COLLECTION OF RAMANUJAN-LIKE SERIES FOR $1/\pi^2$

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Abstract. We write a list of Ramanujan-like series for $1/\pi^2$ and give references for their discovery or proof.

1. List of formulas

Formulas (1) and (2) are proved in [1] and [3] by the WZ-method and formula (3) is proved in [3], again by the WZ-method.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (820n^2+180n+13) = \frac{128}{\pi^2}, \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{1}{2^{4n}} (120n^2+34n+3) = \frac{32}{\pi^2}, \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \frac{(-1)^n}{2^{2n}} (20n^2+8n+1) = \frac{8}{\pi^2}. \tag{3}$$

Formulas (4), (5), (6) and (7) were discovered by the PSLQ algorithm, see [2]. They remain unproven formulas.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (1640n^2+278n+15) = \frac{256\sqrt{3}}{\pi^2}, \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \frac{(-1)^n}{48^n} (252n^2+63n+5) = \frac{48}{\pi^2}, \tag{5}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{80^{3n}} (5418n^2+693n+29) = \frac{128\sqrt{5}}{\pi^2}, \tag{6}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{(1)_n^5} \frac{1}{7^{4n}} (1920n^2+304n+15) = \frac{56\sqrt{7}}{\pi^2}. \tag{7}$$

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Formula (8) was proved in [5] by the WZ-method.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{3}{4}\right)^{3n} (74n^2+27n+3) = \frac{48}{\pi^2}. \tag{8}$$

Formula (9) was discovered by the method described in the last section of [4]: for $k = 8/3$ we numerically guess that $j = 112$. In [4] we conjectured that if k and j are rational then z, a, b, c are algebraic. For $k = 8/3$ we obtain numerical approximations of z, a, b and c. With the help of the PSLQ algorithm we guess that z, a, b, $c \in \mathbb{Q}(\sqrt{5})$.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{15\sqrt{5}-33}{2}\right)^{3n} \times \left[(1220/3 - 180\sqrt{5})n^2 + (303 - 135\sqrt{5})n + (56 - 25\sqrt{5}) \right] = \frac{1}{\pi^2}. \tag{9}$$

It remains an unproven formula.

References

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