)$ and use package EKHAD to obtain $G(n,k)$. Then we have

$$\sum_{n=0}^{\infty} G(n,k) = CONST.$$

To obtain the value of the constant we evaluate at $k = 1/2$. Finally we take $k = 0$.

Modifying WZ pairs

If $(F(n,k),\ G(n,k))$ is a WZ pair, then Zeilberger has proved the identity

$$\sum_{n=0}^{\infty} G(n,0) = \lim_{k\to\infty} \sum_{n=0}^{\infty} G(n,k) + \sum_{k=0}^{\infty} F(0,k).$$

We can define a family of WZ pairs by means of

$$F_x(n,k) = F(n+x,k), \quad G_x(n,k) = G(n+x,k),$$

So, Zeilberger's theorem implies

$$\sum_{n=0}^{\infty} G(n+x,0) = \lim_{k\to\infty} \sum_{n=0}^{\infty} G(n+x,k) + \sum_{k=0}^{\infty} F(x,k).$$

Reduction with WZ1

For $\sum_{n=0}^{\infty} G(n+x,0)$ one obtains

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)_{n+x}^3}{(1)_{n+x}^3} \left[2(n+x) + \frac{1}{2}\right],$$

and for $\lim_{k\to\infty} \sum_{n=0}^{\infty} G(n+x,k) + \sum_{k=0}^{\infty} F(x,k)$, one obtains

$$f(x) = \frac{1}{\pi}\frac{1}{\cos \pi x} + x^2 \frac{4\left(\frac{1}{2}\right)_x^3}{(2x-1)(1)_x^3} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(x+1)_n \left(\frac{3}{2}-x\right)_n},$$

that we will call a reduction. From it, we get the expansion

$$f(x) = \frac{1}{\pi} - \frac{\pi}{2}x^2 + O(x^3).$$

Reduction with WZ2

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{4^{n+x}} \frac{\left(\frac{1}{2}\right)_{n+x}^{3}}{(1)_{n+x}^{3}} \left[\frac{3}{2}(n+x) + \frac{1}{4}\right].$$

Reduction:

$$f(x) = \frac{1}{\pi} \frac{1}{\cos^2 \pi x} + x^2 \frac{4\left(\frac{1}{2}\right)_x^3}{(2x-1)4^x(1)_x^3} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}+x\right)_n}{(1+x)_n \left(\frac{3}{2}-x\right)_n}.$$

We get the expansion

$$f(x) = \frac{1}{\pi} - \pi x^2 + O(x^3).$$

Reduction with WZ3

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{8^{n+x}} \frac{\left(\frac{1}{2}\right)_{n+x}^3}{(1)_{n+x}^3} \left[\frac{3\sqrt{2}}{2}(n+x) + \frac{\sqrt{2}}{4}\right].$$

Reduction:

$$f(x) = \frac{1}{\pi}\frac{1}{\cos \pi x} + x^2 \frac{4\sqrt{2}\left(\frac{1}{2}\right)_x^3}{(2x-1)8^x(1)_x^3} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}+x\right)_n^2}{2^n(1+x)_n\left(\frac{3}{2}-x\right)_n}.$$

We get the expansion

$$f(x) = \frac{1}{\pi} - \frac{3}{2}\pi x^2 + O(x^3).$$

Reduction with WZ4

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+x}}\frac{\left(\frac{1}{2}\right)_{n+x}\left(\frac{1}{4}\right)_{n+x}\left(\frac{3}{4}\right)_{n+x}}{(1)_{n+x}^3}\left[\frac{5}{2}(n+x)+\frac{3}{8}\right].$$

Reduction:

$$f(x)=\frac{1}{\pi}\frac{1}{\cos\pi x}+x^2\frac{6\left(\frac{1}{2}\right)_x\left(\frac{1}{4}\right)_x\left(\frac{3}{4}\right)_x}{(2x-1)4^x(1)_x^3}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}+x\right)_n\left(\frac{1}{2}+2x\right)_n}{4^n(1+x)_n\left(\frac{3}{2}-x\right)_n}.$$

