WEAK TYPE 1 ESTIMATES FOR SPHERICAL MULTIPLIERS ON NONCOMPACT SYMMETRIC SPACES

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FRAMEWORK

G connected unimodular Lie group

K compact subgroup of G

Assume $L^1(K\backslash G/K)$ is commutative

Σ Gelfand spectrum of $L^1(K\backslash G/K)$

$\Sigma \cong \{$ bounded spherical functions $\}$

<u>Spherical transform</u>: $f \in L^1(K\backslash G/K)$

$$\tilde{f}(\lambda) = \int_G f(x)\, \overline{\varphi_\lambda(x)}\, dx \qquad \forall \lambda \in \Sigma$$

$$f(x) = \int_\Sigma \tilde{f}(\lambda)\, \varphi_\lambda(x)\, d\mu(\lambda) \qquad \forall x \in G$$

Extension to K-bi-invariant distributions on G.

$X := G/K$

For $q \in [1, \infty]$, define

$${}^G Op^q(X) := \{ G\text{-invariant bounded linear operators on } L^q(X) \}$$

Given $B \in {}^G Op^q(X)$, there exists a unique K-bi-invariant distribution k_B on G s.t.

$$Bf = f * k_B \qquad \forall f \in C_c^\infty(X)$$

k_B <u>kernel</u> of B

$m_B := \tilde{k}_B$ <u>spherical multiplier</u> associated to B

EXAMPLE

$G = M(n)$

$K = O(n)$

$X = \mathbb{R}^n$

$L^1(K\backslash G/K)$ radial integrable functions on $\mathbb{R}^n$

$\Sigma = [0, \infty)$

$\lambda \in \Sigma$ $\qquad \varphi_\lambda(x) = \dfrac{J_{n/2-1}(2\pi|x|\lambda)}{(2\pi|x|\lambda)^{n/2-1}}$

$$\tilde{f}(\lambda) = \int_{\mathbb{R}^n} f(x)\,\varphi_\lambda(x)\,dx$$

$$= \int_0^\infty f_0(r)\,\frac{J_{n/2-1}(2\pi r\lambda)}{(2\pi r\lambda)^{n/2-1}}\,\omega_n r^{n-1}\,dr$$

${}^G\mathrm{Op}^2(X) = \{$ convolution operators $f \mapsto f * k$ with k radial and $\hat{k}$ bounded $\}$

<u>Theorem</u> (Mihlin–Hörmander) Suppose $B \in {}^G Op^2(X)$ and that m_B satisfies a MH condition of order $[n/2]+1$, i.e.,

$$\left| \lambda^j D^j m_B(\lambda) \right| \leqslant C \qquad \forall \lambda \in \Sigma$$

$j = 0, 1, \ldots, [n/2]+1$. Then B extends to an operator of weak type 1.

SYMMETRIC SPACES OF THE NONCOMPACT TYPE

G connected noncompact semisimple Lie group with finite centre

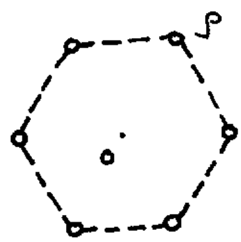
K maximal compact subgroup of G

$X = G/K$ Riemannian <u>symmetric space of the noncompact type</u> and real rank ℓ

