> # Ramanujan series for $\frac{1}{\pi}$ #

> # Automatic proofs. #

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> # Rutgers Exp. Math. Seminar (ZOOM) #

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> # Written in Maple 2023. This version: April 2023 #

> # Title: Ramanujan series for $\frac{1}{\pi}$. Automatic proofs. #

> # Modular equations of $\ell = 2, 3, 4$ got automatically #

>

> # Weber modular polynomials given by Sutherland. #

>

> # Many thanks to Alin Bostan for the great idea of simplifiyng using a rule. It accelerates a lot the computations at u_0 #

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> # Legendre's relation: #

>

> # Let $\ell = 4\sin^2\frac{\pi}{s}$, $\ell \in \{1, 2, 3, 4\}$, #

> # $F_\ell(x) = {}_2F_1\left(\frac{1}{s}, 1-\frac{1}{s}; 1 \,\middle|\, x\right)$, $G_\ell(x) = x\,\frac{dF_\ell(x)}{dx}$, #

>

> # Then, if $\beta = 1 - \alpha$, we have #

>

```
>         # 2α/√ℓ F_ℓ(α) G_ℓ(β) + 2β/√ℓ F_ℓ(β) G_ℓ(α) = 1/π . #
```

$$\#\frac{2\,\alpha}{\sqrt{\ell}}\,F_\ell(\alpha)\,G_\ell(\beta)+\frac{2\,\beta}{\sqrt{\ell}}\,F_\ell(\beta)\,G_\ell(\alpha)=\frac{1}{\pi}\,.\,\#$$

```
>
```

```
> legendre:= proc(lev,alpha) local leve,beta,F,G,leg: local
  s: if lev=1 then s:=6 fi: if lev=2 then s:=4 fi: if lev=3
  then s:=3 fi: if lev=4 then s:=2 fi:
> leve:=4*sin(Pi/s)^2: F:=xx->subs(x=xx,hypergeom(
  [1/s,1-1/s],[1],x)): G:=xx->subs(x=xx,x*diff(F(x),x)):
> beta:=1-alpha: leg:=2*alpha/sqrt(leve)*F(alpha)*G
  (beta)+2*beta/sqrt(leve)*F(beta)*G(alpha):
> print(): print(simplify(leg,hypergeom)=1/Pi): print():
> end:
```

```
>
```

```
> legendre(2,-1/5);
```

$$-\frac{1}{200}\left(9\sqrt{2}\left(\mathrm{hypergeom}\left(\left[\frac{1}{4},\frac{3}{4}\right],[1],-\frac{1}{5}\right)\mathrm{hypergeom}\left(\left[\frac{5}{4},\frac{7}{4}\right],[2],\frac{6}{5}\right)+\mathrm{hypergeom}\left(\left[\frac{1}{4},\frac{3}{4}\right],[1],\frac{6}{5}\right)\mathrm{hypergeom}\left(\left[\frac{5}{4},\frac{7}{4}\right],[2],-\frac{1}{5}\right)\right)\right)=\frac{1}{\pi} \quad (1)$$

```
> legendre(3,5/3);
```

$$-\frac{1}{243}\left(40\sqrt{3}\left(\mathrm{hypergeom}\left(\left[\frac{1}{3},\frac{2}{3}\right],[1],\frac{5}{3}\right)\mathrm{hypergeom}\left(\left[\frac{4}{3},\frac{5}{3}\right],[2],-\frac{2}{3}\right)+\mathrm{hypergeom}\left(\left[\frac{1}{3},\frac{2}{3}\right],[1],-\frac{2}{3}\right)\mathrm{hypergeom}\left(\left[\frac{4}{3},\frac{5}{3}\right],[2],\frac{5}{3}\right)\right)\right)=\frac{1}{\pi} \quad (2)$$

(2)

>

> # Clausen's identity #

>

> # For $x < \frac{1}{2}$, we have ${}_2F_1\left(\left[\frac{1}{s}, 1-\frac{1}{s}\right], [1], x\right)^2$ #

> # = ${}_3F_2\left(\left[\frac{1}{2}, \frac{1}{s}, 1-\frac{1}{s}\right], [1, 1], 4x(1-x)\right)$, #

>

```
> clausen:=proc(lev,x) local R,F,clau,s: if lev=1 then s:=6
  fi: if lev=2 then s:=4 fi: if lev=3 then s:=3 fi: if lev=4
  then s:=2 fi:
> R:=x->hypergeom([1/2,1/s,1-1/s],[1,1],4*x*(1-x)):
> F:=x->hypergeom([1/s,1-1/s],[1],x):
> clau:=x->F(x)^2-R(x):
> print(): print(simplify(clau(x),hypergeom)=0):
> end:
```

```
> clausen(2,1/4);
```

$$\mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], \frac{1}{4}\right)^2 - \mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right], [1, 1], \frac{3}{4}\right) = 0 \quad (3)$$

```
> clausen(2,-2);
```

$$\mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{3}{4}\right], [1], -2\right)^2 - \mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right], \right. \quad (4)$$

$[1, 1], -24\Big) = 0$

> # q #

>

> # Let $\ell = 4\sin^2\frac{\pi}{s}$, $F_\ell(x) = {}_2F_1\left(\frac{1}{s}, 1-\frac{1}{s}; 1 \,\middle|\, x\right)$. #

>

> # If we write $F_\ell(x)^2 = \frac{q}{x(1-x)}\frac{dx}{dq}$, we get #

>

> # $q = C\exp\int\frac{dx}{F\ell(x)^2 x(1-x)}$, (integer coefficients). #

>

> # The mirror map #

>

> # The inverse function $x_\ell(q)$ is the mirror map #

>

> # If we let $\beta = x_\ell(q)$, $\alpha = x_\ell(q^d)$, then #

>

> # $F_\ell(\alpha)^2 = \frac{q^d}{\alpha(1-\alpha)}\frac{d\alpha}{dq^d} = \frac{1}{d}\frac{q}{\alpha(1-\alpha)}\frac{d\alpha}{dq}$, #

>

> # $F_\ell(\beta)^2 = \frac{q}{\beta(1-\beta)}\frac{d\beta}{dq}$. #

> # We deduce #

>

> # $\frac{1}{d}\ \frac{d\alpha}{F_\ell(\alpha)^2\alpha(1-\alpha)} = \frac{d\beta}{F_\ell(\beta)^2\beta(1-\beta)}$ #

>

> # $\Rightarrow A_d(\alpha,\beta) = 0$, (*modular equation*), #

>

> # *If* $m(\alpha,\beta) = \frac{F_\ell(\alpha)}{F_\ell(\beta)}$ (*multiplier*), then #

>

> # $m(\alpha,\beta) = \sqrt{\frac{1}{d}\,\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\,\frac{d\alpha}{d\beta}} = \sqrt{\frac{1}{d}\,\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\,\frac{\alpha'}{\beta'}}$. #

>

> # *Algebraic transformation* #

>

> # $F_\ell(\alpha) = m(\alpha,\beta)\,F_\ell(\beta)$, $A_d(\alpha,\beta) = 0.$ #

>

> # *Russell modular equations corresponding to* $\ell \in \{2,3,4\}$ #

>

> # $u^h = \alpha\,\beta$, $v^h = (1-\alpha)(1-\beta)$, $P(u,v) = 0$ #

>

> # $\ell = 4$, $\frac{d+1}{8} = \frac{n}{m}$, $h = \frac{8}{m}$, $k = n$, *Russell* #

> # $\ell = 3$, $\frac{d+1}{3} = \frac{n}{m}$, $h = \frac{6}{m}$, $k = n$, *Chan & Liaw* #

> # $\ell = 2$, $\frac{d+1}{4} = \frac{n}{m}$, $h = \frac{4}{m}$, $k = 2n$. #

>

```
> russellmodequ:=proc(lev,p) local uu,vv,mm,x,s,val,qq,m
```

```
,coe,k,i,num,aalpha,bbeta,alpha,beta,sb,u,v,suma,ecus,
ecucoef,formu,sol,o,q: global P,hh,ro,BC,h,dpol,dec,pol,
T,llev: dec:=p: o:=time():

> if lev=2 then s:=4: ro:=256: BC:=n->(4*n)!/(n!^4):
  val:=simplify((p+1)/4): h:=4/denom(val): dpol:=2*val:
  fi: if lev=2 and d=3 then h:=1: dpol:=4: fi:
> if lev=3 then s:=3: ro:=108: BC:=n->(3*n)!*(2*n)!/(n!
  ^5): val:=simplify((p+1)/3): h:=6/denom(val): dpol:=
  numer(val): fi:
> if lev=4 then s:=2: ro:=64: BC:=n->((2*n)!)^3/(n!^6):
  val:=simplify((p+1)/8): h:=8/denom(val): dpol:=numer
  (val): fi: if lev=4 and d=3 then h:=1: dpol:=4: fi: if lev=
  4 and d=1 then h:=1: dpol:=2: fi:
> coe:=seq(seq(c[k-i,i],i=0..k),k=1..dpol): num:=nops(
  {coe}):
> qq:=4/ro*exp(int(1/(hypergeom([1/2,1/s,1-1/s],[1,1],
  4*x*(1-x))*x*(1-x)),x));
> hh:=h: Order:=2*num:
> assume(q,real): mm:=convert(solve(series(qq,x)=q,x),
  polynom):
> print(level=lev, degree=p): print(): print(`MODULAR
  EQUATION`):
> aalpha:=subs(q=q^p,mm): bbeta:=mm:

> uu:=convert(series(aalpha^(1/h)*bbeta^(1/h),q,2*
  num),polynom):
> vv:=convert(series((1-aalpha)^(1/h)*(1-bbeta)^(1/h),
  q,2*num),polynom):
> suma:=sum(sum(c[k-i,i]*uu^i*vv^(k-i),i=0..k),k=1..
  dpol): ecus:=series(suma,q,2*num)-1:
> ecucoef:=coeffs(simplify(convert(ecus,polynom)),q):

> formu:=sum(sum(c[k-i,i]*u^i*v^(k-i),i=0..k),k=1..
```

```
   dpol)-1:
> sol:=solve({ecucoef},{coe}): pol:=sort(subs(sol,formu),
   [u,v]): P:=(uu,vv)->subs(u=uu,v=vv,pol):
> dec:=p: llev:=lev:
> print(u^h=alpha*beta,v^h=(1-alpha)*(1-beta)): print
   (pol=0): print(Seconds=time()-o):
> end:

>
> # Weber modular polynomials (given by Sutherland) #
>
> weber5:=proc()
   global Q,R,P:
   Q:=(u,v)->u^3+v^3:
   R:=(u,v)->-u^2*v^2+4:
   P:=(u,v)->Q(u,v)^2-u*v*R(u,v)^2:
   end:

> weber7:=proc()
   global Q,R,P:
   Q:=(u,v)->u^4+v^4+7*u^2*v^2:
   R:=(u,v)->u^3*v^3+8:
   P:=(u,v)->Q(u,v)^2-u*v*R(u,v)^2:
   end:
>
> weber11:=proc()
   global Q,R,P:
   Q:=(u,v)->u^6+v^6:
   R:=(u,v)->u^5*v^5-11*u^4*v^4+44*u^3*v^3-88*
   u^2*v^2+88*u*v-32:
   P:=(u,v)->Q(u,v)^2-u*v*R(u,v)^2:
```

```
end:
```

```
> weber13:=proc() global Q,R,P:
> Q:=(u,v)->u^7+v^7+13*(u^6*v+u*v^6)+52*(u^5*
  v^2+u^2*v^5)+78*(u^4*v^3+u^3*v^4):
> R:=(u,v)->u^6*v^6-64:
> P:=(u,v)->Q(u,v)^2-u*v*R(u,v)^2:
> end:
```

```
> weber17:=proc() global Q,R,P:
  Q:=(u,v)->(u^9+v^9)+17*(u^8*v^5+u^5*v^8)+
  119*(u^6*v^3+u^3*v^6)+272*(u^4*v+u*v^4):
  R:=(u,v)->-(u^8*v^8)-34*(u^7*v+u*v^7)+
  34*(u^6*v^6)+340*(u^4*v^4)+544*(u^2*v^2)-256:
  P:=(u,v)->Q(u,v)^2-(u*v)*R(u,v)^2:
  end:
```

```
> weber41:=proc() global Q,R,P:
  Q:=(u,v)->u^(21)+v^(21)+943*(u^(20)*v^(5)+u^(5)
  *v^(20))+
  123*(u^(20)*v^(17)+u^(17)*v^(20))+
  40713*(u^(19)*v^(10)+u^(10)*v^(19))+72939*
  (u^18*v^3+u^3*v^18)+
  3772*(u^18*v^15+u^15*v^18)+
  733531*(u^17*v^8+u^8*v^17)+15088*(u^16*v+u*
  v^16)+
  339111*(u^16*v^13+u^13*v^16)-
  6494359*(u^15*v^6+u^6*v^15)+3112310*(u^14*
  v^11+u^11*v^14)+
  11736496*(u^13*v^4+u^4*v^13)
  -36004437*(u^12*v^9+u^9*v^12)+10422528*
```

```
(u^11*v^2+u^2*v^11)+
49796960*(u^10*v^7+u^7*v^10)
+86812416*(u^8*v^5+u^5*v^8)+15450112*(u^6*
v^3+u^3*v^6)+
8060928*(u^4*v+u*v^4):
R:=(u,v)->41*(u^(20)*v^(8)+u^(8)*v^(20))-
u^(20)*v^(20)+574*(u^(19)*v+u*v^(19))-
4059*(u^(19)*v^(13)+u^(13)*v^(19))-155554*(u^
(18)*v^(6)+
u^(6)*v^(18))+574*u^(18)*v^(18)-
160310*(u^17*v^11+u^11*v^17)+701100*(u^16*
v^4+u^4*v^16)-
2050*u^16*v^16-1753160*(u^15*v^9+u^9*v^15)-
2488864*(u^14*v^2+u^2*v^14)-462726*u^14*
v^14+
20156994*(u^13*v^7+u^7*v^13)+10496*(u^12+
v^12)-
3571756*u^12*v^12-28050560*(u^11*v^5+u^5*
v^11)+
45567400*u^10*v^10-41039360*(u^9*v^3+u^3*
v^9)-
57148096*u^8*v^8-16625664*(u^7*v+u*v^7)
-118457856*u^6*v^6-
8396800*u^4*v^4+37617664*u^2*v^2-1048576:
P:=(u,v)->Q(u,v)^2-(u*v)*R(u,v)^2:
end:
>
> weber:=proc(dd)
> if dd=5 then weber5() fi:
> if dd=7 then weber7() fi:
> if dd=11 then weber11() fi:
> if dd=13 then weber13() fi:
> if dd=17 then weber17() fi:
> if dd=41 then weber41() fi:
```

```
> print(P(u,v)=0):print(): print(): end:
>
>
```