We get the expansion

$$f(x)=\frac{1}{\pi}-\frac{3}{2}\pi x^2+O(x^3).$$

Reductions with WZ5 and WZ6

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{9^{n+x}} \frac{\left(\frac{1}{2}\right)_{n+x} \left(\frac{1}{4}\right)_{n+x} \left(\frac{3}{4}\right)_{n+x}}{(1)_{n+x}^3} \left[\frac{4\sqrt{3}}{3}(n+x) + \frac{\sqrt{3}}{6} \right].$$

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{64^{n+x}} \frac{\left(\frac{1}{2}\right)_{n+x}^3}{(1)_{n+x}^3} \left[\frac{21}{8}(n+x) + \frac{5}{16} \right].$$

We get the expansions

$$f(x) = \frac{1}{\pi} - 2\pi x^2 + O(x^3), \qquad g(x) = \frac{1}{\pi} - 3\pi x^2 + O(x^3).$$

Reductions with WZ7 and WZ8

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{48^{n+x}}\frac{\left(\frac{1}{2}\right)_{n+x}\left(\frac{1}{4}\right)_{n+x}\left(\frac{3}{4}\right)_{n+x}}{(1)_{n+x}^3}\left[\frac{7\sqrt{3}}{4}(n+x)+\frac{3\sqrt{3}}{16}\right].$$

$$g(x)=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)_{n+x}\left(\frac{1}{6}\right)_{n+x}\left(\frac{5}{6}\right)_{n+x}}{\left(\frac{8}{3}\right)^{3(n+x)}(1)_{n+x}^3}\left[\frac{77\sqrt{2}}{32}(n+x)+\frac{15\sqrt{2}}{64}\right].$$

We get the expansions

$$f(x)=\frac{1}{\pi}-\frac{7}{2}\pi x^2+O(x^3), \qquad g(x)=\frac{1}{\pi}-\frac{7}{2}\pi x^2+O(x^3).$$

Program 2

We have written a program to look for functions $F(n, k)$ characterized by:

1. They have 3 rising factorials in the numerator and in the denominator.
2. All the signs in the rising factorials are positive.
3. The rational part is just n
4. The product of k and the exclusive function of k produces the constant when we take the limit as $k \to \infty$.

The program includes a function of EKHAD to certify if $F(n, k)$ is the first component of a WZ pair.

WZ9. Series and reduction

$$\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)_n^2\left(\frac{1}{2}+k\right)_n}{(1+k)_n^2(1)_n}(4n+2k+1)\frac{\binom{2k}{k}^2}{2^{4k}}=\frac{2}{\pi}.$$

$$\lim_{k\to\infty}\sum_{n=0}^{\infty}G(n,k)=\lim_{k\to\infty}G(0,k)=\lim_{k\to\infty}\frac{\binom{2k}{k}^2}{2^{4k}}(2k+1)=\frac{2}{\pi}.$$

$$f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3}[4(n+x)+1].$$

$$f(x)=2x\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}+x\right)_n}{(x+1)_n^2},\qquad f\left(\frac{1}{2}\right)=2G,$$

where G is the Catalan constant.

WZ10. Series and reduction

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^3}{(1+k)_n^2 (1)_n} (6n+4k+1) \frac{\binom{2k}{k}^2}{2^{4k}} = \frac{4}{\pi}.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3} [6(n+x)+1],$$

$$f(x) = 8x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(x+1)_n^2}, \qquad f\left(\frac{1}{2}\right) = \frac{\pi^2}{2}.$$

