

CHAINS OF SERIES FOR $1/\pi$ ASSOCIATED TO WZ-PAIRS

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INTRODUCTION

If $F(n, k)$ is the first component of a WZ-pair and $F(0, k) = 0$ then $F(sn, k + tn)$ with $s \in \mathbb{N}$ and $t \in \mathbb{Z}$ is also the first component of a WZ-pair verifying $F(s \cdot 0, k + t \cdot 0) = 0$. We use this remark to obtain chains of series for $1/\pi$.

CHAIN-1

We consider the function

$$F(n, k) = \frac{-3^k}{2^{16n}2^{4k}} \frac{(4n+2k)!(6n+2k)!k!}{(3n+k)!^2(2n+k)!n!(2k)!} \frac{n^2}{2k+1}$$

$F(n, k)$ leads to

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{(1/2)_n(1/4)_n(3/4)_n(1/6)_n(5/6)_n}{(1)_n^3(1/3)_n(2/3)_n} \frac{720n^3 + 804n^2 + 236n + 15}{(n+1/3)(n+2/3)} = \frac{128\sqrt{3}}{\pi}.$$

$F(n, k - n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n}3^n} \frac{(1/2)_n(1/4)_n(3/4)_n}{(1)_n^3} (28n+3) = \frac{16\sqrt{3}}{3\pi}.$$

$F(n, k - 2n)$ leads to

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{(1/2)_n(1/4)_n(3/4)_n}{(1)_n^3} (8n+1) = \frac{2\sqrt{3}}{\pi}.$$

$F(2n, k - 3n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \frac{(1/2)_n(1/6)_n^2(5/6)_n^2}{(1)_n^3(1/3)_n(2/3)_n} \frac{44280n^3 + 46620n^2 + 12234n + 565}{(n+1/3)(n+2/3)} = \frac{4608\sqrt{3}}{\pi}.$$

$F(2n, k - 5n)$ leads to

$$\sum_{n=0}^{\infty} \frac{5^{5n}}{2^{6n}3^{5n}} \frac{(1/10)_n(3/10)_n(7/10)_n(9/10)_n}{(1)_n^3(1/2)_n} \frac{2924n^2 + 1668n + 105}{n+1/2} = \frac{432\sqrt{3}}{\pi}.$$

CHAIN-2

We consider the function

$$F(n, k) = \frac{-1}{2^{14n} 2^{4k}} \frac{(4n+2k)!(6n+2k)!k!}{(3n+k)!(2n+k)!^2 n!(2n)!(2k)!} \frac{n^2}{2k+1}$$

$F(n, k)$ leads to

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{3n} \frac{(1/4)_n (3/4)_n (1/6)_n (5/6)_n}{(1)_n^3 (1/2)_n} \frac{296n^2 + 182n + 15}{n + 1/2} = \frac{128}{\pi}.$$

$F(n, k - n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{(1/2)_n (1/4)_n (3/4)_n}{(1)_n^3} (20n + 3) = \frac{8}{\pi}.$$

$F(2n, k - 3n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{6n}}{2^{12n}} \frac{(1/2)_n (1/6)_n^2 (5/6)_n^2}{(1)_n^3 (1/4)_n (3/4)_n} \frac{7720n^3 + 8372n^2 + 2190n + 135}{(n + 1/4)(n + 3/4)} = \frac{2048}{\pi}.$$

CHAIN-3

We consider the function

$$F(n, k) = \frac{(-1)^n}{2^{14n} 2^{2k}} \frac{(4n+2k)!^2 (2n)!k!}{(3n+k)!(2n+k)!^2 n!^3 (2k)!} \frac{n^2}{2k+1}$$

$F(n, k)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{3n}} \frac{(1/2)_n (1/4)_n^2 (3/4)_n^2}{(1)_n^3 (1/3)_n (2/3)_n} \frac{896n^3 + 992n^2 + 296n + 21}{(n + 1/3)(n + 2/3)} = \frac{288}{\pi}.$$