(G,K) is a Gelfand pair

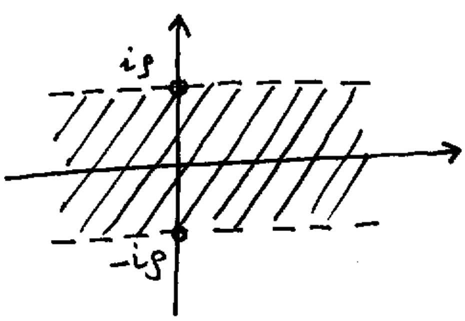
$$W \subseteq \mathbb{R}^{\ell}$$

$\ell = 1$



$\ell = 2$ $(SL(3,\mathbb{R}))$

$$T = \mathbb{R}^{\ell} + iW \subseteq \mathbb{C}^{\ell}$$

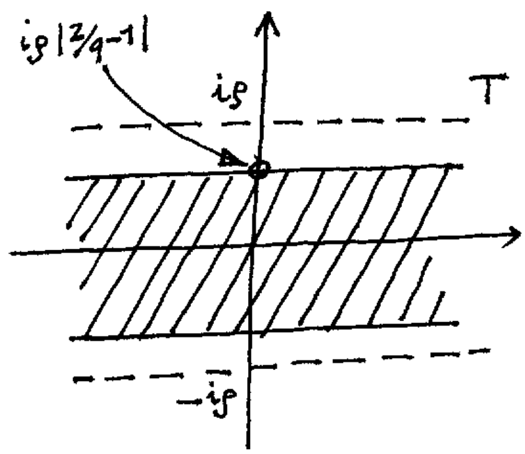


$\overline{T}$ spectrum of $L^1(K\backslash G/K)$

STATEMENT OF THE PROBLEM

AND KNOWN RESULTS

<u>Theorem</u> (J.L. Clerc – E.M. Stein) If $B \in \bigcap_{1<q<\infty} {}^{G}O_p^{q}(X)$, then $m_B \in H(T)^W$, bounded on closed subtubes thereof.



<u>PROBLEM</u> Given $m_B \in H(T)^W$, bounded on closed subtubes of T, find conditions on m_B so that B extends to an operator of weak type 1.

<u>RESULTS</u> • Clerc – Stein

- Stanton – Tomas
- Anker – Lohouè
- Taylor

<u>Theorem</u> (J.-Ph. Anker) Suppose $m_B \in H^\infty(T)^W$ and

(1) $$|D^\alpha m_B(\zeta)| \leqslant C\ (1+|\zeta|)^{-|\alpha|} \qquad \forall \zeta \in T$$

for $|\alpha| \leqslant [(\dim X)/2]+1$. Then B is of weak type 1.

<u>Sketch of the proof.</u> Choose $\psi \in C_c^\infty(K\backslash X)$ s.t. $\psi = 1$ on $B(0,1)$. Write

$$k_B = \underbrace{\psi k_B}_{=:\, k_B^0} + \underbrace{(1-\psi) k_B}_{=:\, k_B^\infty}.$$

Then:

(I) $k_B^\infty \in L^1(K\backslash X) \Rightarrow f \mapsto f * k_B^\infty$ is bounded on $L^1(X)$;

(II) k_B^0 satisfies the following HIC

$$\sup_{0<R<1}\ \sup_{y\in B(0,R)} \int_{B(0,2R)^c} |k_B^0(y^{-1}x) - k_B^0(x)|\, dx < \infty$$

$\Rightarrow f \mapsto f * k_B^0$ is of weak type 1. $\square$

<u>Remark.</u> The integrability of k_B^∞ depends on the boundedness of $D^\alpha m_B$ near $i\rho$.

Recent results of A.D. Ionescu suggest to consider $m_B \in H(T)^W$ such that

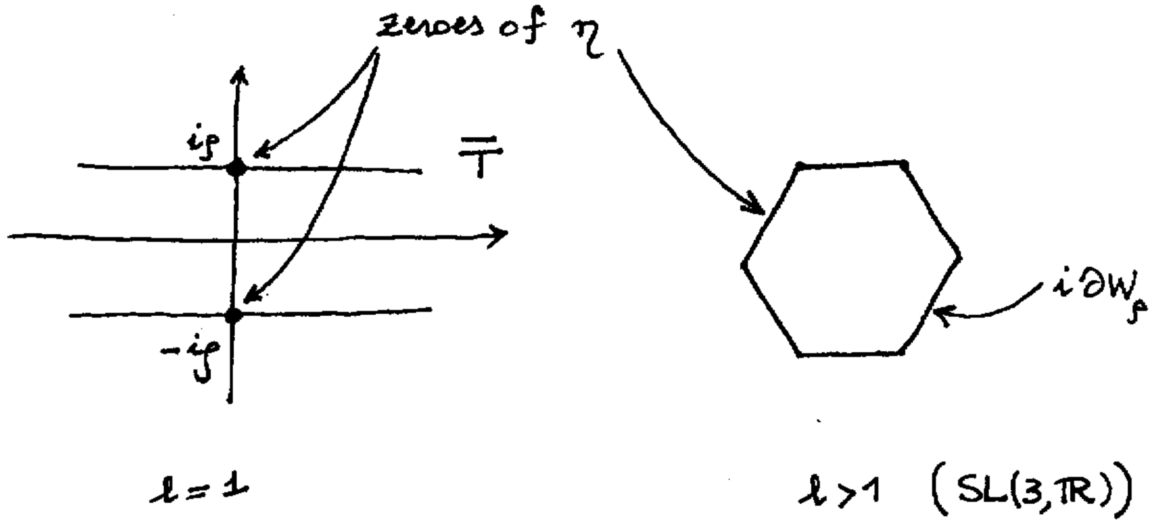
$$(2) \qquad |D^\alpha m_B(\zeta)| \leq c\, \eta(\zeta)^{-|\alpha|} \qquad \forall \zeta \in T \quad \forall \alpha : |\alpha| \leq N,$$

where

$$\eta(\zeta) := \left[|\mathrm{Re}\,\zeta|^2 + \mathrm{dist}(\mathrm{Im}\,\zeta, W_\rho^c)^2 \right]^{1/2} \qquad \forall \zeta \in \overline{T}.$$