> # *Weber modular equations corresponding to $\ell = 1$ #*

```
>
```

> # $\frac{432\,u^{12}}{(u^{12}-16)^3} = \alpha(1-\alpha)$, $\frac{432\,v^{12}}{(v^{12}-16)^3} = \beta(1-\beta)$, #

> # $P(u, v) = 0$. #

```
>
> webermodequ:=proc(lev,dd)
> print(level=lev, degree=dd): print(): print(`MODULAR
  EQUATION`): print(432*u^(12)/(u^(12)-16)^3=alpha*
  (1-alpha), 432*v^(12)/(v^(12)-16)^3=beta*(1-beta)):
> weber(dd):
> end:
>
> modequ:=proc(lev,dd)
> if lev=1 then return webermodequ(lev,dd) else return
  russellmodequ(lev,dd): fi:
> end:

> modequ(1,5);
```

$$level = 1, degree = 5$$

MODULAR EQUATION

$$\frac{432\,u^{12}}{(u^{12}-16)^3} = \alpha\,(1-\alpha),\ \frac{432\,v^{12}}{(v^{12}-16)^3} = \beta\,(1-\beta)$$

$$(u^3+v^3)^2 - u\,v\,(-u^2\,v^2+4)^2 = 0$$

(5)

```
> modequ(3,11);
```

$$level = 3,\ degree = 11$$

MODULAR EQUATION

$$u^6 = \alpha\beta,\ v^6 = (1-\alpha)(1-\beta)$$

$$-u^4 + 15u^3 v + 16u^2 v^2 + 15uv^3 - v^4 + 2u^2 + 12uv + 2v^2 - 1 = 0$$

$$Seconds = 1.703$$ (6)

>

> # Ramanujan series for $\frac{1}{\pi}$ #

>

> # Let $s \in \{2, 3, 4, 6\}$, #

>

> # $\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{s}\right)_n \left(1-\frac{1}{s}\right)_n}{(n!)^3} z_0^n (a+bn)$ #

>

> # $= \sum_{n=0}^{\infty} \frac{B(n)(a+bn)}{(J_0)^n} = \frac{1}{\pi}$ #

>

> # Explicit General Formulas (G.) #

>

> # $\beta = 1 - \alpha$, $A(\alpha, \beta) = 0 \Rightarrow \alpha_{0,}$ $\beta_0 = 1 - \alpha_0$ #

> # $z_0 = 4\,\alpha_0\,\beta_0$, $J_0 = \dfrac{\rho_0}{z_0}$ #

> # $b = \dfrac{1-2\,\alpha_0}{\sqrt{\ell}}\left(m_0\,d + \dfrac{1}{m_0}\right)$, #

> # $a = -2\,\alpha_0\,\beta_0\,m_0\,\dfrac{m'_0}{m_0\,\alpha'_0}\,\dfrac{d}{\sqrt{\ell}}$, #

>

> # $m_0 = \sqrt{\dfrac{1}{d}\,\dfrac{\alpha'_0}{\beta'_0}}$, $\tau_0 = \dfrac{dm_0}{\sqrt{\ell}}\,i$, $q_0 = e^{2\pi i \tau_0}$, and #

>

> # $\dfrac{m'_0}{m_0\alpha'_0} = \dfrac{1}{2\,\alpha'_0}\left(\dfrac{\beta'_0}{\beta_0} - \dfrac{\beta'_0}{1-\beta_0} - \dfrac{\alpha'_0}{\alpha_0} + \dfrac{\alpha'_0}{1-\alpha_0} + \dfrac{\alpha''_0}{\alpha'_0} - \dfrac{\beta''_0}{\beta'_0}\right)$ #

>

> # PROOF #

> # Apply the operator #

> # $a + b\,z_0\,\dfrac{d}{dz}\Big|_{u_0} = a + b\,\dfrac{\alpha_0\,\beta_0}{1-2\,\alpha_0}\,\dfrac{d}{d\alpha}\Big|_{u_0}$ #

>

> # to the Clausen's identity, to obtain #

>

> $\# \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{s}\right)_n \left(1-\frac{1}{s}\right)_n}{(n!)^3} z_0^{\,n} (a+bn) \#$

>

> $\# = a\, F_\ell(\alpha_0)\, F_\ell(\alpha_0) \#$

>

> $\# + \frac{b\,\beta_0}{1-2\,\alpha_0} (1+C)\, F_\ell(\alpha_0) G_\ell(\alpha_0) \#$

>

> $\# + \frac{b\,\beta_0}{1-2\,\alpha_0} (1-C)\, F_\ell(\alpha_0) G_\ell(\alpha_0) \#$

>

> # *Use the algebraic transformation* #

>

> $\# F_\ell(\alpha) = m(\alpha, \beta)\, F_\ell(\beta), \#$

>

> # *and its derivative* #

>

> $\# G_\ell(\alpha) = \alpha \frac{m'}{\alpha'} F_\ell(\beta) + \alpha \frac{m}{\beta} \frac{\beta'}{\alpha'} G_\ell(\beta) \#$

>

> # *to make the suitable substitutions,*
and identify the coefficients of #

>

> $\# F_\ell(\alpha_0)\, F_\ell(\beta_0),\ F_\ell(\alpha_0)\, G_\ell(\beta_0),\ F_\ell(\beta_0)\, G_\ell(\alpha_0), \#$

> $\#$ *to* $0,\ \frac{2\,\alpha_0}{\sqrt{\ell}},\ \frac{2\,\beta_0}{\sqrt{\ell}}.\ \#$

>

> # We get the following system of equations: #

>

> # (1) $a\, m_0 + \dfrac{b\, \beta_0 \alpha_0}{1 - 2\, \alpha_0} + \dfrac{m'_0}{\alpha'_0}(1 - C) = 0$, #

> # (2) $\dfrac{b\, m_0}{1 - 2\, \alpha_0}(1 + C) = \dfrac{2}{\sqrt{\ell}}$, #

> # (3) $\dfrac{b\, m_0}{1 - 2\, \alpha_0} \dfrac{\beta'_0}{\alpha'_0}(1 - C) = \dfrac{2}{\sqrt{\ell}}$. #

>

> # From the above system we obtain the explicit formulas #

>

>

> # We use this for making many algebraic simplifications #

>

```
> fullsimplify:=proc(expression)
combine(evalc(simplify(expand(combine(rationalize
(radnormal(
expand(simplify(rationalize(simplify(
combine(radnormal(expand(expression)),radicals)))))),
radicals)))),radicals):
end:
```

>

> # This procedure solves the system of equations #

>

> # $u^h = v^h$ and $P(u, v) = 0$, #

> # *to obtain* u_0 , v_0 and *therefore* z_0 and J_0 #

>

```
> valuesR:=proc(lev,dd) local o: global hh,h,jjj,www,i,uv0,
  A,G,SOL,nn,AA,div,L0:
> o:=time(): modequ(lev,dd): print(): with(polytools):
  print(`MAPLE FINDS THE SOLUTIONS OF THE  SYSTEM`)
  : print(u^h=v^h,  RMP(u,v)=0): print(J=ro/(4*u^h)):
  print(): print(`VALUES OF J`): print():
> G:=solve({P(u,v),u^hh=v^hh},{u,v}): nn:=nops({G}):
  for i from 1 to nn do: uv0:=evala(simplify(G[i],
  {expanded,symmetry})):  A:=convert(evala(simplify
  (subs(uv0,ro/(4*u^hh)),{eliminated,expanded,
  symmetry})),radical): if A=0 then else AA:=subs
  (RootOf=0,A): if AA=A then print(NUMBER=i): L0:=
  evalc(simplify(evala(subs(uv0,A)))): if evalf(abs(ro/L0),
  100)>1 then div:=1: else div:=0: fi: if div=1 then print
  (DIVERGENT): print(`convergent by analytic
  continuation`): fi: if ro/L0=1 then print(SINGULARITY):
  fi: if  Im(L0)=0 then else print(COMPLEX): fi: print(evalc
  (evala(simplify((L0))))): print(): fi: fi: od:
> print(Seconds=time()-o): end:
```

>

> # *This procedure solves the system of equations* #

>

> # $(u^{12}-16)\, v^4 = (v^{12}-16)\, u^4$, $P(u, v) = 0$, #

> # *to obtain* u_0 , v_0 and *therefore* z_0 and J_0 #

>

```
> valuesCH:=proc(lev,dd) local x,o: global jjj,www,i,A,G,
  SOL,nn,uv0,AA,div,J0:
> o:=time(): modequ(lev,dd): with(polytools):  print
  (`MAPLE FINDS THE SOLUTIONS OF THE  SYSTEM`):
  print((u^(12)-16)*v^4=(v^(12)-16)*u^4, WMP(u,v)=
```

```
0): print(J=(u^(12)-16)^3/u^(12)): print(): print
(`VALUES OF J`): print():
> G:=solve({P(u,v),(u^(12)-16)*v^4=(v^(12)-16)*u^4},
{u,v}): nn:=nops({G}): for i from 2 to nn do: uv0:=
evala(simplify(G[i],{expanded,symmetry})): A:=convert
(evala(simplify(subs(uv0,(u^(12)-16)/u^4),{eliminated,
expanded,symmetry})),radical): if A=0 then else AA:=
subs(RootOf=0,A): if AA=A then print(NUMBER=i): J0:=
evalc(simplify(evala(subs(uv0,A^3)))): if evalf(abs
(1728/J0),100)>1 then div:=1: print(DIVERGENT): print
(`convergent by analytic continuation`): fi: if 1728/J0=1
then print(SINGULARITY): fi: if Im(J0)=0 then else print
(COMPLEX): fi: print(evalc(simplify(evala(simplify(J0)))))
: print(): fi: fi: od: print(Seconds=time()-o):
> end:
>
```