$F(n, k - n)$ leads to

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{(1/2)_n^3}{(1)_n^3} (42n + 5) = \frac{16}{\pi}.$$

$F(n, k - 2n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{(1/2)_n (1/4)_n (3/4)_n}{(1)_n^3} (20n + 3) = \frac{8}{\pi}.$$

$F(2n, k - 3n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \frac{(1/2)_n (1/4)_n (3/4)_n (1/6)_n (5/6)_n}{(1)_n^3 (1/3)_n (2/3)_n} \frac{29520n^3 + 31572n^2 + 8588n + 435}{(n + 1/3)(n + 2/3)} = \frac{6144}{\pi}.$$

CHAIN-4

We consider the function

$$F(n, k) = \frac{1}{2^{12n} 2^{3k}} \frac{(4n + 2k)!^2 k!}{(2n + k)!^3 n!^2 (2k)!} \frac{n^2}{2k + 1}$$

$F(n, k)$ leads to

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(1/4)_n^2 (3/4)_n^2}{(1)_n^3 (1/2)_n} \frac{48n^2 + 32n + 3}{n + 1/2} = \frac{16\sqrt{2}}{\pi}.$$

$F(n, k - n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{(1/2)_n^3}{(1)_n^3} (6n + 1) = \frac{2\sqrt{2}}{\pi}.$$

$F(2n, k - n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{11n}} \frac{(1/2)_n (1/6)_n^2 (5/6)_n^2}{(1)_n^3 (1/3)_n (2/3)_n} \frac{29880n^3 + 32076n^2 + 8842n + 465}{(n + 1/3)(n + 2/3)} = \frac{4608\sqrt{2}}{\pi}.$$

$F(2n, k - 3n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{(1/2)_n (1/6)_n (5/6)_n}{(1)_n^3} (154n + 15) = \frac{32\sqrt{2}}{\pi}.$$

CHAIN-5

We consider the function

$$F(n, k) = \frac{1}{2^{10n} 2^{2k}} \frac{(4n + 2k)!(2n + 2k)!k!(2n)!}{(n + k)!(2n + k)!^2 n!^3 (2k)!} \frac{n^2}{2k + 1}$$

$F(n, k)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{(1/2)_n (1/4)_n (3/4)_n}{(1)_n^3} (20n + 3) = \frac{8}{\pi}.$$

$F(n, k - n)$ leads to

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(1/2)_n^3}{(1)_n^3} (6n + 1) = \frac{4}{\pi}.$$

$F(2n, k - 3n)$ leads to

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{3n} \frac{(1/4)_n (3/4)_n (1/6)_n (5/6)_n}{(1)_n^3 (1/2)_n} \frac{296n^2 + 182n + 15}{n + 1/2} = \frac{128}{\pi}.$$

$F(2n, k - n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{6n}} \frac{(1/2)_n (1/4)_n (3/4)_n (1/6)_n (5/6)_n}{(1)_n^3 (1/3)_n (2/3)_n} \frac{9360n^3 + 10068n^2 + 2792n + 165}{(n + 1/3)(n + 2/3)} = \frac{2304}{\pi}.$$

CHAIN-6

We consider the function

$$F(n, k) = \frac{1}{2^{18n} 2^{3k}} \frac{(8n + 2k)!(4n + 2k)!n!k!}{(4n + k)!(3n + k)!(2n + k)!(2n)!n!^2(2k)!} \frac{n^2}{2k + 1}$$

$F(n, k - n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{(1/2)_n (1/6)_n (5/6)_n}{(1)_n^3} (154n + 15) = \frac{32\sqrt{2}}{\pi}.$$

$F(n, k - 2n)$ leads to

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(1/4)_n^2 (3/4)_n^2}{(1)_n^3 (1/2)_n} \frac{48n^2 + 32n + 3}{n + 1/2} = \frac{16\sqrt{2}}{\pi}.$$

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