ON A FORMULA FOR $\zeta(5)$ CONJECTURED BY ALMKVIST

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Abstract. We justify an expansion conjectured by Gert Almkvist with related supercongruences.

G. Almkvist conjectured in the year 2009 an expansion which is equivalent to the following one:

$$64^x \frac{\left(\frac{1}{2}\right)_x^7}{(1)_x^7} \sum_{n=1}^{\infty} \left(\frac{1}{64}\right)^n \frac{(1-x)_n^7}{\left(\frac{1}{2}-x\right)_n^7} \frac{84(-n+x)^3+88(-n+x)^2+32(-n+x)+4}{(-n+x)^7}$$
$$= 45\zeta(4)+186\zeta(5)x+\mathcal{O}(x^2).$$

We can justify it with the following p-adic supercongruences

$$S(\nu p)-S(\nu)\left(\frac{1}{p}\right)p^3-45T(\nu)\nu^7p^7\zeta_p(4)-186T(\nu)\nu^8p^8\zeta_p(5)\equiv 0 \pmod{p^9},$$

taking into account that $(1/p)=1$, $\zeta_p(4)=0$, and $\zeta_p(5)\equiv\zeta(6-p) \pmod{p}$, and applying an heuristic argument that consist of the possibility of replacing some objects with p-adic analogues (which we conjecture is correct). We know that the functions $S(N)$ and $T(\nu)$ are defined by

$$S(N)=\sum_{n=0}^{N-1} \frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7} 64^n(84n^3+88n+32n+4), \quad T(\nu)=64^\nu \frac{\left(\frac{1}{2}\right)_\nu^7}{(1)_\nu^7}.$$

The coefficient of x leads to the following evaluation

$$\sum_{n=1}^{\infty} \frac{(1)_n^7}{\left(\frac{1}{2}\right)_n^7}\left(\frac{1}{64}\right)^n \left(\frac{4(84n^3-110n^2+48n-7)}{n^8}+\frac{56(7n^2-5n+1)(3n-1)(H_{2n}-H_n)}{n^7}\right)$$
$$=4\pi^4\log(2)+186\zeta(5).$$

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