Note that

$$\eta(\zeta) = 0 \iff \begin{cases} \mathrm{Re}\,\zeta = 0 \\ \mathrm{Im}\,\zeta \in \partial W_\rho \end{cases} \iff \zeta \in i\partial W_\rho$$



Furthermore $\eta(\zeta) \asymp |\zeta|$ as ζ tends to ∞ within T

$\Rightarrow$ (1) and (2) are equivalent at ∞.

A routine adaptation of methods of A.D. Ionescu gives the following

<u>Theorem</u> Suppose $m_B \in H(T)^W$ and satisfies (2). Then B is of weak type 1.

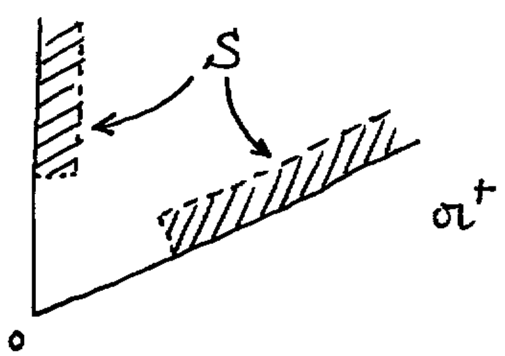
<u>Sketch of the proof.</u> Write

$$k_B = k_B^0 + k_B^\infty$$

as above. Then:

(I) $f \mapsto f * k_B^0$ is of weak type 1

(II) $$\left| k_B^\infty (x(\exp H)\, x') \right| \leq \frac{C e^{-2\rho(H)}}{(1+|H|)^\ell} \quad \forall H \in \mathfrak{a}^+ \setminus S$$



(III) k_B^∞ is integrable on $K(\exp S)K$

(IV) (II + Strömberg) + III $\Rightarrow f \mapsto f * k_B^\infty$ is of weak type 1. $\square$

MAIN RESULT

Motivation. There are natural operators of weak type 1 to which the previous theorem does not apply.

$\mathcal{L}$ Laplace-Beltrami operator on X

$\mathcal{L}^{-\alpha/2}$ $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha \geqslant 0$

Proposition (J.-Ph. Anker) If $0 \leqslant \operatorname{Re} \alpha \leqslant 2$, then $\mathcal{L}^{-\alpha/2}$ is of weak type 1.

Define $Q(\zeta) := \langle \zeta, \zeta \rangle + \langle \rho, \rho \rangle \quad \forall \zeta \in \mathbb{C}^{\ell}$.
Then

$$m_{\mathcal{L}^{-\alpha/2}}(\zeta) = Q(\zeta)^{-\alpha/2}.$$

Remark. $m_{\mathcal{L}^{-\alpha/2}}$ satisfies (2) if and only if either $\ell = 1$ and $\operatorname{Re} \alpha = 0$, or $\ell > 1$ and $\alpha = 0$.

Clearly if $m_{\mathcal{L}^{-\alpha/2}}$ satisfies (2), then $\operatorname{Re}\alpha = 0$.

Suppose $\operatorname{Re}\alpha = 0$, i.e., $\alpha = -2iu$, $u \in \mathbb{R}$. Note that

$$\eta(\zeta)\,\left|\nabla m_{\mathcal{L}^{iu}}(\zeta)\right| = 2|u|\,\frac{|\zeta|}{\left|Q(\zeta)^{1-iu}\right|}\,\eta(\zeta)$$

