A NEW FORMULA FOR COMPUTING THE CATALAN CONSTANT

JESÚS GUILLERA

Let

$$U(n,k) = \frac{(1)_n^3}{\left(\frac{1}{2}+k\right)_n^3}(-1)^n(-1)^k.$$

The function

$$F(n,k) = U(n,k)\frac{n+2k+1}{(2n+2k+1)^3},$$

has a companion $G(n,k)$ (it can be automatically found using the Wilf-Zeilberger algorithm):

$$G(n,k) = U(n,k)\frac{4n+2k+3}{2(2n+2k+1)^3},$$

such that the pair formed by $F(n,k)$ and $G(n,k)$ is a WZ pair. This means that

$$G(n,k+1) - G(n,k) = F(n+1,k) - F(n,k).$$

An important property due to Wilf and Zeilberger is

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{k=0}^{\infty} F(0,k).$$

Applying it, we obtain

$$\sum_{n=0}^{\infty}(-1)^n \frac{(1)_n^3}{\left(\frac{1}{2}\right)_n^3}\frac{4n+3}{2(2n+1)^3} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = G.$$

where G is the Catalan constant. Another important property is that all the functions $F(in, k+jn)$, where $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ have companions. Using the Wilf-Zeilberger algorithm to get the companion $G_2(n,k)$ of $F_2(n,k) = F(2n, k+n)$, and applying the identity

$$\sum_{n=0}^{\infty} G_2(n,0) = \sum_{k=0}^{\infty} F_2(0,k) = \sum_{k=0}^{\infty} F(0,k) = G,$$

we obtain

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 (1)_n^3}{\left(\frac{1}{6}\right)_n^3 \left(\frac{5}{6}\right)_n^3}\left(\frac{-64}{19683}\right)^n \frac{165 - 3160n + 21240n^2 - 57184n^3 + 45136n^4}{1024\, n^3(1-2n)^3} = G,$$

which converge to G giving approximately

$$\log_{10}\frac{19683}{64} \approx 2.4879$$

digits per term.

Department of Mathematics, University of Zaragoza, 50009 Zaragoza, SPAIN
E-mail address: jguillera@gmail.com

Date: May 8, 2019.