Expansions related to Ramanujan series and alike

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1 Series for $1/\pi$

We conjectured in [2] and later proved in [3] that if

$$\sum_{n=0}^{\infty} u^n B_n = \frac{1}{\pi}, \qquad \text{with} \quad u=1 \quad \text{or} \quad u=-1 \tag{1}$$

is a Ramanujan-type series for $1/\pi$, then as $x \to 0$ the following expansion holds

$$\sum_{n=0}^{\infty} u^n B_{n+x} = \frac{1}{\pi} - \frac{k\pi}{2} x^2 + O(x^3), \tag{2}$$

in which k is rational. It is very curious that for the series like those in the chains of [4], we can also conjecture an expansion of the form (2).

2 Series for $1/\pi^2$

We conjectured in [3] that if

$$\sum_{n=0}^{\infty} u^n B_n = \frac{1}{\pi^2}, \qquad \text{with} \quad u=1 \quad \text{or} \quad u=-1 \tag{3}$$

is a Ramanujan-like series for $1/\pi^2$ (see [1]), then as $x \to 0$ the following expansion holds

$$\sum_{n=0}^{\infty} u^n B_{n+x} = \frac{1}{\pi^2} - k\frac{x^2}{2!} + j\pi^2 \frac{x^4}{4!} + O(x^5), \tag{4}$$

in which k and j are rational. In [2], we conjectured a weaker expansion. It is very curious, that our new unproved series

$$\sum_{n=0}^{\infty} \frac{1}{64^n} \frac{\left(\frac{1}{4}\right)_n^3 \left(\frac{3}{4}\right)_n^3}{\left(\frac{1}{2}\right)_n (1)_n^5} \frac{672n^3 + 472n^2 + 78n + \frac{9}{2}}{n + \frac{1}{2}} = \frac{64\sqrt{2}}{\pi^2}, \tag{5}$$

leads also to an expansion of the form (4).

3 Series for $1/\pi^3$

We now conjecture that if

$$\sum_{n=0}^{\infty} u^n B_n = \frac{1}{\pi^3}, \qquad \text{with} \quad u = 1 \quad \text{or} \quad u = -1 \tag{6}$$

is a Ramanujan-like series for $1/\pi^3$ (the only known example, see [1], is unproved and was discovered by B. Gourevitch), then as $x \to 0$ the following expansion holds

$$\sum_{n=0}^{\infty} u^n B_{n+x} = \frac{1}{\pi^3} - k\frac{1}{\pi}\frac{x^2}{2!} + j\pi\frac{x^4}{4!} - l\pi^3\frac{x^6}{6!} + O(x^7), \tag{7}$$

in which k, j and l are rational.

References

[1] J. Guillera, About a new kind of Ramanujan type series. *Exp. Math.* **12** (2003) 507-510.

[2] J. Guillera, A new method to obtain series for $1/\pi$ and $1/\pi^2$, *Exp. Math.* **15** (2006) 83-89.

[3] J. Guillera, A Matrix form of Ramanujan-type series for $1/\pi$. Submitted for publication.

In my personal journal:

[4] J. Guillera, Chains of series for $1/\pi$ associated to WZ-pairs.

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