(*)

$$= 2|u|\,e^{-u\arg Q(\zeta)}\,\frac{|\zeta|}{|Q(\zeta)|}\,\eta(\zeta).$$

Write $\zeta = a+ib$, where $a \in \mathbb{R}^{\ell}$ and $b \in W_{\rho}$.
Then

$$Q(a+ib) = \langle a,a\rangle + \langle \rho,\rho\rangle - \langle b,b\rangle + 2i\langle a,b\rangle$$

$$\eta(a+ib) = \left[\langle a,a\rangle + d(b, W_{\rho}^{c})^{2}\right]^{1/2}.$$

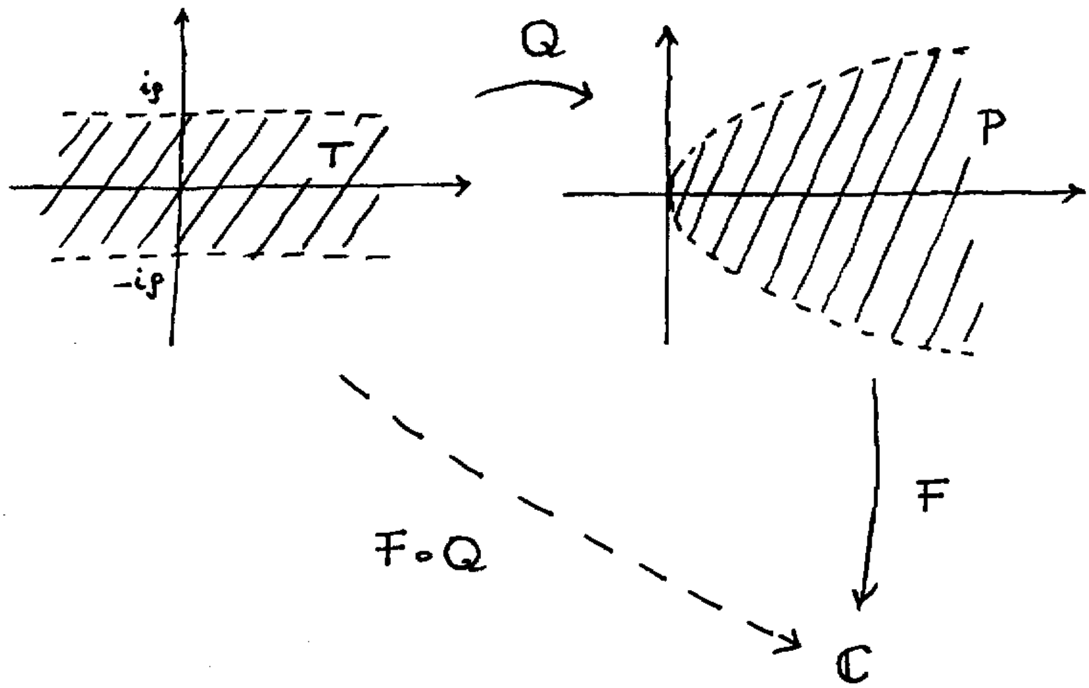
Fix $a \neq 0$, and let b tend to ρ within W_{ρ}.
Then the RHS of (*) tends to

$$2|u|\,e^{-u\arg Q(a+i\rho)}\,\frac{|a+i\rho|\,|a|}{\left|\langle a,a\rangle + 2i\langle a,\rho\rangle\right|}$$

$$\asymp \frac{|a|}{\left|\,|a|^{2} + 2i\langle a,\rho\rangle\right|} \qquad \text{as } a \to 0.$$

This stays bounded iff $\ell = 1$: if $\ell > 1$ and $a \in \rho^{\perp}$ $\dfrac{|a|}{\left|\,|a|^{2} + 2i\langle a,\rho\rangle\right|} \asymp \dfrac{1}{|a|}$.

$P := Q(T)$



$$\mathcal{X}(P;N) = \left\{ F \in H(P) : \quad |F^{(j)}(z)| \leqslant \frac{C}{\min\left(|z|^{1+j}, |z|^{j}\right)} \quad j = 0, 1, \dots, N \right\}.$$

<u>Remark.</u> If $F \in \mathcal{X}(P;N)$, then

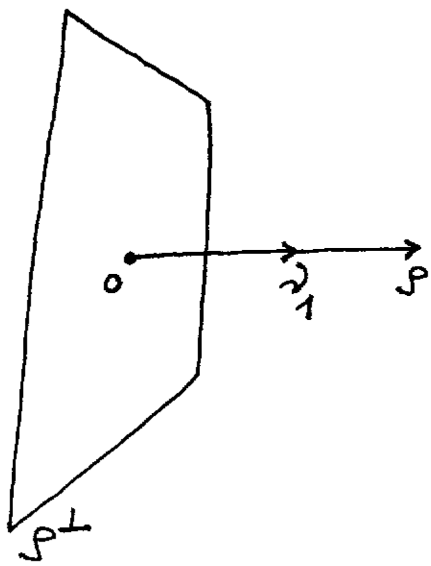
$$\left| D^{I}(F \circ Q)(\zeta) \right| \leqslant \frac{C}{\min\left(|Q(\zeta)|^{1+|I|}, |Q(\zeta)|^{|I|/2} \right)} \tag{3}$$

$$\forall \zeta \in T, \quad \forall I : \ |I| \leqslant N.$$

MORE NOTATION.

