SERIES CLOSELY RELATED TO RAMANUJAN FORMULAS FOR PI

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Abstract

We consider an extension of Ramanujan-type series for π by using a free variable k and experimentally find the value of them at $k = 1/2$, then we conjecture its relation to modular functions.

1 The formulas

Here is a list of the formulas I have found with my computer. Most of the series for $f(1/2)$ are new. In next section of the paper we explain the experimental method used.

Alternate series

$$F(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}+k\right)_n \left(\frac{3}{4}+k\right)_n}{(1+k)_n^3} [20(n+k)+3] \tag{1}$$

$$F(0) = \frac{8}{\pi} \quad , \quad F\left(\frac{1}{2}\right) = 2 \cdot 2^2 \cdot \ln(2^2)$$

$$F(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n \cdot 4^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}+k\right)_n \left(\frac{3}{4}+k\right)_n}{(1+k)_n^3} [28(n+k)+3] \tag{2}$$

$$F(0) = \frac{4 \cdot 4}{\sqrt{3} \cdot \pi} \quad , \quad F\left(\frac{1}{2}\right) = 2 \cdot 4^2 \cdot \ln\left(\frac{3^3}{2^4}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{18^{2n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n}{(1+k)_n^3}[260(n+k)+23] \tag{3}$$

$$F(0)=\frac{4\cdot 18}{\pi}\quad,\quad F\left(\frac{1}{2}\right)=2\cdot 18^2\cdot\ln\left(\frac{3^4}{2^6}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{5^n\cdot 72^{2n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n}{(1+k)_n^3}[644(n+k)+41] \tag{4}$$

$$F(0)=\frac{4\cdot 72}{\sqrt{5}\cdot\pi}\quad,\quad F\left(\frac{1}{2}\right)=2\cdot 72^2\cdot\ln\left(\frac{2^{18}}{3^4\cdot 5^5}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{882^{2n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n}{(1+k)_n^3}[21460(n+k)+1123] \tag{5}$$

$$F(0)=\frac{4\cdot 882}{\pi}\quad,\quad F\left(\frac{1}{2}\right)=2\cdot 882^2\cdot\ln\left(\frac{2^2\cdot 3^{20}}{7^{12}}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{80^{3n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{6}+k\right)_n\left(\frac{5}{6}+k\right)_n}{(1+k)_n^3}[5418(n+k)+263] \tag{6}$$

$$F(0)=\frac{80^2}{2\sqrt{15}\cdot\pi}\quad,\quad F\left(\frac{1}{2}\right)=2\cdot 80^3\cdot\ln\left(\frac{2^2\cdot 3^9}{5^7}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{440^{3n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{6}+k\right)_n\left(\frac{5}{6}+k\right)_n}{(1+k)_n^3}[261702(n+k)+10177] \tag{7}$$

$$F(0)=\frac{3\cdot 440^2}{\sqrt{330}\cdot\pi}\quad,\quad F\left(\frac{1}{2}\right)=6\cdot 440^3\cdot\ln\left(\frac{2^{13}\cdot 11^5}{3^3\cdot 5^{11}}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{53360^{3n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{6}+k\right)_n\left(\frac{5}{6}+k\right)_n}{(1+k)_n^3}\cdot \tag{8}$$

$$\cdot[545140134(n+k)+13591409]$$

$$F(0)=\frac{3\cdot 53360^2}{2\sqrt{10005}\cdot\pi}\quad,\quad F\left(\frac{1}{2}\right)=6\cdot 53360^3\cdot\ln\left(\frac{3^{21}\cdot 5^{13}\cdot 29^5}{2^{38}\cdot 23^{11}}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{2n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n}{(1+k)_n^3}[51(n+k)+7] \tag{9}$$

$$F(0)=\frac{12\cdot\sqrt{3}}{\pi} \quad , \quad F\left(\frac{1}{2}\right)=108\cdot\ln\left(\frac{2^2}{3}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{80^n}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n}{(1+k)_n^3}[9(n+k)+1] \tag{10}$$