WZ11. Series and reduction

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{\left(\frac{1}{2}+2k\right)_n \left(\frac{1}{2}\right)_n^2}{(1+k)_n^2 (1)_n} (6n+4k+1) \frac{\binom{3k}{k}\binom{4k}{k}}{2^{6k}} = \frac{2\sqrt{2}}{\pi}.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3} [6(n+x)+1],$$

$$f(x) = 4x \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}+\frac{1}{4}\right)_n \left(\frac{x}{2}+\frac{3}{4}\right)_n}{(x+1)_n^2}, \qquad f\left(\frac{1}{2}\right) = 4G.$$

WZ12. Series and reduction

We make the transformation $F(n,k) = F_9(n, k+n)$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{2}+k\right)_n}{(1)_n (1+k)_n \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n} \frac{\binom{2k}{k}^2}{2^{4k}} \times \frac{(2n+2k+1)(20n+4k+3)-16kn}{2n+k+1} = \frac{8}{\pi}.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(x+\frac{1}{2}\right)_n \left(x+\frac{1}{4}\right)_n \left(x+\frac{3}{4}\right)_n}{(x+1)_n^3} [20(n+x)+3],$$

$$f(x) = 16x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(x+\frac{1}{2}\right)_n}{(x+1)_n (2x+1)_n}, \qquad f\left(\frac{1}{2}\right) = 16 \ln 2.$$

WZ13. Series and reduction

We make the transformation $F(n,k) = F_{10}(n, k+n)$.

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}+k\right)_n^2 \left(\frac{1}{2}\right)_n^3}{\left(1+\frac{k}{2}\right)_n^2 \left(\frac{1}{2}+\frac{k}{2}\right)_n^2 (1)_n} R(n,k) \frac{\binom{2k}{k}^2}{2^{4k}} = \frac{16}{\pi},$$

$$R(n,k) = \frac{(2n+2k+1)^2(42n+4k+5) - 32kn(4n+3k+2)}{(2n+k+1)^2}.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3} [42(n+x)+5],$$

$$f(x) = 32x \sum_{n=0}^{\infty} \frac{\left(x+\frac{1}{2}\right)_n^2}{(2x+1)_n^2}, \qquad f\left(\frac{1}{2}\right) = \frac{8}{3}\pi^2.$$

WZ14. Series and reduction

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n \left(\frac{1}{2}+2k\right)_n}{(1)_n (1+k)_n \left(1+\frac{k}{2}\right)_n \left(\frac{1}{2}+\frac{k}{2}\right)_n} \frac{\binom{3k}{k}\binom{4k}{k}}{2^{6k}}$$

$$\times \frac{(2n+4k+1)(154n+16k+15)-384kn}{2n+k+1} = \frac{32\sqrt{2}}{\pi}.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{\left(x+\frac{1}{2}\right)_n \left(x+\frac{1}{6}\right)_n \left(x+\frac{5}{6}\right)_n}{(x+1)_n^3} [154(n+x)+15],$$

$$f(x) = 128x \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}+\frac{1}{4}\right)_n \left(\frac{x}{2}+\frac{3}{4}\right)_n}{(x+1)_n (2x+1)_n}, \qquad f\left(\frac{1}{2}\right) = 128 \ln 2.$$

The PSLQ algorithm

If $(x_1, \dots x_n)$ is a vector of real numbers, then the PSLQ algorithm finds a vector $(a_1, \dots, a_n)$ of integers (with $a_j \neq 0$ for some j), such that (with the number of decimals that we are using)

$$a_1 x_1 + \cdots + a_n x_n = 0.$$

The vector it finds has the smallest possible norm.

The PSLQ algorithm is very useful to discover identities but we need another method to obtain rigourous proofs of them.