> *# Main procedure 1 #*

>

> *# It shows the corresponding modular equation, #*

> *# and gets all the possible values of* u_0, v_0, z_0, J_0 *#*

>

>

```
> valuesJ:=proc(lev,dd)
> if lev=1 then return valuesCH(lev,dd): else return
valuesR(lev,dd): fi:
> end:
>
```

> *# Here we choose the NUMBER #*

>

```
> mychoose:=proc(lev,dd,kk) global u0v0,u0,v0,P1,P2,
rulepol:
```

```
> u0v0:=evala(simplify(G[kk],{expanded,symmetry})):
  P1:=evala(Minpoly(op(2,u0v0[1]),x)): P2:=evala
  (Minpoly(op(2,u0v0[2]),x)): u0:=fullsimplify(convert(op
  (2,u0v0[1]),radical)): v0:=fullsimplify(convert(op(2,
  u0v0[2]),radical)): rulepol:={subs(x=u,P1),subs(x=v
  (u),P2)}:
> end:
>
```

> *#This computes* $\frac{dv}{du}$, and $\frac{d^2v}{du^2}$,
>
> and *evaluates them at* u_0 #

```
>
> dvddv:=proc(dd) global T,dv,ddv,dv0,ddv0,atu0:
> T:=u->P(u,v(u)): atu0:={u=u0,v(u)=v0,diff(v(u),u)=
  dv0,diff(diff(v(u),u),u)=ddv0}:
> dv:=simplify(factor(solve(diff(T(u),u),diff(v(u),u)))):
  ddv:=diff(dv,u):
> dv0:=fullsimplify(subs(atu0,simplify(dv,rulepol))):
  ddv0:=fullsimplify(subs(atu0,simplify(ddv,rulepol))):
> end:
>
```

> *#This calculates* α, β, $\frac{d\alpha}{du}$, $\frac{d\beta}{du}$, $\frac{d^2\alpha}{du^2}$,
>
> $\frac{d^2\beta}{du^2}$ *at* u_0 (*Russell*) #

```
>
> alpha0beta0russel:=proc(dd) global T,alpha0,beta0,z0,
  dalpha0,dbeta0,ddalpha0,ddbeta0:
> T:=u->subs(v=v(u),P(u,v)): alpha0:=solve(alpha*(1-
  alpha)=expand(u0^h),alpha)[1]:  beta0:=1-alpha0:
```

```
> z0:=fullsimplify(4*alpha0*beta0):
> dalpha0:=evalc(fullsimplify(expand(subs(u=u0,subs
  (beta(u)=beta0,subs(alpha(u)=alpha0,subs(v(u)=v0,
  subs(diff(v(u),u)=dv0,simplify(expand((alpha(u)*(h*u^
  (h-1)-h*v(u)^(h-1)*diff(v(u),u))-h*u^(h-1))/(alpha(u)-
  beta(u)))))))))))):
> dbeta0:=evalc(fullsimplify(expand(h*u0^(h-1)-h*v0^
  (h-1)*dv0-dalpha0))):
> ddalpha0:=fullsimplify(expand(solve(h*(h-1)*u0^(h-2)
  -2*dalpha0*dbeta0-x*beta0-alpha0*(h*(h-1)*u0^(h-2)-
  h*(h-1)*v0^(h-2)*dv0^2-h*v0^(h-1)*ddv0-x),x))):
> ddbeta0:=fullsimplify(expand(h*(h-1)*u0^(h-2)-h*
  (h-1)*v0^(h-2)*dv0^2-h*v0^(h-1)*ddv0-ddalpha0)):
  end:
```

> #This calculates α, β, $\frac{d\alpha}{du}$, $\frac{d\beta}{du}$, $\frac{d^2\alpha}{du^2}$, $\frac{d^2\beta}{du^2}$ at u_0 (Weber)#

```
>
> alpha0beta0weber:=proc(lev,dd) local q: global u0,v0,
  alpha0,beta0,dalpha0,dbeta0,ddalpha0,ddbeta0,m0,
  dm0,atu0,tau0,q0:
> atu0:={u=u0,v(u)=v0,diff(v(u),u)=dv0,diff(diff(v(u),u),
  u)=ddv0}:
> alpha0:=solve(al*(1-al)=simplify(432*w0),al)[1]:
  beta0:=1-alpha0:
> dalpha0:=fullsimplify(432*subs(atu0,simplify(dw,
  rulepol))/(1-2*alpha0)):
  dbeta0:=fullsimplify(rationalize(432*subs(atu0,simplify
  (dy,rulepol))/(1-2*beta0))):
  m0:=fullsimplify(sqrt(fullsimplify(1/dd*dalpha0/dbeta0))
```

```
 ):
 print(m[0]=m0): print(abs(m[0])=abs(m0)): tau0:=
 evalc(expand(fullsimplify(m0*dd/sqrt(lev)*I))): q0:=exp
 (2*Pi*I*tau0):
> ddalpha0:=fullsimplify(2*dalpha0^2+432*subs(atu0,
 simplify(ddw,rulepol)))/(1-2*alpha0):
 ddbeta0:=fullsimplify(2*dbeta0^2+432*fullsimplify(subs
 (atu0,simplify(ddy,rulepol))))/(1-2*beta0): dm0:=
 fullsimplify(expand(rationalize(simplify(expand(m0/2*
 (dbeta0/beta0-dbeta0/alpha0-dalpha0/alpha0+
 dalpha0/beta0+ddalpha0/dalpha0-ddbeta0/dbeta0)))))):
> end:

>

> # This proves the Ramanujan series of ℓ ∈ {2, 3, 4} #

>

> rama:=proc(lev,dd) local diver, q,o,BB: global q0,zz0,rh,
 u0,v0,alpha0,beta0,z0,T,dv0,dalpha0,dbeta0,m0,polu,
 ddv0,ddalpha0,ddbeta0,dm0,L0,b,a,vap,uu,cuu,sigma,
 uf,dv,ddv,atu0,tau0,SDIV,FC,imtau0: o:=time(): atu0:=
 {u=u0,v(u)=v0,diff(v(u),u)=dv0,diff(diff(v(u),u),u)=
 ddv0}:  T:=u->P(u,v(u) ):
> dv:=simplify(factor(solve(diff(T(u),u),diff(v(u),u)))):
 dv0:=subs(u=u0,subs(v(u)=v0,dv)):
> ddv:=diff(dv,u): ddv0:=fullsimplify(expand(rationalize
 (solve(subs(u=u0,subs(v(u)=v0,subs(diff(v(u),u)=dv0,
 subs(diff(diff(v(u),u),u)=x,diff(diff(T(u),u),u))))),x)))):
 alpha0beta0russel(dd):
> m0:=fullsimplify(sqrt(rationalize(1/dec*dalpha0/dbeta0)
 )):
> dm0:=fullsimplify(expand(rationalize(simplify(expand
 (m0/2*(dbeta0/beta0-dbeta0/alpha0-dalpha0/alpha0+
 dalpha0/beta0+ddalpha0/dalpha0-ddbeta0/dbeta0)))))):
```

```
> L0:=simplify(evala(ro/z0)):
> b:=simplify(evala(fullsimplify((1-2*alpha0)*1/sqrt(llev)*
  (m0*dec+1/m0)))):
> a:=simplify(evala(fullsimplify(-2*alpha0*beta0*
  dm0/dalpha0*dec/sqrt(llev)))):
> print(level=llev, degree=dec): print(): print(m[0]=m0):
  print(abs(m[0])=abs(m0)): tau0:=evalc(expand
  (fullsimplify(m0*dd/sqrt(lev)*I))): assume(abs(ro/L)<1)
  : BB:=convert(sum(BC(n)*(b*n+a)/L^n,n=0..infinity),
  hypergeometric): if abs(evalf(ro/L0,100))>1 then FC:=
  round(evalf(subs(L=L0,Pi*BB))): imtau0:=Im(tau0/FC):
  print(Im(tau[0])=imtau0): print(abs(q[0])=simplify(exp
  (-2*Pi*imtau0))): print(): print(): print(Sum(BC(n)*
  (b/FC*n+a/FC)/L0^n,n=0..infinity)): print(DIVERGENT):
  print(`ANALYTIC CONTINUATION`): else FC:=1: print
  (tau[0]=tau0): print(q[0]=simplify(exp(2*Pi*I*tau0))):
  print(): print(): print(Sum(BC(n)*(b/FC*n+a/FC)/L0^n,
  n=0..infinity)= 1/Pi): print(): fi: print(): print
  (`HYPERGEOMETRIC FORM`): print(): print(subs(L=L0,
  BB/FC)=1/Pi): print(Seconds=time()-o):
> end:
>

> # This proves the Ramanujan series of ℓ=1 #
>
> chudnovsky:=proc(lev,dd) local o,BC,FC,SDIV,SD,BB;
  global u0,v0,T,w0,dv0,alpha0,beta0,z,dalpha0,dbeta0,
  m0,ddv0,ddalpha0,rootone,dy,ddy,a1,a2,
  ddbeta0,b0,a0,J0, atu0,w,y,y0,dv,ddv,dw,ddw,zp,z0,
  dm0,diver,tau0,imtau0,SDIVER:
> o:=time(): print(level=lev,degree=dd): print(): BC:=
  n->(6*n)!/((3*n)!*(n!^3)):
```

```
atu0:={u=u0,v(u)=v0,diff(v(u),u)=dv0,diff(diff(v(u),u),
u)=ddv0}:
> dvddv(dd):
w:=u^(12)/(u^(12)-16)^3: y:=v(u)^(12)/(v(u)^(12)
-16)^3:
> dw:=diff(w,u): ddw:=diff(dw,u):
dy:=subs(diff(v(u),u)=dv,diff(y,u)): ddy:=subs(diff(v
(u),u)=dv,diff(dy,u)):
w0:=fullsimplify(subs(atu0,simplify(w,rulepol))):
T:=u->P(u,v(u)):
alpha0beta0weber(lev,dd): z0:=expand(4*alpha0*
beta0): J0:=simplify(evala(fullsimplify(12^3/z0))):
b0:=simplify(evala(fullsimplify((1-2*alpha0)*(m0*
dd+1/m0)*1/sqrt(lev)))):
> a0:=simplify(evala(fullsimplify(-2*alpha0*beta0*
dm0/dalpha0*dd/sqrt(lev)))): assume(abs(1728/J)<1):
BB:=convert(sum(BC(n)*(b0*n+a0)/J^n,n=0..infinity),
hypergeometric): if abs(evalf(1728/J0,100))>1 then
diver:=1: FC:=round(evalf(subs(J=J0,Pi*BB))):
imtau0:=Im(tau0/FC): print(Im(tau[0])=imtau0): print
(abs(q[0])=simplify(exp(-2*Pi*imtau0))): print(): print
(Sum(BC(n)*(b0/FC*n+a0/FC)/J0^n,n=0..infinity)):
print(DIVERGENT): print(): print(`ANALYTIC
CONTINUATION`): else FC:=1: print(tau[0]=tau0): print
(q[0]=simplify(exp(2*Pi*I*tau0))): print(): print(): print
(Sum(BC(n)*(b0/FC*n+a0/FC)/J0^n,n=0..infinity)=1/Pi)
: fi: print():  print(`HYPERGEOMETRIC FORM`): print():
print(subs(J=J0,BB/FC)=1/Pi): print(Seconds=time()-o):
end:
>
> # Main procedure 2 #
>
> # It proves the Ramanujan series of degree d #
```

```
>
> ramapi:=proc(lev,dd,ee) global llev,dec:
> mychoose(lev,dd,ee):
> if lev=1 then return chudnovsky(lev,dd) else llev:=lev:
  dec:=dd: return rama(lev,dd) fi:
> end:

>
> # EXAMPLES OF LEVEL 1 #
>
> valuesJ(1,13);
```

$$level = 1, degree = 13$$

MODULAR EQUATION

$$\frac{432\,u^{12}}{\left(u^{12}-16\right)^{3}} = \alpha\,(1-\alpha),\ \frac{432\,v^{12}}{\left(v^{12}-16\right)^{3}} = \beta\,(1-\beta)$$

$$\left(u^{7}+13\,u^{6}v+52\,u^{5}v^{2}+78\,u^{4}v^{3}+78\,u^{3}v^{4}+52\,u^{2}v^{5}+13\,u\,v^{6}+v^{7}\right)^{2}-u\,v\left(u^{6}v^{6}-64\right)^{2}=0$$