$$F(0)=\frac{4\cdot\sqrt{15}}{5\pi} \quad , \quad F\left(\frac{1}{2}\right)=12\cdot\ln\left(\frac{3^9}{2^2\cdot 5^5}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{500^{2n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n}{(1+k)_n^3}[14151(n+k)+827] \tag{11}$$

$$F(0)=\frac{1500\cdot\sqrt{3}}{\pi} \quad , \quad F\left(\frac{1}{2}\right)=500\cdot 15^3\cdot\ln\left(\frac{5^6}{2^6\cdot 3^5}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n}{(1+k)_n^3}[1230(n+k)+106] \tag{12}$$

$$F(0)=\frac{192\cdot\sqrt{3}}{\pi} \quad , \quad F\left(\frac{1}{2}\right)=13824\cdot\ln\left(\frac{2^8}{3^5}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3024^n}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n}{(1+k)_n^3}[330(n+k)+26] \tag{13}$$

$$F(0)=\frac{216\cdot\sqrt{7}}{7\pi} \quad , \quad F\left(\frac{1}{2}\right)=1944\cdot\ln\left(\frac{7^7}{2^{10}\cdot 3^6}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n\left(\frac{1}{6}+k\right)_n\left(\frac{5}{6}+k\right)_n}{(1+k)_n^5}\cdot \tag{14}$$

$$\cdot[1640(n+k)^2+278(n+k)+15]$$

$$F(0)=\frac{256\cdot\sqrt{3}}{3\pi^2} \quad , \quad F\left(\frac{1}{2}\right)=3072\cdot\ln\left(\frac{3^9}{2^{14}}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{48^n}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n}{(1+k)_n^5}\cdot \tag{15}$$

$$\cdot[252(n+k)^2+63(n+k)+5]$$

$$F(0)=\frac{48}{\pi^2} \quad , \quad F\left(\frac{1}{2}\right)=288\cdot\ln\left(\frac{2^{10}}{3^6}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{80^{3n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n\left(\frac{1}{6}+k\right)_n\left(\frac{5}{6}+k\right)_n}{(1+k)_n^5}\cdot \tag{16}$$

$$\cdot[5418(n+k)^2+693(n+k)+29]$$

$$F(0)=\frac{128\cdot\sqrt{5}}{\pi^2} \quad , \quad F\left(\frac{1}{2}\right)=115200\cdot\ln\left(\frac{2^{74}\cdot 5^5}{3^{54}}\right)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\frac{\left(\frac{1}{2}+k\right)_n^3}{(1+k)_n^3}[6(n+k)+1] \tag{17}$$

$$F(0)=\frac{2\sqrt{2}}{\pi} \quad , \quad F\left(\frac{1}{2}\right)=4\cdot G$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}\frac{\left(\frac{1}{2}+k\right)_n^5}{(1+k)_n^5}[20(n+k)^2+8(n+k)+1] \tag{18}$$

$$F(0)=\frac{8}{\pi^2} \quad , \quad F\left(\frac{1}{2}\right)=7\cdot\zeta(3)$$

$$F(k)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\frac{\left(\frac{1}{2}+k\right)_n^5}{(1+k)_n^5}[820(n+k)^2+180(n+k)+13] \tag{19}$$

$$F(0)=\frac{128}{\pi^2} \quad , \quad F\left(\frac{1}{2}\right)=256\cdot\zeta(3)$$

Series of positive terms

$$F(k)=\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\frac{\left(\frac{1}{2}+k\right)_n^3}{(1+k)_n^3}[6(n+k)+1] \tag{20}$$

$$F(0)=\frac{4}{\pi}\quad,\quad F\left(\frac{1}{2}\right)=\frac{\pi^2}{2}$$

$$F(k)=\sum_{n=0}^{\infty}\frac{1}{2^{6n}}\frac{\left(\frac{1}{2}+k\right)_n^3}{(1+k)_n^3}[42(n+k)+5] \tag{21}$$

