Endpoint results for the heat maximal operator associated with multidimensional Laguerre functions

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For low parameter values, the maximal operator of the heat kernel for Laguerre function expansions is not bounded on all L^p spaces. We shall see that the endpoint results here depend strongly on the dimension and involve logarithmic factors.

The Laguerre functions $\mathcal{L}_k^\alpha(x)$, $k = 0, 1, \ldots$

$\alpha > -1$, are orthogonal in $L^2(\mathbb{R}_+; dx)$

and are eigenfunctions of

$$\Delta^\alpha = -\left(x \frac{d^2}{dx^2} + \frac{d}{dx} - \frac{x}{4} - \frac{\alpha^2}{4x}\right).$$

d-dimensional extension:

$$\mathcal{L}_k^\alpha = \bigotimes_{i=1}^{d} \mathcal{L}_{k_i}^{\alpha_i}, \quad k \in \mathbb{N}^d,$$

$\alpha = (\alpha_1, \ldots, \alpha_d) \in (-1, \infty)^d$,

orthogonal basis in $L^2(\mathbb{R}_+^d; dx)$,

eigenfunctions of $\Delta^\alpha = \sum_1^d \Delta_{x_i}^{\alpha_i}$.

Δ^α has a self-adjoint extension.

The heat semigroup $T_t^\alpha = e^{-t\Delta^\alpha}$, $t > 0$,

is given by integration against a kernel

$$H_t^\alpha = \bigotimes_1^d H_t^{\alpha_i}$$

For $f \in L^2(\mathbb{R}_+^d)$ or $f \in L^p$ does $T_t^\kappa f \to f$ a.e. as $t \to 0$?

Define the maximal operator

$$M^\kappa f(x) = \sup_{t>0} |T_t^\kappa f(x)|$$

M^κ behaves well when $\alpha_i \geq 0$ for all i, but not when some $\alpha_i < 0$, since

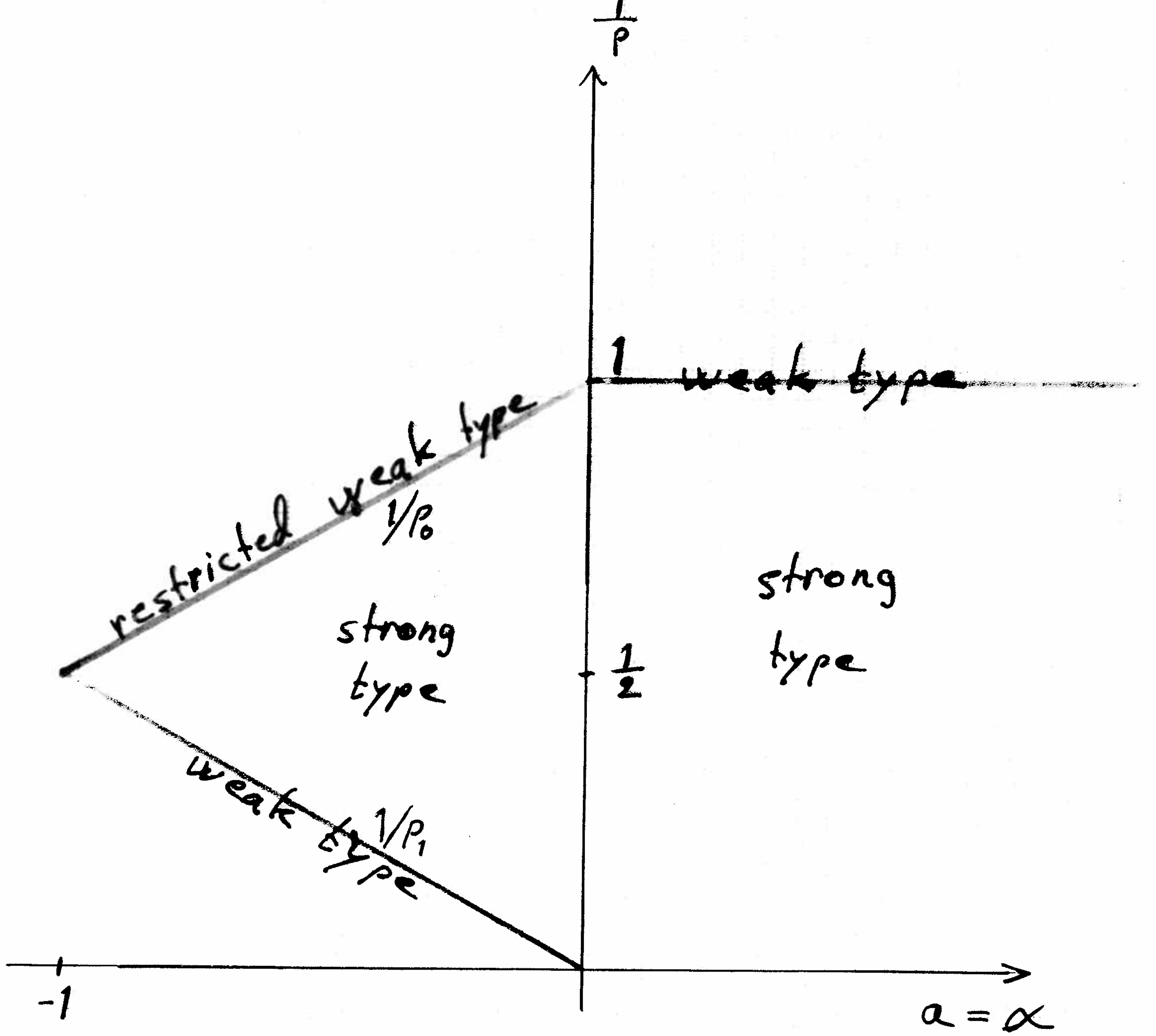
$$H_t^{\alpha_i}(\xi, \eta) \sim \xi^{\frac{\alpha_i}{2}} \eta^{\frac{\alpha_i}{2}} \quad \text{as } \xi, \eta \to 0$$