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$\left(u^{12}-16\right)v^{4} = \left(v^{12}-16\right)u^{4},\ WMP(u, v) = 0$$

$$J = \frac{\left(u^{12}-16\right)^{3}}{u^{12}}$$

VALUES OF J

$$NUMBER = 2$$

$$3448440000 - 956448000\sqrt{13}$$

$$NUMBER = 3$$

$$3448440000 + 956448000\sqrt{13}$$

$$NUMBER = 4$$

$$3448440000 + 956448000\sqrt{13}$$

$$NUMBER = 5$$

$$3448440000 - 956448000\sqrt{13}$$

$$NUMBER = 6$$

$$SINGULARITY$$

$$1728$$

$$NUMBER = 7$$

$$SINGULARITY$$

$$1728$$

$$NUMBER = 8$$

$$SINGULARITY$$

$$1728$$

$$NUMBER = 9$$

$SINGULARITY$

1728

$NUMBER = 10$

$SINGULARITY$

1728

$NUMBER = 11$

287496

$NUMBER = 12$

$$3448440000 - 956448000\sqrt{13}$$

$NUMBER = 14$

$SINGULARITY$

1728

$NUMBER = 15$

287496

$NUMBER = 19$

287496

$NUMBER = 20$

-884736000

$NUMBER = 21$

$$-884736000$$

$$NUMBER = 22$$

$$-884736000$$

$$NUMBER = 23$$

$$287496$$

$$Seconds = 9.969 \tag{7}$$

> ramapi(1,13,3);

$$level = 1,\ degree = 13$$

$$m_0 = \frac{\sqrt{13}}{13}$$

$$|m_0| = \frac{\sqrt{13}}{13}$$

$$\tau_0 = \mathrm{I}\sqrt{13}$$

$$q_0 = \mathrm{e}^{-2\pi\sqrt{13}}$$

$$\sum_{n=0}^{\infty} \Bigg((6\,n)\,!\,\Bigg(\frac{12\sqrt{63821550 - 183885\sqrt{13}}\;n}{13225}$$

$$- \frac{1}{76760148435150} \Big(\big(339197\sqrt{13}$$

$$- 1342161\big)$$

$$\sqrt{21207206720085870+5777534127061830\sqrt{13}}\,\Big)\Big)\Big)$$

$$\Big/ \left((3\,n)!\; n!^3 \left(3448440000+956448000\sqrt{13}\right)^n \right) = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$-\frac{1}{76760148435150}\Big(\big(339197\sqrt{13}$$

$$-1342161\big)$$

$$\sqrt{21207206720085870+5777534127061830\sqrt{13}}$$

$$\mathrm{hypergeom}\left(\left[\frac{1}{6},\frac{1}{2},\frac{5}{6},-\frac{125\sqrt{13}}{20124}+\frac{1651}{1548}\right],\left[1,1,\right.\right.$$

$$\left.\left.-\frac{125\sqrt{13}}{20124}+\frac{103}{1548}\right],\frac{1728}{3448440000+956448000\sqrt{13}}\right)\Big)$$

$$=\frac{1}{\pi}$$

$$Seconds = 3.687 \quad \textbf{(8)}$$

```
> ramapi(1,13,19);
```

$$level = 1,\ degree = 13$$

$$m_0 = \frac{2}{13}+\frac{3\,I}{13}$$

$$\left|m_0\right| = \frac{\sqrt{13}}{13}$$

$$\tau_0 = -3 + 2\,\mathrm{I}$$

$$q_0 = \mathrm{e}^{-4\pi}$$

$$\sum_{n=0}^{\infty} \frac{(6\,n)!\left(\frac{84\sqrt{33}\,n}{121} + \frac{20\sqrt{33}}{363}\right)}{(3\,n)!\,n!^3\,287496^n} = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\frac{20\sqrt{33}\,\mathrm{hypergeom}\left(\left[\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{68}{63}\right], \left[\frac{5}{63}, 1, 1\right], \frac{8}{1331}\right)}{363}$$

$$= \frac{1}{\pi}$$

$$\mathit{Seconds} = 2.282 \qquad (9)$$

```
> valuesJ(1,17);
```

$$\mathit{level} = 1, \mathit{degree} = 17$$

MODULAR EQUATION

$$\frac{432\,u^{12}}{\left(u^{12} - 16\right)^3} = \alpha\,(1 - \alpha), \frac{432\,v^{12}}{\left(v^{12} - 16\right)^3} = \beta\,(1 - \beta)$$

$$\left(17\,u^8\,v^5 + 17\,u^5\,v^8 + u^9 + 119\,u^6\,v^3 + 119\,u^3\,v^6 + v^9\right.$$

$$+ 272 u^4 v + 272 u v^4\big)^2 - u v \big(-u^8 v^8 + 34 u^6 v^6$$

$$- 34 u^7 v + 340 u^4 v^4 - 34 u v^7 + 544 u^2 v^2 - 256\big)^2 = 0$$

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$\left(u^{12} - 16\right) v^4 = \left(v^{12} - 16\right) u^4, WMP(u, v) = 0$$

$$J\sim = \frac{\left(u^{12} - 16\right)^3}{u^{12}}$$

VALUES OF J

NUMBER = 2

SINGULARITY

1728

NUMBER = 3

SINGULARITY

1728

NUMBER = 4

8000

NUMBER = 5

8000

$$NUMBER = 6$$

$$\left(-5433650000 - 1317854000\sqrt{17}\right)\sqrt{26 + 10\sqrt{17}} + 10805632000\sqrt{17} + 44552760000$$

$$NUMBER = 7$$

$$COMPLEX$$

$$44552760000 - 10805632000\sqrt{17} + \mathrm{I}\left(-5433650000 + 1317854000\sqrt{17}\right)\sqrt{-26 + 10\sqrt{17}}$$

$$NUMBER = 8$$

$$287496$$

$$NUMBER = 9$$

$$SINGULARITY$$

$$1728$$

$$NUMBER = 10$$

$$3448440000 + 956448000\sqrt{13}$$

$$NUMBER = 11$$

$$3448440000 - 956448000\sqrt{13}$$

$$NUMBER = 13$$

$$8000$$

$$NUMBER = 14$$
$$287496$$

$$NUMBER = 15$$
$$DIVERGENT$$
convergent by analytic continuation
$$26125000 - 18473000\sqrt{2}$$

$$NUMBER = 16$$
$$DIVERGENT$$
convergent by analytic continuation
$$41113158120 - 29071392966\sqrt{2}$$

$$NUMBER = 17$$
$$-884736$$

$$NUMBER = 18$$
$$-884736$$

$$NUMBER = 19$$
$$-884736000$$

$$NUMBER = 20$$
$$-884736000$$

$$NUMBER = 21$$
$$-147197952000$$

$$NUMBER = 22$$
$$-147197952000$$

$$NUMBER = 23$$
$$-884736$$

$$NUMBER = 24$$
$$-884736000$$

$$NUMBER = 25$$
$$-147197952000$$

$$NUMBER = 26$$
$$3448440000 - 956448000\sqrt{13}$$

$$NUMBER = 27$$
$$26125000 + 18473000\sqrt{2}$$

$$NUMBER = 28$$
$$41113158120 + 29071392966\sqrt{2}$$

$$Seconds = 13.875 \quad (10)$$

```
> ramapi(1,17,15);
```

$$level = 1, degree = 17$$

$$m_0 = -\frac{3\,I}{17} + \frac{2\sqrt{2}}{17}$$

$$|m_0| = \frac{\sqrt{17}}{17}$$

$$\Im(\tau_0) = \frac{2\sqrt{2}}{3}$$

$$|q_0| = e^{-\frac{4\pi\sqrt{2}}{3}}$$

$$\sum_{n=0}^{\infty} \left((6\,n)!\, \left(\frac{28\,I\sqrt{19348750 + 18729360\sqrt{2}}\; n}{39675} + \frac{4\,I\sqrt{9674375 + 9364680\sqrt{2}}\,\left(182\sqrt{2} + 125\right)}{9429425} \right) \right) \Big/ \left((3\,n)!\; n!^3 \left(26125000 - 18473000\sqrt{2}\right)^n \right)$$

DIVERGENT

ANALYTIC CONTINUATION

HYPERGEOMETRIC FORM

$$\frac{4\,I}{9429425} \sqrt{9674375 + 9364680\sqrt{2}}\,\left(182\sqrt{2} + 125\right) \operatorname{hypergeom}\left(\left[\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{375\sqrt{2}}{9982} + \frac{791}{713} \right], \left[1, 1, \right.\right.$$

$$\frac{375\sqrt{2}}{9982} + \frac{78}{713}\Big], \frac{1728}{26125000 - 18473000\sqrt{2}}\Big) = \frac{1}{\pi}$$

$$Seconds = 5.156 \qquad \textbf{(11)}$$

```
> ramapi(1,17,16);
```

$$level = 1, degree = 17$$

$$m_0 = \frac{4}{17} - \frac{I}{17}$$

$$|m_0| = \frac{\sqrt{17}}{17}$$

$$\Im(\tau_0) = 1$$

$$|q_0| = e^{-2\pi}$$

$$\sum_{n=0}^{\infty} \left((6n)! \left(\frac{462\sqrt{272369841 + 174499344\sqrt{2}}\, n}{1168561} + \frac{20\,(7 + 3\sqrt{2})\sqrt{272369841 + 174499344\sqrt{2}}}{3505683} \right) \right) \Bigg/ \left((3n)!\, n!^3\, (41113158120 - 29071392966\sqrt{2})^n \right)$$