$\nu_1, \nu_2, \dots, \nu_\ell$ orthonormal basis in $\mathbb{R}^\ell$

$\nu_1 = \rho/|\rho|$, $(\nu_2, \dots, \nu_\ell \in \rho^\perp$



$\xi = (\xi_1, \xi_2, \dots, \xi_\ell)$

$\eta = (\eta_1, \eta_2, \dots, \eta_\ell)$

$\zeta = (\zeta_1, \zeta_2, \dots, \zeta_\ell)$

$\zeta_j = \xi_j + i\eta_j$

$I = (i_1, \dots, i_\ell)$

$$D^I = \frac{\partial^{|I|}}{\partial \zeta_1^{i_1} \dots \partial \zeta_\ell^{i_\ell}}$$

$|I| = i_1 + \dots + i_\ell$

$|I|' = 2i_1 + i_2 + \dots + i_\ell$ <u>anisotropic length</u> of I

$T^+ := \mathbb{R}^\ell + i\,(W \cap \overline{\Omega^+})$.

<u>Theorem</u> (S.M. – M. Vallarino) Suppose $m_B \in H(T)^W$:

(i) if for some $\sigma \in [0,1)$, and all $|I| \leq N$

$$(3') \quad |D^I m_B(\zeta)| \leq \frac{C}{\min\left(|Q(\zeta)|^{\sigma+|I|/2}, |Q(\zeta)|^{|I|/2}\right)} \quad \forall \zeta \in T^+,$$

then B is of weak type 1;

(ii) if $F \in \mathcal{H}(P;N)$, then $F(\mathcal{L})$ is of weak type 1.

<u>Remarks.</u> (a) If $F \in \mathcal{H}(P;N)$, then $m_{F(\mathcal{L})}$ satisfies (3′) with $\sigma = 1$.

(b) If m_B satisfies (3′) for some $\sigma > 1$, then B is not necessarily of weak type 1, even if B is of the form $F(\mathcal{L})$.

Indeed, $\mathcal{L}^{-\sigma}$ is not of weak type 1 if $\sigma > 1$, and $m_{\mathcal{L}^{-\sigma}}$ satisfies (3′).

(c) We do not know whether (i) holds with $\sigma = 1$.

(d) Part (ii) applies to $\mathcal{L}^{-\alpha/2}$ when $0 \leq \operatorname{Re} \alpha \leq 2$.

<u>Remarks on the proof.</u> $\psi \in C_c^\infty(K\backslash G/K)$,

$\psi = 1$ near the origin

$$k_B^{(0)} = k_B \, \psi$$

$$k_B^{(\infty)} = k_B \, (1-\psi)$$

(I) $k_B^{(0)}$ satisfies a Hörmander type integral condition

$$\Rightarrow \quad f \mapsto f * k_B^{(0)} \text{ is of weak type } 1$$

(II) $$k_B^{(\infty)} = k_B^{(\infty,1)} + \underbrace{k_B^{(\infty,2)}}_{\in L^1(K\backslash G/K)}$$

(*) $$\left| k_B^{(\infty,1)}(\exp H) \right| \leqslant \frac{c}{1+\delta(H)^{l+1-2\sigma}} e^{-2\rho(H)} \quad \forall H \in \mathfrak{a}^+$$

in case (i), and

(*)' $$\left| k_B^{(\infty,1)}(\exp H) \right| \leqslant \frac{c}{1+\delta(H)^{l-1}} e^{-\rho(H) - |\rho|\,|H|} \quad \forall H \in \mathfrak{a}^+$$

in case (ii). Here

$$\sigma(H) = \left(H_1^2 + |H'|^4\right)^{1/4},$$

where $H = (H_1, H') \in \mathbb{R} \times \mathbb{R}^{\ell-1}$.

(III) $f \mapsto f * k_B^{(\infty,1)}$ is of weak type 1 (Strömberg). $\square$