Evaluations at $x = 1/2$

Let $B(n,x) = \left(\frac{1}{2}+x\right)_n \left(\frac{1}{4}+x\right)_n \left(\frac{3}{4}+x\right)_n (1+x)_n^{-3}$, and

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{882^2}\right)^{n+x+\frac{1}{2}} \cdot B(n,x) \left[5365(n+x) + \frac{1123}{4}\right],$$

$$g(x) = \sum_{n=0}^{\infty} \left(\frac{1}{99^4}\right)^{n+x+\frac{1}{2}} \cdot B(n,x) \left[\frac{13195}{\sqrt{2}}(n+x) + \frac{1103}{2\sqrt{2}}\right],$$

$f(0) = g(0) = 1/\pi$. With the PSLQ algorithm, we find that

$$f(1/2) = \ln 2 + 10 \ln 3 - 6 \ln 7,$$

$$g(1/2) = \frac{13}{2}\pi - 16 \arctan \frac{\sqrt{2}}{2} - 24 \arctan \frac{\sqrt{2}}{3}.$$

Conjecture

Let us consider the function ($u = 1$ or $u = -1$)

$$R(x) = \sum_{n=0}^{\infty} u^n B(n+x) z^{n+x} [a + b(n+x)].$$

If $R(0) = 1/\pi$ with z, a, b algebraic numbers, then there is a rational number k, such that

$$R(x) = \frac{1}{\pi} - \frac{k\pi}{2} x^2 + O(x^3).$$

It implies that u, $B(n)$ and k determine z, a and b by solving (numerically and then using identify) the equations

$$R(0) = \frac{1}{\pi}, \quad R'(0) = 0 \quad \text{and} \quad R''(0) = -k\pi.$$

How to solve the equations

If we define the functions of x (and also of z):

$$S(x) = \sum_{n=0}^{\infty} u^n B(n+x) z^{n+x}, \quad T(x) = \sum_{n=0}^{\infty} u^n B(n+x) z^{n+x} (n+x).$$

Then z, a and b are the solutions of the equations

$$aS(0) + bT(0) = \frac{1}{\pi}, \qquad aS'(0) + bT'(0) = 0,$$

$$aS''(0) + bT''(0) = -k\pi.$$

We have conjectured that the last equation is equivalent to

$$S'(0) = -\pi\sqrt{N} S(0),$$

Series of positive terms. Example $k = 8$

The sum is always $1/\pi$.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3} (97 - 56\sqrt{3})^n \left(\sqrt{78\sqrt{3} - 135} + 6\sqrt{14\sqrt{3} - 24} \cdot n \right),$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \left(\frac{1}{81}\right)^n \left(\frac{2\sqrt{2}}{9} + \frac{20\sqrt{2}}{9} n \right),$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3} \left(\frac{13\sqrt{7} - 34}{54}\right)^n \left(\frac{7\sqrt{7} - 10}{27} + \frac{13\sqrt{7} - 7}{9} n \right),$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} \left(\frac{4}{125}\right)^n \left(\frac{2\sqrt{15}}{25} + \frac{22\sqrt{15}}{25} n \right).$$

Ramanujan-Sato's series

$$\sum_{n=0}^{\infty} B_n \left(\frac{\sqrt{5}-1}{2}\right)^{12n} (20n+10-3\sqrt{5}) = \frac{20\sqrt{3}+9\sqrt{15}}{6\pi},$$

where B_n are the Apéry numbers, defined by

$$B_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

which satisfy the recurrence:

$$n^3 B_n - (2n-3)(17n^2-17n+5)B_{n-1} + (n-1)^3 B_{n-2} = 0.$$

A method to obtain families of series

Let $S(z)$ and $W(z)$ be the functions

$$S(z) = \sum_{n=0}^{\infty} B_n z^n, \qquad W(z) = \sum_{n=0}^{\infty} B_n^{'} z^n.$$

Then if $z(q)$ is the solution of the equation

$$q = \pm e^{-\pi\sqrt{N}} = z \exp \frac{W(z)}{S(z)},$$

the functions $b(q)$ and $a(q)$ are given by

$$b = \sqrt{N} \frac{q}{zS} \frac{dz}{dq}, \qquad a = \frac{1}{S} \left[\frac{1}{\pi} - \frac{q\sqrt{N}}{S} \frac{dS}{dq} \right].$$