DIVERGENT

ANALYTIC CONTINUATION

HYPERGEOMETRIC FORM

$$\frac{1}{3505683}\Bigg(20\,(7+3\sqrt{2})\sqrt{272369841+174499344\sqrt{2}}\ \mathrm{hypergeom}\Bigg(\Bigg[\frac{1}{6},\frac{1}{2},\frac{5}{6},\frac{109}{99}+\frac{10\sqrt{2}}{231}\Bigg],\Bigg[1,1,\frac{10}{99}+\frac{10\sqrt{2}}{231}\Bigg],\frac{1728}{41113158120-29071392966\sqrt{2}}\Bigg)\Bigg)=\frac{1}{\pi}$$

$$Seconds = 3.906 \tag{12}$$

\>

\> ramapi(1,17,22);

$$level = 1,\ degree = 17$$

$$m_0 = \frac{\sqrt{67}}{34} - \frac{I}{34}$$

$$|m_0| = \frac{\sqrt{17}}{17}$$

$$\tau_0 = \frac{I\sqrt{67}}{2} + \frac{1}{2}$$

$$q_0 = -e^{-\pi\sqrt{67}}$$

$$\sum_{n=0}^{\infty} \frac{(6n)!\left(\frac{43617\sqrt{330}\,n}{96800}+\frac{10177\sqrt{330}}{580800}\right)}{(3n)!\,n!^{3}\,(-147197952000)^{n}} = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\frac{1}{580800}\left(10177\sqrt{330}\ \mathrm{hypergeom}\left(\left[\frac{1}{6},\frac{1}{2},\frac{5}{6},\frac{271879}{261702}\right],\left[\frac{10177}{261702},1,1\right],-\frac{1}{85184000}\right)\right) = \frac{1}{\pi}$$

$$Seconds = 8.250 \qquad (13)$$

```
> ramapi(1,17,28);
```

$$level = 1,\ degree = 17$$

$$m_0 = \frac{4}{17} - \frac{I}{17}$$

$$|m_0| = \frac{\sqrt{17}}{17}$$

$$\tau_0 = 1 + 4I$$

$$q_0 = e^{-8\pi}$$

$$\sum_{n=0}^{\infty}\left((6n)!\left(\frac{1848\sqrt{272369841-174499344\sqrt{2}}\,n}{1168561} - \frac{80\left(-7+3\sqrt{2}\right)\sqrt{272369841-174499344\sqrt{2}}}{3505683}\right)\right)\Big/$$

$$\left((3n)!\, n!^3 \left(41113158120 + 29071392966\sqrt{2}\right)^n\right) = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$-\frac{1}{3505683}\left(80\left(-7 + 3\sqrt{2}\right)\sqrt{272369841 - 174499344\sqrt{2}}\ \mathrm{hypergeom}\left(\left[\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{109}{99} - \frac{10\sqrt{2}}{231}\right], \left[1, 1, \frac{10}{99} - \frac{10\sqrt{2}}{231}\right], \frac{1728}{41113158120 + 29071392966\sqrt{2}}\right)\right) = \frac{1}{\pi}$$

$$Seconds = 14.515 \quad \textbf{(14)}$$

```
> valuesJ(1,41);
```

level = 1, *degree* = 41

MODULAR EQUATION

$$\frac{432\, u^{12}}{\left(u^{12} - 16\right)^3} = \alpha\,(1 - \alpha),\ \frac{432\, v^{12}}{\left(v^{12} - 16\right)^3} = \beta\,(1 - \beta)$$

$$\left(123\, u^{20} v^{17} + 123\, u^{17} v^{20} + 3772\, u^{18} v^{15} + 3772\, u^{15} v^{18} + 40713\, u^{19} v^{10} + 339111\, u^{16} v^{13} + 339111\, u^{13} v^{16}\right.$$

$$+ 40713 u^{10} v^{19} + 943 u^{20} v^{5} + 733531 u^{17} v^{8}$$
$$+ 3112310 u^{14} v^{11} + 3112310 u^{11} v^{14} + 733531 u^{8} v^{17}$$
$$+ 943 u^{5} v^{20} + u^{21} + 72939 u^{18} v^{3} - 6494359 u^{15} v^{6}$$
$$- 36004437 u^{12} v^{9} - 36004437 u^{9} v^{12} - 6494359 u^{6} v^{15}$$
$$+ 72939 u^{3} v^{18} + v^{21} + 15088 u^{16} v + 11736496 u^{13} v^{4}$$
$$+ 49796960 u^{10} v^{7} + 49796960 u^{7} v^{10} + 11736496 u^{4} v^{13}$$
$$+ 15088 u v^{16} + 10422528 u^{11} v^{2} + 86812416 u^{8} v^{5}$$
$$+ 86812416 u^{5} v^{8} + 10422528 u^{2} v^{11} + 15450112 u^{6} v^{3}$$
$$+ 15450112 u^{3} v^{6} + 8060928 u^{4} v + 8060928 u v^{4}\big)^{2}$$
$$- u v \big(-u^{20} v^{20} + 574 u^{18} v^{18} - 4059 u^{19} v^{13}$$
$$- 2050 u^{16} v^{16} - 4059 u^{13} v^{19} + 41 u^{20} v^{8} - 160310 u^{17} v^{11}$$
$$- 462726 u^{14} v^{14} - 160310 u^{11} v^{17} + 41 u^{8} v^{20}$$
$$- 155554 u^{18} v^{6} - 1753160 u^{15} v^{9} - 3571756 u^{12} v^{12}$$
$$- 1753160 u^{9} v^{15} - 155554 u^{6} v^{18} + 574 u^{19} v$$
$$+ 701100 u^{16} v^{4} + 20156994 u^{13} v^{7} + 45567400 u^{10} v^{10}$$
$$+ 20156994 u^{7} v^{13} + 701100 u^{4} v^{16} + 574 u v^{19}$$
$$- 2488864 u^{14} v^{2} - 28050560 u^{11} v^{5} - 57148096 u^{8} v^{8}$$
$$- 28050560 u^{5} v^{11} - 2488864 u^{2} v^{14} + 10496 u^{12}$$
$$- 41039360 u^{9} v^{3} - 118457856 u^{6} v^{6} - 41039360 u^{3} v^{9}$$
$$+ 10496 v^{12} - 16625664 u^{7} v - 8396800 u^{4} v^{4}$$

$$- 16625664\, u\, v^{7} + 37617664\, u^{2} v^{2} - 1048576 \Big)^{2} = 0$$

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$\left(u^{12} - 16\right) v^{4} = \left(v^{12} - 16\right) u^{4},\ WMP(u, v) = 0$$

$$J\sim = \frac{\left(u^{12} - 16\right)^{3}}{u^{12}}$$

VALUES OF J

NUMBER = 2

SINGULARITY

1728

NUMBER = 3

SINGULARITY

1728

NUMBER = 4

8000

NUMBER = 5

8000

NUMBER = 6

$$22015749613248 - 9845745509376\sqrt{5}$$

$$NUMBER = 7$$

$$22015749613248 - 9845745509376\sqrt{5}$$

$$NUMBER = 8$$

$$212846400 + 95178240\sqrt{5}$$

$$NUMBER = 9$$

$$212846400 - 95178240\sqrt{5}$$

$$NUMBER = 12$$

$$287496$$

$$NUMBER = 13$$

SINGULARITY

$$1728$$

$$NUMBER = 14$$

DIVERGENT

convergent by analytic continuation

$$632000 - 282880\sqrt{5}$$

$$NUMBER = 15$$

$$632000 + 282880\sqrt{5}$$

$$NUMBER = 16$$

$$19830091900536000 + 3260047059360000\sqrt{37}$$

$$NUMBER = 17$$

$$19830091900536000 - 3260047059360000\sqrt{37}$$

$$NUMBER = 18$$

$$8000$$

$$NUMBER = 19$$

DIVERGENT

convergent by analytic continuation

$$26125000 - 18473000\sqrt{2}$$

$$NUMBER = 20$$

$$287496$$

$$NUMBER = 21$$

DIVERGENT

convergent by analytic continuation

$$41113158120 - 29071392966\sqrt{2}$$

$$NUMBER = 22$$

$$22015749613248 - 9845745509376\sqrt{5}$$

$$NUMBER = 24$$

$$-884736000$$

$$NUMBER = 25$$
$$-884736000$$

$$NUMBER = 26$$
$$-26253741264076800$$

$$NUMBER = 27$$
$$-26253741264076800$$

$$NUMBER = 28$$
$$-884736000$$

$$NUMBER = 29$$
$$26125000 + 18473000\sqrt{2}$$

$$NUMBER = 30$$
$$-26253741264076800$$

$$NUMBER = 31$$
$$212846400 - 95178240\sqrt{5}$$

$$NUMBER = 32$$
$$45298879956160200 - 32031145197162000\sqrt{2} + 20258274977314560\sqrt{5} - 14324763611604600\sqrt{5}\sqrt{2}$$

$$NUMBER = 33$$

$$\left(-29145028935750 - 20608647741500\sqrt{2}\right)\sqrt{274 + 194\sqrt{2}} + 482593381088000\sqrt{2} + 682490104577000$$

$$NUMBER = 34$$

DIVERGENT

convergent by analytic continuation

$$632000 - 282880\sqrt{5}$$

$$NUMBER = 35$$

$$41113158120 + 29071392966\sqrt{2}$$

$$NUMBER = 36$$

$$19830091900536000 - 3260047059360000\sqrt{37}$$

$$Seconds = 646.625 \quad (15)$$

```
> ramapi(1,41,26);
```

$$level = 1, degree = 41$$

$$m_0 = \frac{\sqrt{163}}{82} + \frac{I}{82}$$

$$\left|m_0\right| = \frac{\sqrt{41}}{41}$$

$$\tau_0 = \frac{I\sqrt{163}}{2} - \frac{1}{2}$$

$$q_0 = -e^{-\pi\sqrt{163}}$$

$$\sum_{n=0}^{\infty} \frac{(6n)!\left(\frac{90856689\sqrt{10005}\,n}{711822400} + \frac{13591409\sqrt{10005}}{4270934400}\right)}{(3n)!\,n!^3\,(-262537412640768000)^n}$$

$$= \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\frac{1}{4270934400}\left(13591409\sqrt{10005}\ \mathrm{hypergeom}\left(\left[\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{558731543}{545140134}\right], \left[\frac{13591409}{545140134}, 1, 1\right], -\frac{1}{151931373056000}\right)\right) = \frac{1}{\pi}$$

$$\mathit{Seconds} = 17.672 \quad \textbf{(16)}$$

```
> ramapi(1,41,25);
```

$$\mathit{level} = 1, \mathit{degree} = 41$$

$$m_0 = \frac{\sqrt{43}}{82} + \frac{11\,I}{82}$$

$$|m_0| = \frac{\sqrt{41}}{41}$$

$$\tau_0 = \frac{I\sqrt{43}}{2} - \frac{11}{2}$$

$$q_0 = -e^{-\sqrt{43}\,\pi}$$

$$\sum_{n=0}^{\infty} \frac{(6\,n)!\left(\frac{2709\sqrt{15}\,n}{1600} + \frac{263\sqrt{15}}{3200}\right)}{(3\,n)!\,n!^3\,(-884736000)^n} = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\frac{1}{3200}\left(263\sqrt{15}\,\mathrm{hypergeom}\left(\left[\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{5681}{5418}\right], \left[\frac{263}{5418}, 1, 1\right], -\frac{1}{512000}\right)\right) = \frac{1}{\pi}$$

Seconds = 16.063 **(17)**

```
> #EXAMPLES OF LEVEL 2#
>
> valuesJ(2,19);
```

level = 2, *degree* = 19

MODULAR EQUATION

$$u^4 = \alpha\,\beta,\ v^4 = (1-\alpha)\,(1-\beta)$$

$$u^{10} + 1069652\,u^9 v - 473435\,u^8 v^2 + 40369104\,u^7 v^3 + 180680938\,u^6 v^4 + 277747448\,u^5 v^5 + 180680938\,u^4 v^6 + 40369104\,u^3 v^7 - 473435\,u^2 v^8 + 1069652\,u v^9 + v^{10} - 5\,u^8 + 1945520\,u^7 v + 18143244\,u^6 v^2 + 63330832\,u^5 v^3 + 92621602\,u^4 v^4 + 63330832\,u^3 v^5 + 18143244\,u^2 v^6 + 1945520\,u v^7 - 5\,v^8 + 10\,u^6 + 1028856\,u^5 v + 18054142\,u^4 v^2 + 37644400\,u^3 v^3 + 18054142\,u^2 v^4 + 1028856\,u v^5 + 10\,v^6 - 10\,u^4 + 147888\,u^3 v - 1120948\,u^2 v^2 + 147888\,u v^3 - 10\,v^4 + 5\,u^2 + 2388\,u v + 5\,v^2 - 1 = 0$$

$$Seconds = 43.812$$

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$u^4 = v^4,\ RMP(u, v) = 0$$

$$J\sim = \frac{64}{u^4}$$

VALUES OF J

$$NUMBER = 1$$

$$2509056$$

$$NUMBER = 2$$

$$\frac{234291200\left(3116448+344850\sqrt{114}\right)^{1/3}\sqrt{114}}{3357}$$

$$-\frac{327065600\left(3116448+344850\sqrt{114}\right)^{1/3}}{1119}$$

$$+\frac{257383168}{3}$$

$$-\frac{29177139200\left(3116448+344850\sqrt{114}\right)^{2/3}\sqrt{114}}{61355889}$$