In one dimension, T_t^a and M^a are bounded on L^p only for $p_0 < p < p_1$, where $p_0 = \frac{2}{2+a}$ and $p_1 = p_0' = -\frac{2}{a}$

$d=1$, L^p properties of M^a

Macías, Segovia and Torrea

Stempak



Restricted weak type (p_0, p_0) for $d=1$ (Macias, Segovia, Torrea) means that

$$|\{ M^a \chi_E > \lambda \}| \leq C \lambda^{-p_0} |E|, \qquad \lambda > 0, \; E \subset \mathbb{R}_+$$

or equivalently

$$M^a : L^{p_0,1} \to L^{p_0,\infty}$$

where the Lorentz space $L^{p_0,1}$ is defined by

$$\int_0^\infty f^*(s) \, s^{\frac{1}{p_0}} \frac{ds}{s} < \infty$$

and f^* is the decreasing rearrangement of $|f|$

$d > 1$ (Nowak and S.)

Let $a = \min \alpha_i$, and assume $-1 < a < 0$

p_0, p_1 are given by a as before.

M^α is strong type (p, p) for $p_0 < p < p_1$.

Let $\tilde{d} = \#\{i : \alpha_i = a\}$.

If $\tilde{d} = 1$, the one-dimensional results remain true.

Theorem (2006). For $d = 2$ and 3,

$$\left|\{ M^{\alpha}\chi_E > \lambda \}\right| \le C \lambda^{-p_0} |E| \left[\log\left(2 + \frac{1}{|E|}\right)\right]^{\frac{p_0}{p_1}(\tilde{d}-1)}$$

$\lambda > 0$, $E \subset \mathbb{R}^d_+$

or equivalently

$$M^{\alpha} : L^{p_0, 1}_{\frac{\tilde{d}-1}{p_1}} \longrightarrow L^{p_0, \infty}$$

where $L^{p_0,1}_{\frac{\tilde{d}-1}{p_1}}$ is defined by

$$\int_0^{\infty} f^*(s)\, s^{\frac{1}{p_0}} \left[\log\left(2 + \frac{1}{s}\right)\right]^{\frac{\tilde{d}-1}{p_1}} \frac{ds}{s} < \infty .$$

Theorem (October, 2007). Let $\tilde{d} = d$, i.e. all $\alpha_i = a$. Then for all d

$$\left|\{ M^{\alpha}\chi_E > \lambda \}\right| \le C |E| \lambda^{-p_0} \left[\log\left(2 + \frac{1}{\lambda}\right)\right]^{\frac{p_0}{p_1}(d-1)},$$

$\lambda > 0$, $E \subset \mathbb{R}^d_+$.

November, 2007

$\tilde{d} = d$

Estimates for $\left|\{x : M^{\alpha}\chi_E(x) > \lambda\}\right|$, $E \subset \mathbb{R}^d_+$, $|E| < \infty$

$\lambda > 1$	$C\,|E|\,\lambda^{-p_0}$	
$\lambda < 1$	$C\,|E|\,\lambda^{-p_0}\left[\log\left(2+\frac{1}{\lambda}\right)\right]^{\frac{p_0}{p_1}\left(\left[\frac{d}{2}\right]-1\right)}$	$C\,|E|\,\lambda^{-p_0}\left[\log\left(2+\frac{1}{\lambda}\right)\right]^{\frac{p_0}{p_1}(d-1)}$
	$|E| > 1$	$|E| < 1$

$d=1$, idea of proof

$$M^a f(\xi) = \sup_{t>0} \left| \int_0^\infty H_t^a(\xi,\eta)\, f(\eta)\, d\eta \right|$$