Example

For $B_n = \sum_{j=0}^{n} \binom{2j}{j}^2 \binom{2n-2j}{n-j}^2$, we get

$$z = q - 8q^2 + 44q^3 - 192q^4 + 718q^5 - 2400q^6 + 7352q^7 - 20992q^8 + \cdots\ ,$$

$$S = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + 28q^8 + \cdots\ .$$

These expansions seem to correspond to

$$z = \frac{\theta_2^4(q)}{16\theta_3^4(q)}, \qquad S = \theta_3^4(q),$$

$$\theta_2(q) = \sum_{n=-\infty}^{n=\infty} q^{(n+1/2)^2}, \qquad \theta_3(q) = \sum_{n=-\infty}^{n=\infty} q^{n^2}.$$

To prove that z and S are right use Yifan Yang's method.

Similar series for $1/\pi^2$

$$\sum_{n=0}^{\infty} z^n \frac{C(n)}{(1)_n^5}(an^2+bn+c) = \frac{1}{\pi^2}, \qquad -1 \le z < 1,$$

where z, a, b and c are algebraic numbers and $C(n)$ is the product of 5 Pochhammer symbols obtained joining blocks:

$$(1/2)_n, \qquad (1/3)_n(2/3)_n, \qquad (1/4)_n(3/4)_n, \qquad (1/6)_n(5/6)_n,$$

$$(1/5)_n(2/5)_n(3/5)_n(4/5)_n, \qquad (1/10)_n(3/10)_n(7/10)_n(9/10)_n,$$

$$(1/8)_n(3/8)_n(5/8)_n(7/8)_n, \qquad (1/12)_n(5/12)_n(7/12)_n(11/12)_n.$$

$$(1/8)_n(3/8)_n(5/8)_n(7/8)_n = \frac{C_8(n)}{C_4(n)} = \frac{1}{4^{8n}}\frac{(8n)!}{(4n)!}.$$

We have proved some of them with the WZ-method.

Series for $1/\pi^2$. WZ15

$$F(n,k) = \frac{\left(\frac{1}{2}\right)_n^5}{2^{2n}(1)_n(1+k)_n^4} \cdot 8n(2n+4k+1)\frac{\binom{2k}{k}^4}{2^{8k}}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{2^{2n}(1)_n(1+k)_n^4}(20n^2+8n+1+24kn+8k^2+4k)\frac{\binom{2k}{k}^4}{2^{8k}} = \frac{8}{\pi^2}$$

$k \to \infty$ determine the constant $8/\pi^2$. For $k=0$, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5}(20n^2+8n+1) = \frac{8}{\pi^2}.$$

Reduction obtained with WZ15

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(x+\frac{1}{2}\right)_n^5}{(x+1)_n^5} \left[20(n+x)^2 + 8(n+x) + 1\right].$$

$$f(x) = 8x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^4}{(x+1)_n^4} (4n+2x+1), \qquad f\left(\frac{1}{2}\right) = 7\zeta(3).$$

Series for $1/\pi^2$. WZ16 y WZ17

$$F(n,k)=\frac{1}{2^{4n}}\frac{\left(\frac12\right)_n^3\left(\frac14\right)_n\left(\frac34\right)_n\left(\frac12+k\right)_n^2}{(1)_n(1+k)_n^2\left(1+\frac{k}{2}\right)_n^2\left(\frac12+\frac{k}{2}\right)_n^2}\frac{\binom{2k}{k}^4}{2^{8k}}\cdot 32n(4n+4k+1)$$

$$\sum_{n=0}^{\infty}\frac{1}{2^{4n}}\frac{\left(\frac12\right)_n^3\left(\frac14\right)_n\left(\frac34\right)_n}{(1)_n^5}(120n^2+34n+3)=\frac{32}{\pi^2}.$$

$$F(n,k)=\frac{(-1)^n}{2^{10n}}\frac{\left(\frac12\right)_n^5\left(\frac12+k\right)_n^4}{\left(1+\frac{k}{2}\right)_n^4\left(\frac12+\frac{k}{2}\right)_n^4(1)_n}\frac{\binom{2k}{k}^4}{2^{8k}}\cdot 128n(6n+4k+1)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\frac{\left(\frac12\right)_n^5}{(1)_n^5}(820n^2+180n+13)=\frac{128}{\pi^2}.$$