$$+\frac{458208051200\left(3116448+344850\sqrt{114}\right)^{2/3}}{61355889}$$

$$NUMBER = 3$$

SINGULARITY

256

$$NUMBER = 4$$

$$45142272+10948608\sqrt{17}$$

$$NUMBER = 5$$

$$-82944$$

$$NUMBER = 6$$

$$-199148544$$

$$NUMBER = 7$$

COMPLEX

$$-\frac{22252544}{3}$$

$$-\frac{3715477504\left(2517457958+276004554\sqrt{29}\sqrt{3}\right)^{1/3}}{992793}$$

$$+\frac{1}{328545980283}\left(4322256480256\left(2517457958\right.\right.$$

$$\left.\left.+276004554\sqrt{29}\sqrt{3}\right)^{2/3}\right)$$

$$-\frac{1}{328545980283}\left(419254097920\left(2517457958\right.\right.$$

$$\left.\left.+276004554\sqrt{29}\sqrt{3}\right)^{2/3}\sqrt{29}\sqrt{3}\right)$$

$$+\frac{1}{992793}\left(627802112\left(2517457958\right.\right.$$

$$\left.\left.+276004554\sqrt{29}\sqrt{3}\right)^{1/3}\sqrt{29}\sqrt{3}\right)+I\Bigg($$

$$-\frac{1}{992793}\left(3715477504\left(2517457958\right.\right.$$

$$\left.\left.+276004554\sqrt{29}\sqrt{3}\right)^{1/3}\sqrt{3}\right)$$

$$+\frac{1}{109515326761}\left(419254097920\left(2517457958\right.\right.$$

$$\left.\left.+276004554\sqrt{29}\sqrt{3}\right)^{2/3}\sqrt{29}\right)$$

$$- \frac{1}{328545980283} \left(4322256480256 \left(2517457958 + 276004554 \sqrt{29} \sqrt{3}\right)^{2/3} \sqrt{3}\right) + \frac{1}{330931} \left(627802112 \left(2517457958 + 276004554 \sqrt{29} \sqrt{3}\right)^{1/3} \sqrt{29}\right)\Bigg)$$

$$Seconds = 49.453 \tag{18}$$

```
> ramapi(2,19,6);
```

$$level = 2, degree = 19$$

$$m_0 = \frac{\sqrt{74}}{38} + \frac{I\sqrt{2}}{38}$$

$$|m_0| = \frac{\sqrt{19}}{19}$$

$$\tau_0 = \frac{I\sqrt{37}}{2} - \frac{1}{2}$$

$$q_0 = -e^{-\sqrt{37}\,\pi}$$

$$\sum_{n=0}^{\infty} \frac{(4n)! \left(\frac{5365 n}{882} + \frac{1123}{3528}\right)}{n!^4 (-199148544)^n} = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\frac{1}{3528}\left(1123\,\mathrm{hypergeom}\left(\left[\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{22583}{21460}\right],\left[\frac{1123}{21460},1,1\right],-\frac{1}{777924}\right)\right)=\frac{1}{\pi}$$

$$Seconds = 1.594 \quad (19)$$

```
> ramapi(2,19,5);
```

$$level = 2,\ degree = 19$$

$$m_0 = \frac{\sqrt{26}}{38} + \frac{5\,I\sqrt{2}}{38}$$

$$|m_0| = \frac{\sqrt{19}}{19}$$

$$\tau_0 = \frac{I\sqrt{13}}{2} - \frac{5}{2}$$

$$q_0 = -e^{-\pi\sqrt{13}}$$

$$\sum_{n=0}^{\infty} \frac{(4n)!\left(\frac{65n}{18}+\frac{23}{72}\right)}{n!^4\,(-82944)^n} = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\frac{23\,\mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{283}{260}\right], \left[\frac{23}{260}, 1, 1\right], -\frac{1}{324}\right)}{72}$$

$$= \frac{1}{\pi}$$

$$Seconds = 1.406 \quad (20)$$

> valuesJ(2,29);

$$level = 2,\ degree = 29$$

MODULAR EQUATION

$$u^2 = \alpha\beta,\ v^2 = (1-\alpha)(1-\beta)$$

$$u^{15} + 7592191322114338383\,u^{14} v$$
$$- 1396662256859735369480\,7\,u^{13} v^{2}$$
$$+ 98434397648371904165297\,35\,u^{12} v^{3}$$
$$- 880508561952176961197135\,79\,u^{11} v^{4}$$
$$+ 264881102454684464005109883\,u^{10} v^{5}$$
$$- 339110438936155583303166131\,u^{9} v^{6}$$
$$+ 327822432596177883480907299\,u^{8} v^{7}$$
$$+ 327822432596177883480907299\,u^{7} v^{8}$$
$$- 339110438936155583303166131\,u^{6} v^{9}$$
$$+ 264881102454684464005109883\,u^{5} v^{10}$$
$$- 880508561952176961197135\,79\,u^{4} v^{11}$$
$$+ 98434397648371904165297\,35\,u^{3} v^{12}$$

$$
\begin{aligned}
&- 139666225685973536948 07\,u^{2} v^{13} \\
&+ 759219132211433838 3\,u\, v^{14} + v^{15} - 15\, u^{14} \\
&+ 441511003932968509 26\,u^{13} v \\
&- 801159466133909520128 85\,u^{12} v^{2} \\
&+ 865125449177985194290 2828\,u^{11} v^{3} \\
&- 496876859070322701834 45927\,u^{10} v^{4} \\
&+ 732541219677338639529 1122\,u^{9} v^{5} \\
&+ 912734483936820578936 054283\,u^{8} v^{6} \\
&- 749739327649930298909 414424\,u^{7} v^{7} \\
&+ 912734483936820578936 054283\,u^{6} v^{8} \\
&+ 732541219677338639529 1122\,u^{5} v^{9} \\
&- 496876859070322701834 45927\,u^{4} v^{10} \\
&+ 865125449177985194290 2828\,u^{3} v^{11} \\
&- 801159466133909520128 85\,u^{2} v^{12} \\
&+ 441511003932968509 26\,u\, v^{13} - 15\, v^{14} + 105\, u^{13} \\
&+ 111337818333508248789\,u^{12} v \\
&- 746747209361822980929 30\,u^{11} v^{2} \\
&- 135759767088963526856 8626\,u^{10} v^{3} \\
&+ 175519360241020438387 389123\,u^{9} v^{4} \\
&- 225782350367252329130 12001\,u^{8} v^{5}
\end{aligned}
$$

$$+ 685632655975243084716955860\,u^{7} v^{6}$$
$$+ 685632655975243084716955860\,u^{6} v^{7}$$
$$- 225782350367252329130120 01\,u^{5} v^{8}$$
$$+ 175519360241020438387389123\,u^{4} v^{9}$$
$$- 135759767088963526856862 6\,u^{3} v^{10}$$
$$- 746747209361822980929 30\,u^{2} v^{11}$$
$$+ 111337818333508248789\,u\,v^{12} + 105\,v^{13} - 455\,u^{12}$$
$$+ 159561487140115731244\,u^{11} v$$
$$+ 174695780104364520734642\,u^{10} v^{2}$$
$$- 11101813741117032499783812\,u^{9} v^{3}$$
$$+ 201075954132759831412608055\,u^{8} v^{4}$$
$$+ 349154723555304342205053272\,u^{7} v^{5}$$
$$+ 120037255935309848803783612\,u^{6} v^{6}$$
$$+ 349154723555304342205053272\,u^{5} v^{7}$$
$$+ 201075954132759831412608055\,u^{4} v^{8}$$
$$- 11101813741117032499783812\,u^{3} v^{9}$$
$$+ 174695780104364520734642\,u^{2} v^{10}$$
$$+ 159561487140115731244\,u\,v^{11} - 455\,v^{12} + 1365\,u^{11}$$
$$+ 143135164756761661287\,u^{10} v$$
$$+ 283824735558969676013571\,u^{9} v^{2}$$

$$
\begin{aligned}
&+ 28412836968529641849555593\, u^{8} v^{3} \\
&- 56670680773339657795710 54\, u^{7} v^{4} \\
&+ 191014637062060944039908838\, u^{6} v^{5} \\
&+ 191014637062060944039908838\, u^{5} v^{6} \\
&- 5667068077333965779571054\, u^{4} v^{7} \\
&+ 28412836968529641849555593\, u^{3} v^{8} \\
&+ 28382473555896967601357 1\, u^{2} v^{9} \\
&+ 14313516475676166 1287\, u v^{10} + 1365\, v^{11} - 3003\, u^{10} \\
&+ 834850853140014440 82\, u^{9} v \\
&- 141493436473926277032 63\, u^{8} v^{2} \\
&+ 195905607005587365219516 0\, u^{7} v^{3} \\
&- 975516250436904707339636 22\, u^{6} v^{4} \\
&- 465313755154527491729363 40\, u^{5} v^{5} \\
&- 975516250436904707339636 22\, u^{4} v^{6} \\
&+ 195905607005587365219516 0\, u^{3} v^{7} \\
&- 141493436473926277032 63\, u^{2} v^{8} \\
&+ 834850853140014440 82\, u v^{9} - 3003\, v^{10} + 5005\, u^{9} \\
&+ 318818280396548845 97\, u^{8} v \\
&- 167490774279805805587756\, u^{7} v^{2} \\
&+ 28199128087131218413412 52\, u^{6} v^{3}
\end{aligned}
$$

$$
\begin{aligned}
&- 2616731238291742368065 1674\, u^{5} v^{4} \\
&- 26167312382917423680651674\, u^{4} v^{5} \\
&+ 28199128087131218413 41252\, u^{3} v^{6} \\
&- 16749077427980580558775 6\, u^{2} v^{7} \\
&+ 3188182803965488459 7\, u v^{8} + 5005 v^{9} - 6435 u^{8} \\
&+ 784452919711303524 0\, u^{7} v \\
&- 309889524078737529604 68\, u^{6} v^{2} \\
&+ 311665434115872429962 7864\, u^{5} v^{3} \\
&+ 19246994916022486646 85678\, u^{4} v^{4} \\
&+ 311665434115872429962 7864\, u^{3} v^{5} \\
&- 309889524078737529604 68\, u^{2} v^{6} \\
&+ 784452919711303524 0\, u v^{7} - 6435 v^{8} + 6435 u^{7} \\
&+ 11960713001216746 77\, u^{6} v \\
&+ 204392743189812085111 35\, u^{5} v^{2} \\
&+ 712650893196916561319 241\, u^{4} v^{3} \\
&+ 712650893196916561319 241\, u^{3} v^{4} \\
&+ 204392743189812085111 35\, u^{2} v^{5} \\
&+ 11960713001216746 77\, u v^{6} + 6435 v^{7} - 5005 u^{6} \\
&+ 10560849049485003 4\, u^{5} v \\
&- 12955192665094062654 11\, u^{4} v^{2}
\end{aligned}
$$

$$+ 16061879206164618424444 u^3 v^3$$
$$- 12955192665094062654 11 u^2 v^4$$
$$+ 1056084904948500 34 u v^5 - 5005 v^6 + 3003 u^5$$
$$+ 4830887024610855 u^4 v$$
$$+ 141113497039864831 50 u^3 v^2$$
$$+ 141113497039864831 50 u^2 v^3$$
$$+ 4830887024610855 u v^4 + 3003 v^5 - 1365 u^4$$
$$+ 94634825344812 u^3 v - 1754805264002431 8 u^2 v^2$$
$$+ 94634825344812 u v^3 - 1365 v^4 + 455 u^3$$
$$+ 549304034965 u^2 v + 549304034965 u v^2 + 455 v^3$$
$$- 105 u^2 + 368941806 u v - 105 v^2 + 15 u + 15 v - 1 = 0$$
$$Seconds = 228.656$$