Enough consider $f \geq 0$, $0<t\leq 1$

Lemma: For $0<t\leq 1$, $-1<a<0$

$$H_t^a(\xi,\eta) \leq C e^{-c\frac{\xi+\eta}{t}} \xi^{-\frac{1}{p_1}} \eta^{-\frac{1}{p_1}} t^{\frac{2}{p_1}-1} +$$

$$+ C e^{-ct(\xi+\eta)} \frac{1}{\sqrt{t\xi}} \exp\left(-c\frac{(\xi-\eta)^2}{t\xi}\right) =$$

$$I + II,$$

$$\xi, \eta > 0$$

$d=1$

p_0 : $\displaystyle\int_0^\infty H_t^a(\xi,\eta)\, f(\eta)\, d\eta \le$

$$\underbrace{C\, e^{-\frac{\xi}{t}}\, \xi^{-\frac{1}{p_1}}\, t^{\frac{2}{p_1}-1}}_{\le C\left(\frac{\xi}{t}\right)^{\frac{2}{p_1}-1}} \int_0^\infty f(\eta)\, \eta^{-\frac{1}{p_1}}\, d\eta + C\, e^{-ct\xi} M_{HL} f(\cdot$$

$$\le \int_0^\infty f^*(s)\, s^{-\frac{1}{p_1}}\, ds$$

$$= \int_0^\infty f^*(s)\, s^{\frac{1}{p_0}}\, \frac{ds}{s}$$

$$\le C\, \underbrace{e^{-c\frac{\xi}{t}}\, \xi^{-\frac{1}{p_0}}}_{\in L^{p_0,\infty}}\, \|f\|_{L^{p_0,1}} + \underbrace{C\, e^{-ct\xi} M_{HL} f(\cdot}_{\in L^{p_0} \subset L^{p_0,\infty}}$$

$$\int f(\eta)\, g(\eta)\, d\eta \le \int_0^\infty f^*(s)\, g^*(s)\, ds$$

$d > 1 \qquad \tilde{d} = d \qquad H_t^{\alpha} = \bigotimes_1^d H_t^a$

Use $\quad H_t^a(x_i, y_i) \leq I + II$

in each variable, to get 2^d terms.

"II only": trivial

"I only": Get at most

$$\exp\left(-c\,\frac{\sum x_i}{t}\right) t^{\left(\frac{2}{p_1}-1\right)d} \prod x_i^{-\frac{1}{p_1}} \cdot$$

$$\cdot \underbrace{\int_E \prod y_i^{-\frac{1}{p_1}} \exp\left(-c\,\frac{\sum y_i}{t}\right) dy}_{\leq C |E|^{\frac{1}{p_0}} \left[\log\left(2 + \frac{t^d}{|E|}\right)\right]^{\frac{d-1}{p_1}}}$$

max for $t \sim \sum x_i$

$$\sup_{t>0} \ldots \quad \leq C \left(\sum x_i\right)^{\left(\frac{2}{p_1}-1\right)d} \prod x_i^{-\frac{1}{p_1}} \cdot$$

$$\cdot |E|^{\frac{1}{p_0}} \left[\log\left(2 + \frac{\sum x_i}{|E|}\right)\right]^{\frac{d-1}{p_1}}$$

Elementary lemma:

$$\Big|\Big\{ x \in R_+^d : \prod_1^d x_i^{-1} \underbrace{\exp\Big(-\frac{\sum x_i}{t}\Big)}_{\text{or } \chi_{[0,t]^d}} > \lambda \Big\}\Big| \leq$$

$$\leq C_d \frac{1}{\lambda} \big[\log(2 + t^d \lambda)\big]^{d-1} \qquad \lambda > 0$$

The level set where this $> \lambda$ has measure at most

$$C\,|E|\,\lambda^{-p_0}\left[\log\left(2+\frac{1}{\lambda}\right)\right]^{\frac{p_0}{p_1}(d-1)}.$$

Mixed terms: ...

I for x_i, y_i, $i = 1, \dots, d'$

II for $i = d'+1, \dots, d$

$$x = (x', x'') \in \mathbb{R}^{d'} \times \mathbb{R}^{d-d'}$$

M'' max operator in x''

We get

$$\sup_t \int \prod_1^{d'} H_t^{\alpha}(x_i, y_i)\, \underbrace{M''\chi_E(y', x'')}_{\sim\, \chi_{E_1} + \dots}\, dy'$$

$$|E_1| \sim |E|$$

Now argue as before, in the x' variables with x'' fixed and with $E_1^{x''} = \{y' : (y', x'') \in E_1\}$ instead of E.

Get

$$\left|\left\{ x' : \sup_t \ldots (x', x'') > \lambda \right\}\right| \leq$$

$$\leq C \, |E_1^{x''}| \, \lambda^{-p_0} \left[\log\left(2 + \frac{1}{\lambda}\right)\right]^{\frac{p_0}{p_1}(d'-1)}$$

Then integrate in x'' :

$$\left|\left\{ x : \sup_t \ldots (x) > \lambda \right\}\right| \leq$$

$$\leq C \, |E| \, \lambda^{-p_0} \left[\log\left(2 + \frac{1}{\lambda}\right)\right]^{\frac{p_0}{p_1}(d'-1)}$$