$$F(0)=\frac{16}{\pi}\quad,\quad F\left(\frac{1}{2}\right)=\frac{8\pi^2}{3}$$

$$F(k)=\sum_{n=0}^{\infty}\frac{1}{3^{2n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n}{(1+k)_n^3}[8(n+k)+1] \tag{22}$$

$$F(0)=\frac{2\sqrt{3}}{\pi}\quad,\quad F\left(\frac{1}{2}\right)=\sqrt{3}\cdot\pi$$

$$F(k)=\sum_{n=0}^{\infty}\frac{1}{3^{4n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n}{(1+k)_n^3}[10(n+k)+1] \tag{23}$$

$$F(0)=\frac{9}{2\pi\sqrt{2}}\quad,\quad F\left(\frac{1}{2}\right)=\frac{81}{2\sqrt{2}}\cdot\left[\frac{\pi}{2}-4\arcsin\left(\frac{1}{3}\right)\right]$$

$$F(k)=\sum_{n=0}^{\infty}\frac{1}{7^{4n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n}{(1+k)_n^3}[40(n+k)+3] \tag{24}$$

$$F(0)=\frac{49}{3\pi\sqrt{3}}\quad,\quad F\left(\frac{1}{2}\right)=\frac{49^2}{3\sqrt{3}}\cdot\left[-\frac{\pi}{6}+4\arcsin\left(\frac{1}{7}\right)\right]$$

$$F(k)=\sum_{n=0}^{\infty}\frac{1}{99^{2n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n}{(1+k)_n^3}[280(n+k)+19] \tag{25}$$

$$F(0)=\frac{198}{\pi\sqrt{11}}\quad,\quad F\left(\frac{1}{2}\right)=\frac{2\cdot 99^2}{\sqrt{11}}\cdot\left[-\frac{\pi}{2}+4\arcsin\left(\frac{7}{18}\right)\right]$$

$$F(k)=\sum_{n=0}^{\infty}\frac{1}{99^{4n}}\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n}{(1+k)_n^3}[26390(n+k)+1103] \qquad (26)$$

$$F(0)=\frac{9801}{2\pi\sqrt{2}} \quad , \quad F\left(\frac{1}{2}\right)=\frac{99^4}{2\sqrt{2}}\cdot\arcsin\frac{8668855388657}{3^8\cdot 11^{12}}$$

$$F(k)=\sum_{n=0}^{\infty}\frac{1}{2^{4n}}\frac{\left(\frac{1}{2}+k\right)_n^3\left(\frac{1}{4}+k\right)_n\left(\frac{3}{4}+k\right)_n}{(1+k)_n^5}\cdot \qquad (27)$$

$$\cdot[120(n+k)^2+34(n+k)+3]$$

$$F(0)=\frac{32}{\pi^2} \quad , \quad F\left(\frac{1}{2}\right)=\frac{16\cdot\pi^2}{3}$$

$$F(k)=\sum_{n=0}^{\infty}\frac{1}{2^{6n}}\frac{\left(\frac{1}{2}+k\right)_n^7}{(1+k)_n^7}[168(n+k)^3+76(n+k)^2+14(n+k)+1] \qquad (28)$$

$$F(0)=\frac{32}{\pi^3} \quad , \quad F\left(\frac{1}{2}\right)=\frac{\pi^4}{2}$$

The formulas of type 3F2 for $F(0)$ have been proved by Srinivasa Ramanujan [9], Peter and Jonathan Borwein [2],[3], David and Gregory Chudnovsky [2],[5] and Heng Huat Chan, Wen Chin Liaw and Victor Tan [4]. Formulas (18), (19) and (27) for $f(0)$ are hypergeometric of type 5F4 and have been proved by me [6],[7]. Formula (19) for $\zeta(3)$ has been proved by Theodros Amdeberhan [1]. Formula (28) for $F(0)$ is hypergeometric of type 7F6 and was found experimentally by Boris Gourevitch with the help of the software PARI-GP by using the function *lindep*, which looks for integer relations. Most of the formulas for $F(1/2)$ in this paper are unproved [8].