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$u^2 = v^2, RMP(u, v) = 0$$

$$J\sim = \frac{64}{u^2}$$

VALUES OF J

$$NUMBER = 1$$

2304

$$NUMBER = 2$$

$$2509056$$

$$NUMBER = 3$$

$$24591257856$$

$$NUMBER = 4$$

$$347648256 + 141926400\sqrt{6}$$

$$NUMBER = 5$$

$$DIVERGENT$$

convergent by analytic continuation

$$COMPLEX$$

$$\frac{30330982400\left(611668 + 478686\sqrt{2}\right)^{1/3}}{2191}$$

$$- \frac{425492918272\left(611668 + 478686\sqrt{2}\right)^{2/3}}{685783}$$

$$+ \frac{228395501568\left(611668 + 478686\sqrt{2}\right)^{2/3}\sqrt{2}}{685783}$$

$$+ 3539618048$$

$$- \frac{46643970048\left(611668 + 478686\sqrt{2}\right)^{1/3}\sqrt{2}}{2191} + \mathrm{I}\left(\right.$$

$$- \frac{30330982400\sqrt{3}\left(611668 + 478686\sqrt{2}\right)^{1/3}}{2191}$$

$$- \frac{425492918272\sqrt{3}\,(611668 + 478686\sqrt{2})^{2/3}}{685783}$$

$$+ \frac{228395501568\sqrt{3}\,(611668 + 478686\sqrt{2})^{2/3}\sqrt{2}}{685783}$$

$$+ \frac{46643970048\,(611668 + 478686\sqrt{2})^{1/3}\sqrt{3}\sqrt{2}}{2191}\Bigg)$$

$NUMBER = 6$

-12288

$NUMBER = 7$

$-34412544 - 5990400\sqrt{33}$

$NUMBER = 8$

DIVERGENT

convergent by analytic continuation

$-1776660480 + 671514624\sqrt{7}$

$NUMBER = 9$

$-9995065344 - 1323878400\sqrt{57}$

$Seconds = 229.734$ **(21)**

```
> ramapi(2,29,3);
```

$level = 2, degree = 29$

$$m_0 = \frac{\sqrt{29}}{29}$$

$$|m_0| = \frac{\sqrt{29}}{29}$$

$$\tau_0 = \frac{I}{2}\sqrt{58}$$

$$q_0 = e^{-\pi\sqrt{58}}$$

$$\sum_{n=0}^{\infty} \frac{(4n)!\left(\frac{52780\sqrt{2}\,n}{9801} + \frac{2206\sqrt{2}}{9801}\right)}{n!^4\, 24591257856^n} = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\frac{1}{9801}\left(2206\sqrt{2}\ \text{hypergeom}\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{27493}{26390}\right], \left[\frac{1103}{26390}, 1, 1\right], \frac{1}{96059601}\right)\right) = \frac{1}{\pi}$$

Seconds = 1.141 **(22)**

```
> ramapi(2,29,9);
```

level = 2, *degree* = 29

$$m_0 = \frac{\sqrt{114}}{58} + \frac{I\sqrt{2}}{58}$$

$$\left|m_0\right| = \frac{\sqrt{29}}{29}$$

$$\tau_0 = \frac{I\sqrt{57}}{2} - \frac{1}{2}$$

$$q_0 = -e^{-\sqrt{57}\,\pi}$$

$$\sum_{n=0}^{\infty} \left((4n)! \left(\left(\left(-\frac{65\sqrt{19}}{1568} + \frac{20995\sqrt{3}}{4704} \right) n + \frac{7331\sqrt{3}}{18816} - \frac{513\sqrt{19}}{6272} \right) \right) \Big/ \left(n!^4 \left(-9995065344 - 1323878400\sqrt{57} \right)^n \right) \right) = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\left(\frac{7331\sqrt{3}}{18816} - \frac{513\sqrt{19}}{6272} \right) \mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, -\frac{101\sqrt{57}}{17290} + \frac{1977}{1820} \right], \left[1, 1, -\frac{101\sqrt{57}}{17290} + \frac{157}{1820} \right], \frac{256}{-9995065344 - 1323878400\sqrt{57}} \right) = \frac{1}{\pi}$$

Seconds = 14.969 **(23)**

```
> ramapi(2,29,8);
```

$$level = 2, degree = 29$$

$$m_0 = \frac{7\sqrt{2}}{58} + \frac{3\,\mathrm{I}\sqrt{2}}{58}$$

$$|m_0| = \frac{\sqrt{2}\,\sqrt{58}}{58}$$

$$\Im(\tau_0) = \frac{7}{10}$$

$$|q_0| = \mathrm{e}^{-\frac{7\pi}{5}}$$

$$\sum_{n=0}^{\infty} \left((4n)! \left(\frac{\mathrm{I}\sqrt{-26019 + 14168\sqrt{7}}\; n}{180} + \frac{\mathrm{I}\sqrt{-3717 + 2024\sqrt{7}}\,\left(1344 + 773\sqrt{7}\right)}{1049040} \right) \right) \Big/ \left(n!^4 \left(-1776660480 + 671514624\sqrt{7}\right)^n \right)$$

DIVERGENT

ANALYTIC CONTINUATION

HYPERGEOMETRIC FORM

$$\frac{\mathrm{I}}{1049040} \sqrt{-3717 + 2024\sqrt{7}}\, \Big(1344 + 773\sqrt{7}\Big)\, \mathrm{hypergeom}\left(\left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{48\sqrt{7}}{1457} + \frac{6601}{5828} \right], \right.$$

$$\left[1, 1, \frac{48\sqrt{7}}{1457} + \frac{773}{5828}\right],$$

$$\left.\frac{256}{-1776660480 + 671514624\sqrt{7}}\right) = \frac{1}{\pi}$$

$$Seconds = 5.000 \quad (24)$$

```
> valuesJ(2,3);
```

$$level = 2, degree = 3$$

MODULAR EQUATION

$$u^4 = \alpha\beta, v^4 = (1-\alpha)(1-\beta)$$

$$u^2 + 4uv + v^2 - 1 = 0$$

$$Seconds = 0.422$$

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$u^4 = v^4, RMP(u, v) = 0$$

$$J\sim = \frac{64}{u^4}$$

VALUES OF J

$$NUMBER = 1$$

$$2304$$

$$NUMBER = 2$$

SINGULARITY

$$256$$

$$NUMBER = 3$$

$$-1024$$

$$Seconds = 0.547 \tag{25}$$

```
> ramapi(2,3,1);
```

$$level = 2, degree = 3$$

$$m_0 = \frac{\sqrt{3}}{3}$$

$$|m_0| = \frac{\sqrt{3}}{3}$$

$$\tau_0 = \frac{I}{2}\sqrt{6}$$

$$q_0 = e^{-\pi\sqrt{6}}$$

$$\sum_{n=0}^{\infty} \frac{(4n)!\left(\frac{4\sqrt{3}\,n}{3} + \frac{\sqrt{3}}{6}\right)}{n!^4\, 2304^n} = \frac{1}{\pi}$$

$$HYPERGEOMETRIC\ FORM$$

$$\frac{\sqrt{3}\,\mathrm{hypergeom}\left(\left[\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{9}{8}\right],\left[\frac{1}{8},1,1\right],\frac{1}{9}\right)}{6}=\frac{1}{\pi}$$

Seconds = 0.735 **(26)**

> #EXAMPLES OF LEVEL 3#

>

> valuesJ(3,23):

level = 3, *degree* = 23

MODULAR EQUATION

$$u^6 = \alpha\beta,\ v^6 = (1-\alpha)(1-\beta)$$

$$-u^8 + 606u^7v - 1201u^6v^2 - 450u^5v^3 + 2460u^4v^4 - 450u^3v^5 - 1201u^2v^6 + 606uv^7 - v^8 + 4u^6 + 828u^5v - 348u^4v^2 - 936u^3v^3 - 348u^2v^4 + 828uv^5 + 4v^6 - 6u^4 + 657u^3v - 642u^2v^2 + 657uv^3 - 6v^4 + 4u^2 + 96uv + 4v^2 - 1 = 0$$

Seconds = 8.047

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$u^6 = v^6,\ RMP(u, v) = 0$$

$$J\sim = \frac{27}{u^6}$$

VALUES OF J

$$NUMBER = 1$$

$$8983440 + 5186592\sqrt{3} + \frac{77194296\sqrt{-845 + 492\sqrt{3}}}{23} + \frac{44568144\sqrt{3}\sqrt{-845 + 492\sqrt{3}}}{23}$$

$$NUMBER = 2$$

$$-311040\sqrt{13} - 1121472$$

$$NUMBER = 3$$

$$83808 - 48600\sqrt{3}$$

$$NUMBER = 4$$

$$5553738 + 3206385\sqrt{3}$$

$$NUMBER = 5$$

DIVERGENT

convergent by analytic continuation

$$-27$$

$$NUMBER = 6$$

$$3375$$

$$NUMBER = 7$$

$$-1728$$

$$NUMBER = 8$$

$$-27000000$$

$$Seconds = 10.515 \tag{27}$$

```
> ramapi(3,23,8);
```

$$level = 3, degree = 23$$

$$m_0 = \frac{I\sqrt{3}}{46} + \frac{\sqrt{89}}{46}$$

$$|m_0| = \frac{\sqrt{23}}{23}$$

$$\tau_0 = -\frac{1}{2} + \frac{I\sqrt{267}}{6}$$

$$q_0 = -e^{-\frac{\sqrt{267}\,\pi}{3}}$$

$$\sum_{n=0}^{\infty} \frac{(3n)!\,(2n)!\left(\frac{4717\sqrt{3}\,n}{1500} + \frac{827\sqrt{3}}{4500}\right)}{n!^5\,(-27000000)^n} = \frac{1}{\pi}$$

$$HYPERGEOMETRIC\ FORM$$

$$\frac{1}{4500}\Big(827\sqrt{3}\ \mathrm{hypergeom}\Big(\Big[\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{14978}{14151}\Big], \Big[\frac{827}{14151},$$

$$1,\ 1\Big],\ -\frac{1}{250000}\Big)\Big) = \frac{1}{\pi}$$

$$Seconds = 1.703 \tag{28}$$

```
> ramapi(3,23,7);
```

$$level = 3,\ degree = 23$$

$$m_0 = -\frac{5\,I\sqrt{3}}{46} + \frac{\sqrt{17}}{46}$$

$$|m_0| = \frac{\sqrt{23}}{23}$$

$$\tau_0 = \frac{5}{2} + \frac{I\sqrt{51}}{6}$$

$$q_0 = -e^{-\frac{\pi\sqrt{51}}{3}}$$

$$\sum_{n=0}^{\infty} \frac{(3\,n)!\,(2\,n)!\left(\frac{17\sqrt{3}\,n}{12} + \frac{7\sqrt{3}}{36}\right)}{n!^{5}\,(-1728)^{n}} = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\frac{7\sqrt{3}\ \mathrm{hypergeom}\left(\left[\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{58}{51}\right], \left[\frac{7}{51}, 1, 1\right], -\frac{1}{16}\right)}{36}$$

$$= \frac{1}{\pi}$$

$$Seconds = 1.453 \qquad (29)$$

```
> ramapi(3,23,5);
```

$$level = 3, degree = 23$$

$$m_0 = -\frac{I\sqrt{3}}{23} + \frac{2\sqrt{5}}{23}$$

$$|m_0| = \frac{\sqrt{23}}{23}$$

$$\Im(\tau_0) = \frac{\sqrt{15}}{6}$$

$$|q_0| = e^{-\frac{\pi\sqrt{15}}{3}}$$

$$\sum_{n=0}^{\infty} \frac{(3n)!\,(2n)!\left(\frac{5\sqrt{3}\,n}{3} + \frac{4\sqrt{3}}{9}\right)}{n!^5\,(-27)^n}$$