Reduction obtained with WZ17

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\frac{\left(x+\frac{1}{2}\right)_n^5}{(x+1)_n^5}\left[820(n+x)^2+180(n+x)+13\right].$$

$$f(x)=128x\sum_{n=0}^{\infty}\frac{\left(x+\frac{1}{2}\right)_n^4}{(2x+1)_n^4}(4n+6x+1), \qquad f\left(\frac{1}{2}\right)=256\zeta(3).$$

(This series for $\zeta(3)$ was first obtained by T. Amdeberhan).

Expansions in powers series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2(n+x)}} \frac{\left(\frac{1}{2}\right)_{n+x}^5}{(1)_{n+x}^5} [20(n+x)^2+8(n+x)+1] = \frac{8}{\pi^2} - 4x^2 + O(x^4),$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10(n+x)}} \frac{\left(\frac{1}{2}\right)_{n+x}^5}{(1)_{n+x}^5} [820(n+x)^2 + 180(n+x) + 13]$$
$$= \frac{128}{\pi^2} - 320x^2 + O(x^4).$$

The above expansions have been found by experimental methods and suggest the following conjecture.

Conjecture

Let us consider the function ($u = 1 \,\mathrm{or}\, u = -1$)

$$R(x) = \sum_{n=0}^{\infty} u^n B(n+x) z^{n+x} [a(n+x)^2 + b(n+x) + c],$$

If $R(0) = 1/\pi^2$ with a, b, c, z algebraic, then there is a rational number k, such that

$$R(x) = \frac{1}{\pi^2} - \frac{k}{2} x^2 + O(x^4).$$

It implies that $u, B(n), k$ determine z, a, b, c by means of

$$R(0) = \frac{1}{\pi^2}, \quad R'(0) = 0, \quad R''(0) = -k, \quad R'''(0) = 0.$$

Conjecture: Examples $k = 1, 2, 3$

For $k = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} (20n^2 + 8n + 1) = \frac{8}{\pi^2}.$$

For $k = 2$

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} (120n^2 + 34n + 3) = \frac{32}{\pi^2},$$

For $k = 3$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{n!^5 48^n} (252n^2 + 63n + 5) = \frac{48}{\pi^2}.$$

Conjecture: Examples $k = 5, 7, 8$

For $k = 5$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} (820n^2 + 180n + 13) = \frac{128}{\pi^2}.$$

For $k = 7$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^5 2^{10n}} (1640n^2 + 278n + 15) = \frac{256\sqrt{3}}{3\pi^2},$$

For $k = 8$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{n!^5 7^{4n}} (1920n^2 + 304n + 15) = \frac{56\sqrt{7}}{\pi^2},$$

Conjecture: Example $k = 15$

For $k = 15$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^5 80^{3n}} (5418n^2+693n+29) = \frac{128\sqrt{5}}{\pi^2}.$$

$$\left(\frac{1}{6}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{5}{6}\right)_n = C_6(n) = \frac{1}{6^{6n}} \frac{(6n)!}{n!}.$$

$$\sum_{n=0}^{\infty} \frac{(6n)!}{n!^6} \frac{(-1)^n}{2880^{3n}} (5418n^2 + 693n + 29) = \frac{128\sqrt{5}}{\pi^2}.$$

Series for $1/\pi$	Series for $1/\pi^2$
Theory of modular functions	Modular-like theory?
Equations for z, a and b The eq. for z is reducible	Equations for z, a, b and c Is the eq. for z reducible?
WZ method, all?	WZ method, all?

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