Dirichlet and Fejér families of operators in Banach spaces

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Introduction

$(L^p(\mathbb{R}), \|\ \ \|_p)$ usual Lebesgue space $1 \leq p < \infty$.

$g \in L^p(\mathbb{R})$ with $1 < p < \infty$ and $\varepsilon > 0$.

Partial Hilbert transforms, $H_\varepsilon : L^p(\mathbb{R}) \to L^p(\mathbb{R})$,

$$(H_\varepsilon g)(t) := \frac{i}{\pi} \int_{\varepsilon \leq |s|} \frac{g(t-s)}{s} ds, \qquad g \in L^p(\mathbb{R}).$$

Riesz's theorem $Hg := \text{lím}_{\varepsilon \to 0^+} H_\varepsilon g$, $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$

Introduction

Dirichlet kernel, $(d_s)_{s\geq 0}$,

$$d_s(r) := \frac{\sin(sr)}{\pi r}, \qquad r \in \mathbb{R}\backslash\{0\}, \qquad s \geq 0$$

$\mathcal{D}(s) : L^p(\mathbb{R}) \to L^p(\mathbb{R})$, Dirichlet operators, $1 < p < \infty$,

$$\mathcal{D}(s)(g) := d_s * g, \qquad g \in L^p(\mathbb{R}),$$

Riesz's theorem implies $(\mathcal{D}(s))_{s\geq 0} \subset \mathcal{B}(L^p(\mathbb{R}))$,
$\sup_{s>0} \|\mathcal{D}(s)\| < \infty$, $\mathcal{D}(s)g \to g$ when $s \to \infty$ for $g \in L^p(\mathbb{R})$.

Introduction

Fejér kernel,

$$f_s(r) := \frac{1-\cos(sr)}{\pi r^2}, \qquad s>0, r\in\mathbb{R}\backslash\{0\},$$

Fejér operators, $\mathcal{F}(s): L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $\mathcal{F}(s)g := f_s * g$, .

$f_s \in L^1(\mathbb{R})$, $\mathcal{F}(s)\in\mathcal{B}(L^p(\mathbb{R}))$, $\sup_{s>0}\frac{1}{s}\|\mathcal{F}(s)\| < \infty$, $\frac{1}{s}\mathcal{F}(s)g \to g$,

$s\mapsto f_s$ NOT Lipschitz in $L^1(\mathbb{R})$ but YES in $\mathcal{B}(L^p(\mathbb{R}))$

Note $f_s(r) = \int_0^s d_t(r)dt$, with $r\in\mathbb{R}$ and $s>0$.

$$f_s * f_t = 2\int_0^s f_u du + (t-s)f_s, \qquad 0<s\leq t.$$

1. UMD spaces

X Banach space, Hilbert transform characterizes *UMD*-property,

$\varepsilon \in (0,1)$, $p \in (1,\infty)$ and $N > 1$, $H_{\varepsilon,N} \in \mathcal{B}(L^p(\mathbb{R};X))$

$$(H_{\varepsilon,N}g)(t) := \frac{i}{\pi}\int_{\varepsilon \le |s| \le N} \frac{g(t-s)}{s}\,ds, \qquad g \in L^p(\mathbb{R};X).$$

X is UMD-space $\Leftrightarrow$ $\text{lím}_{\varepsilon,N} H_{\varepsilon,N}g =: H(g)$ and $H \in \mathcal{B}(L^p(\mathbb{R};X))$

Truncated Dirichlet operators, $(\mathcal{D}_N(s))_{s,N\geq 0} \subset \mathcal{B}(L^p(\mathbb{R};X))$

$$\mathcal{D}_N(s)g(t) := \int_{-N}^{N} \frac{\sin(sr)}{r} g(t-r)\frac{dr}{\pi}, \qquad t \in \mathbb{R}.$$

$$\mathcal{F}(s)(g) := f_s * g, \quad g \in L^p(\mathbb{R};X).$$

Theorem Let X be a Banach space. Conditions are equivalent.

