Some challenging formulas for π

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Abstract. In this note we show some Ramanujan-like series for π of hypergeometric type which remain unproven.

1. Introduction

In the last decade I have discovered and proved by the WZ-method four series for $1/\pi^2$ of a new kind, similar to those found for Ramanujan for the constant $1/\pi$. In this note we show a list of hypergeometric series of the same style which remain unproven.

Formulae for $1/\pi^2$ by Jesús Guillera (Spain)

October 2003 in Exp. Math.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{2^{10n}} (1640n^2 + 278n + 15) = \frac{256\sqrt{3}}{\pi^2}, \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{(-1)^n}{48^n} (252n^2 + 63n + 5) = \frac{48}{\pi^2}, \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \frac{(-1)^n}{80^{3n}} (5418n^2 + 693n + 29) = \frac{128\sqrt{5}}{\pi^2}, \tag{3}$$

and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{8}\right)_n \left(\frac{3}{8}\right)_n \left(\frac{5}{8}\right)_n \left(\frac{7}{8}\right)_n}{(1)_n^5} \frac{1}{7^{4n}} (1920n^2 + 304n + 15) = \frac{56\sqrt{7}}{\pi^2}. \tag{4}$$

J. Guillera, About a new kind of Ramanujan type series. *Exp. Math.* **12**, 507-510, (2003).

July 2010 in arXiv, March 2012 in Exp. Math.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} (-1)^n \left(\frac{3}{4}\right)^{6n} (1930n^2 + 549n + 45) = \frac{384}{\pi^2}, \tag{5}$$

and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{15\sqrt{5}-33}{2}\right)^{3n} \times \tag{6}$$

$$\left[(1220/3 - 180\sqrt{5})n^2 + (303 - 135\sqrt{5})n + (56 - 25\sqrt{5}) \right] = \frac{1}{\pi^2}.$$

J. Guillera, Mosaic supercongruences of Ramanujan-type. *Exp. Math.* **21**, 65–68, (2012), (e-print arXiv:1007.2290).

FORMULA BY GERT ALMKVIST (SWEDEN) AND JESÚS GUILLERA (SPAIN)

September 2010 in arXiv, 2012 in Exp. Math.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \left(\frac{3}{5}\right)^{6n} (532n^2+126n+9) = \frac{375}{\pi^2}. \tag{7}$$

G. Almkvist and J. Guillera, Ramanujan-like series and String theory, *Exp. Math.*, (e-print arXiv:1009.5202).

FORMULA FOR $1/\pi^3$ BY BORIS GOUREVITCH (RUSSIA)

October 2003 in Exp. Math.

$$\frac{1}{32} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7} \frac{1}{2^{6n}} (168n^3+76n^2+14n+1) = \frac{1}{\pi^3}. \tag{8}$$

J. Guillera, About a new kind of Ramanujan type series. *Exp. Math.* **12**, 507-510, (2003).

FORMULA FOR $1/\pi^4$ BY JIM CULLEN (U.S.A.)

February 2011 in Russian Math. Surveys.

$$\frac{1}{2048} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^9} \frac{1}{2^{12n}} \times (43680n^4+20632n^3+4340n^2+466n+21) = \frac{1}{\pi^4}. \tag{9}$$

W. Zudilin, Arithmetic hypergeometric series *Russian Math. Surveys* 66:2 (2011), 369–420. *Russian version in Uspekhi Mat. Nauk* 66:2 (2011), 163–216.