DIVERGENT

ANALYTIC CONTINUATION

HYPERGEOMETRIC FORM

$$\frac{4\sqrt{3}\,\mathrm{hypergeom}\left(\left[\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{19}{15}\right], \left[\frac{4}{15}, 1, 1\right], -4\right)}{9} = \frac{1}{\pi}$$

$$Seconds = 1.500 \quad (30)$$

```
> valuesJ(3,5);
```

$$level = 3, degree = 5$$

MODULAR EQUATION

$$u^6 = \alpha\beta, v^6 = (1-\alpha)(1-\beta)$$

$$u^2 + 3uv + v^2 - 1 = 0$$

$$Seconds = 0.422$$

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$u^6 = v^6, RMP(u, v) = 0$$

$$J\sim = \frac{27}{u^6}$$

VALUES OF J

$$NUMBER = 1$$

$$3375$$

$$NUMBER = 2$$

DIVERGENT

convergent by analytic continuation

$$-27$$

$$NUMBER = 3$$

$$216$$

$$NUMBER = 4$$

$$-1728$$

$$Seconds = 0.657 \quad (31)$$

```
> ramapi(3,5,2);
```

$$level = 3, degree = 5$$

$$m_0 = \frac{\sqrt{5}}{5}$$

$$|m_0| = \frac{\sqrt{5}}{5}$$

$$\Im(\tau_0) = \frac{\sqrt{15}}{6}$$

$$|q_0| = e^{-\frac{\pi\sqrt{15}}{3}}$$

$$\sum_{n=0}^{\infty} \frac{(3n)!\,(2n)!\left(\frac{5\sqrt{3}\,n}{3} + \frac{4\sqrt{3}}{9}\right)}{n!^5\,(-27)^n}$$

DIVERGENT

ANALYTIC CONTINUATION

HYPERGEOMETRIC FORM

$$\frac{4\sqrt{3}\,\mathrm{hypergeom}\left(\left[\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{19}{15}\right],\left[\frac{4}{15},1,1\right],-4\right)}{9}=\frac{1}{\pi}$$

$$Seconds = 2.109 \tag{32}$$

```
> #EXAMPLES OF LEVEL 4#
>
> valuesJ(4,19);
```

$$level = 4,\ degree = 19$$

MODULAR EQUATION

$$u^4 = \alpha\beta,\ v^4 = (1-\alpha)(1-\beta)$$

$$u^5 + 5u^4v + 10u^3v^2 + 10u^2v^3 + 5uv^4 + v^5 - 5u^4 + 92u^3v - 62u^2v^2 + 92uv^3 - 5v^4 + 10u^3 + 62u^2v + 62uv^2 + 10v^3 - 10u^2 + 92uv - 10v^2 + 5u + 5v - 1 = 0$$

$$Seconds = 1.250$$

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$u^4 = v^4,\ RMP(u, v) = 0$$

$$J{\sim} = \frac{16}{u^4}$$

VALUES OF J

$$NUMBER = 1$$

$$256$$

$$NUMBER = 2$$

$$295168 + 198912 \left(46 + 6\sqrt{57}\right)^{1/3} - 17664 \left(46 + 6\sqrt{57}\right)^{1/3}\sqrt{57} + 194304 \left(46 + 6\sqrt{57}\right)^{2/3} - 23808 \left(46 + 6\sqrt{57}\right)^{2/3}\sqrt{57}$$

$$NUMBER = 3$$

$$96256 + 43008\sqrt{5}$$

$$NUMBER = 4$$

$$-64$$

$$NUMBER = 5$$

DIVERGENT

convergent by analytic continuation

$$-10304 + 4608\sqrt{5}$$

$$NUMBER = 6$$

$$-307264 - 125440\sqrt{6}$$

$$Seconds = 1.875 \tag{33}$$

```
> ramapi(4,19,6);
```

$$level = 4, degree = 19$$

$$m_0 = \frac{I}{19} + \frac{3\sqrt{2}}{19}$$

$$|m_0| = \frac{\sqrt{19}}{19}$$

$$\tau_0 = -\frac{1}{2} + \frac{3\,I\sqrt{2}}{2}$$

$$q_0 = -e^{-3\pi\sqrt{2}}$$

$$\sum_{n=0}^{\infty} \frac{(2n)!^3 \left(\left(210 - 84\sqrt{6}\right)n + \frac{177}{2} - 36\sqrt{6}\right)}{n!^6 \left(-307264 - 125440\sqrt{6}\right)^n} = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\left(\frac{177}{2} - 36\sqrt{6}\right) \mathrm{hypergeom}\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{4} - \frac{\sqrt{6}}{14}\right], \left[1, 1, \frac{1}{4} - \frac{\sqrt{6}}{14}\right], \frac{64}{-307264 - 125440\sqrt{6}}\right) = \frac{1}{\pi}$$

$$Seconds = 2.671 \qquad (34)$$

```
> ramapi(4,19,2);
```

$$level = 4,\ degree = 19$$

$$m_0 = \frac{\sqrt{19}}{19}$$

$$|m_0| = \frac{\sqrt{19}}{19}$$

$$\tau_0 = \frac{I}{2}\sqrt{19}$$

$$q_0 = e^{-\pi\sqrt{19}}$$

$$\sum_{n=0}^{\infty} \left((2n)!^3 \left(\left(\frac{(6\sqrt{57}-38)(46+6\sqrt{57})^{1/3}}{8} + \frac{19}{2} + \frac{(-3\sqrt{57}+19)(46+6\sqrt{57})^{2/3}}{8} \right) n + \frac{35}{12} + \frac{(7\sqrt{57}-47)(46+6\sqrt{57})^{1/3}}{24} + \frac{(-5\sqrt{57}+29)(46+6\sqrt{57})^{2/3}}{48} \right) \middle/ \left(n!^6 \left((198912-17664\sqrt{57})(46+6\sqrt{57})^{1/3} + 295168 + (194304-23808\sqrt{57})(46+6\sqrt{57})^{2/3} \right)^n \right) \right) = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\mathrm{hypergeom}\Bigg(\Bigg[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{\left(4\sqrt{57}-44\right)\left(46+6\sqrt{57}\right)^{1/3}}{684} + \frac{431}{342} + \frac{\left(5\sqrt{57}-41\right)\left(46+6\sqrt{57}\right)^{2/3}}{684}\Bigg], \Bigg[1, 1, \frac{\left(4\sqrt{57}-44\right)\left(46+6\sqrt{57}\right)^{1/3}}{684} + \frac{89}{342} + \frac{\left(5\sqrt{57}-41\right)\left(46+6\sqrt{57}\right)^{2/3}}{684}\Bigg], 64\Big/\Big(\big(198912 - 17664\sqrt{57}\big)\left(46+6\sqrt{57}\right)^{1/3} + 295168 + \big(194304 - 23808\sqrt{57}\big)\left(46+6\sqrt{57}\right)^{2/3}\Big)\Bigg) \left(\frac{\left(14\sqrt{57}-94\right)\left(46+6\sqrt{57}\right)^{1/3}}{48} + \frac{35}{12} + \frac{\left(-5\sqrt{57}+29\right)\left(46+6\sqrt{57}\right)^{2/3}}{48}\right) = \frac{1}{\pi}$$

$$\mathit{Seconds} = 12.953 \tag{35}$$

> valuesJ(4,7);

$$\mathit{level} = 4, \mathit{degree} = 7$$

MODULAR EQUATION

$$u^8 = \alpha\beta, v^8 = (1-\alpha)(1-\beta)$$

$$u + v - 1 = 0$$

$$\mathit{Seconds} = 0.422$$

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$u^8 = v^8, RMP(u, v) = 0$$

$$J\sim = \frac{16}{u^8}$$

VALUES OF J

$NUMBER = 1$

$$4096$$

$NUMBER = 2$

$$256$$

$NUMBER = 3$

$$-1088 - 768\sqrt{2}$$

$Seconds = 0.562$ **(36)**

```
> ramapi(4,7,3);
```

$level = 4, degree = 7$

$$m_0 = \frac{\sqrt{6}}{7} + \frac{I}{7}$$

$$|m_0| = \frac{\sqrt{7}}{7}$$

$$\tau_0 = \frac{I\sqrt{6}}{2} - \frac{1}{2}$$

$$q_0 = -e^{-\pi\sqrt{6}}$$

$$\sum_{n=0}^{\infty} \frac{(2n)!^3\left(\left(-6+6\sqrt{2}\right)n - \frac{5}{2} + 2\sqrt{2}\right)}{n!^6\left(-1088-768\sqrt{2}\right)^n} = \frac{1}{\pi}$$

HYPERGEOMETRIC FORM

$$\left(-\frac{5}{2}+2\sqrt{2}\right)\text{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{5}{4}-\frac{\sqrt{2}}{12}\right],\left[1,1,\frac{1}{4}-\frac{\sqrt{2}}{12}\right],\frac{64}{-1088-768\sqrt{2}}\right) = \frac{1}{\pi}$$

$$Seconds = 2.110 \quad \textbf{(37)}$$

> valuesJ(4,3);

$$level = 4,\ degree = 3$$

MODULAR EQUATION

$$u^4 = \alpha\beta,\ v^4 = (1-\alpha)(1-\beta)$$

$$u + v - 1 = 0$$

$$Seconds = 0.281$$

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$u^4 = v^4, RMP(u, v) = 0$$

$$J\sim = \frac{16}{u^4}$$

VALUES OF J

NUMBER = 1

256

NUMBER = 2

−64

Seconds = 0.328 **(38)**

```
> valuesJ(4,31);
```

level = 4, *degree* = 31

MODULAR EQUATION

$$u^8 = \alpha\beta, v^8 = (1-\alpha)(1-\beta)$$

$$-u^4 + 4u^3v - 6u^2v^2 + 4uv^3 - v^4 + 4u^3 + 4v^3 - 6u^2 - 6v^2 + 4u + 4v - 1 = 0$$

Seconds = 0.735

MAPLE FINDS THE SOLUTIONS OF THE SYSTEM

$$u^8 = v^8, RMP(u, v) = 0$$

$$J\sim = \frac{16}{u^8}$$

$$VALUES\ OF\ J$$

$$NUMBER = 1$$

$$13164544 - 2245632 \left(108 + 12\sqrt{93}\right)^{1/3} + 1863168 \left(108 + 12\sqrt{93}\right)^{2/3} - 156160 \left(108 + 12\sqrt{93}\right)^{2/3}\sqrt{93} + 457728 \left(108 + 12\sqrt{93}\right)^{1/3}\sqrt{93}$$

$$NUMBER = 2$$

$$96256 + 43008\sqrt{5}$$

$$NUMBER = 3$$

$$256$$

$$NUMBER = 5$$

DIVERGENT

convergent by analytic continuation

$$-1088 + 768\sqrt{2}$$

$$NUMBER = 6$$

DIVERGENT

convergent by analytic continuation

$$-1254464 + 887040\sqrt{2}$$

$$Seconds = 3.282 \tag{39}$$

```
> ramapi(4,31,5);
```

$$level = 4,\ degree = 31$$

$$m_0 = \frac{\sqrt{6}}{31} - \frac{5\,I}{31}$$

$$|m_0| = \frac{\sqrt{31}}{31}$$

$$\Im(\tau_0) = \frac{\sqrt{6}}{6}$$

$$|q_0| = e^{-\frac{\pi\sqrt{6}}{3}}$$

$$\sum_{n=0}^{\infty} \frac{(2n)!^3 \left(\frac{(6+6\sqrt{2})\,n}{3} + \frac{5}{6} + \frac{2\sqrt{2}}{3} \right)}{n!^6 \left(-1088 + 768\sqrt{2}\right)^n}$$

$$DIVERGENT$$

$$ANALYTIC\ CONTINUATION$$

$$HYPERGEOMETRIC\ FORM$$

$$\frac{1}{3}\left(\left(\frac{5}{2} + 2\sqrt{2}\right) \mathrm{hypergeom}\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{4} + \frac{\sqrt{2}}{12}\right], \left[1, 1, \frac{1}{4} + \frac{\sqrt{2}}{12}\right], \frac{64}{-1088 + 768\sqrt{2}} \right) \right) = \frac{1}{\pi}$$

$$Seconds = 3.968 \qquad \textbf{(40)}$$