2 The experimental method

We write

$$F(k)=\sum_{n=0}^{\infty}\frac{B(n,k)}{a^n}\cdot P(n,k)$$

where $P(n,k)$ is the polynomial part. We consider the following sums:

$$F_0(k)=\sum_{n=0}^{\infty}\frac{B(n,k)}{a^n}, \quad F_1(k)=\sum_{n=0}^{\infty}\frac{B(n,k)}{a^n}\cdot(n+k), \quad F_2(k)=\sum_{n=0}^{\infty}\frac{B(n,k)}{a^n}\cdot(n+k)^2.$$

and in the alternate case look for integer relations between

$$F_0(k), \quad F_1(k), \quad F_2(k), \quad \log(2), \quad \log(3), \quad \log(5), \quad \log(7),$$

$$\log(11), \quad \log(13), \quad \log(17), \quad \log(19), \quad \log(23), \quad \log(29), \quad \log(31).$$

This means that we want to find integers

$$b_0, \quad b_1, \quad b_2, \quad a_2, \quad a_3, \quad a_5, \quad a_7, \quad a_{11}, \quad a_{13}, \quad a_{17}, \quad a_{19}, \quad a_{23}, \quad a_{29}, \quad a_{31},$$

such that

$$b_0 \cdot F_0(k) + b_1 \cdot F_1(k) + b_2 \cdot F_2(k) +$$
$$+ a_2 \cdot \log(2) + a_3 \cdot \log(3) + \cdots + a_{31} \cdot \log(31) = 0.$$

The software used has been PARI-GP which has the function *lindep* which looks for integer relations. For other series we need constants such as π, $\zeta(3)$, $G, \cdots$. For series such as (24) we have guessed the presence of trigonometric inverse functions.
I have also used the software MAPLE + (Simon Plouffe's inverter package) to find the following two interesting evaluations:

For function (1) we have

$$F\left(\frac{1}{4}\right) = 2\left[K\left(\frac{1}{\sqrt{2}}\right)\right]^2$$

For function (2) we have

$$F\left(\frac{1}{4}\right) = \frac{8\sqrt{2}}{3\sqrt[4]{3}} \cdot \left[K\left(\frac{1}{\sqrt{2}}\right)\right]^2$$

In our list of series we have limited to integer values of a, although there are also series for some rational values or even for algebraic values. Examples are:

$$F(k) = \sum_{n=0}^{\infty} \left(\frac{32}{81}\right)^n \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}+k\right)_n \left(\frac{3}{4}+k\right)_n}{(1+k)_n^3} [7(n+k)+1] \qquad (29)$$

$$F(0) = \frac{9}{2\pi} \quad , \quad F\left(\frac{1}{2}\right) = \frac{81}{8\sqrt{2}} \cdot \left[\frac{\pi}{2} - 2\arcsin\left(\frac{1}{3}\right)\right]$$

$$F(k) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{4}\right)^{2n} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{3}+k\right)_n \left(\frac{2}{3}+k\right)_n}{(1+k)_n^3} [5(n+k)+1] \qquad (30)$$

$$F(0) = \frac{4 \cdot \sqrt{3}}{3\pi} \quad , \quad F\left(\frac{1}{2}\right) = \frac{4}{3} \cdot \ln\left(\frac{3^3}{2^2}\right)$$

$$F(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(42+24\sqrt{3})^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}+k\right)_n \left(\frac{3}{4}+k\right)_n}{(1+k)_n^3} \cdot \tag{31}$$

$$\cdot[(756+448\sqrt{3})\cdot(n+k)+(51+32\sqrt{3})]$$

$$F(0) = \frac{4\cdot(42+24\sqrt{3})}{\pi} \quad , \quad F\left(\frac{1}{2}\right) = 2\cdot(42+24\sqrt{3})^2 \cdot \ln\left[\frac{42+24\sqrt{3}}{81}\right]^2$$

3 Conclusion

We have found formulas which are closely related to Ramanujan type formulas for π. I believe this suggest that the new series are related to the theory of modular functions. Another fact which supports this conjecture is that it seems to be a corresponding one-one between the series at 0 and at 1/2, in the sense that they happen for the same values of the base in the powers to n.

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