(i) $H \in \mathcal{B}(L^p(\mathbb{R};X))$ with $1 < p < \infty$.

(ii) Dirichlet operators $(\mathcal{D}(s))_{s\geq 0} \subset \mathcal{B}(L^p(\mathbb{R};X))$, $1 < p < \infty$

$$\sup_{s>0} \|\mathcal{D}(s)\| := C < \infty,$$

where $\mathcal{D}(s)f := \lim_{N\to+\infty} \mathcal{D}_N(s)f$.

(iii) The Fejér family $(\mathcal{F}(s))_{s\geq 0}$ is a Lipschitz function.

1. UMD spaces

Shift group $(T(s))_{s\in\mathbb{R}}$ in $L^p(\mathbb{R};X)$, $T(s)g = g(\cdot - s)$, then

$$H(g) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{g(\cdot - s)}{s} ds = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{T(s)g}{s} ds.$$

C_0-groups in UMD-spaces, $T \equiv (T(s))_{s\in\mathbb{R}} \subset \mathcal{B}(X)$,

the Hilbert transform, H_0^T, where

$$H_0^T(x) := \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{T(s)x}{s} ds, \qquad x \in X,$$

$$H_r^T(x) := \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{e^{-irs}T(s)x}{s} ds, \qquad x \in X,$$

have been studied by Berkson, Gillespie, Muhly; Monniaux

Objectives

Our main aims (in this talk and in this paper) are

(i) to give definitions of Hilbert transforms, Dirichlet and Fejér operators independent of C_0-groups.

(ii) prove results about Hilbert, Dirichlet and Fejér families in these abstract settings.

(iii) recover known cases, in particular in the UMD case.

(iv) give several examples to illustrate our results.

2. Hilbert families of operators

Definition A Hilbert family $H : \mathbb{R} \to \mathcal{B}(X)$:

(i) $\sup\{\|H(s)\| : s \in \mathbb{R}\} < \infty$;

(ii) $H(s)H(t) = H(t)H(s) = H(s) - H(t) + I$, for $s < t$;

(iii) the function $s \mapsto H(s)x$ has $H(s^-)x$ and $H(s^+)x$;

(iv) $H(s)x = \dfrac{H(s^+)x + H(s^-)x}{2}$ for $s \in \mathbb{R}$ and $x \in X$;

(v) $H(s)x \to -x$ and $H(s)x \to x$ as $s \to -\infty$ and as $s \to +\infty$.

The operator $H(0)$ will be called the generalized Hilbert transform.

2. Hilbert families of operators

Proposition Let $H \equiv (H(s))_{s\in\mathbb{R}}$ be a Hilbert family. Then

(i) $H^3(s) = H(s)$ for $s \in \mathbb{R}$.

(ii) $H(s)x - H(s^-)x = x - H^2(s)x = H(s^+)x - H(s)x$.

(iii) $H^2(s^+)x = H^2(s^-)x = x$.

2. Hilbert families of operators

Recall a *spectral family* is a projection-valued $E : \mathbb{R} \to \mathcal{B}(X)$:

(i) $\sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\} < \infty$;

(ii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\text{mín}\{\lambda, \mu\})$, for $\lambda, \mu \in \mathbb{R}$;

(iii) $E(\cdot)x$ is right continuous on $\mathbb{R}$;

(iv) $E(\cdot)x$ has a left-hand limit at each point of $\mathbb{R}$, $E(s^-)x$;

(v) $E(\lambda)x \to 0$, $E(\lambda)x \to x$ as $\lambda \to -\infty$, $\lambda \to +\infty$.

$$Ax \equiv \int_{\mathbb{R}} \lambda dE(\lambda), \qquad x \in D(A).$$

self-adjoint, scalar-type operators, and well-bounded operators

2. Hilbert families of operators

Theorem Let $E \equiv (E(\lambda))_{\lambda\in\mathbb{R}}$ a spectral family. Then $H^E \equiv (H^E(s))_{s\in\mathbb{R}}$,

$$H^E(s)x := E(s)x + E(s^-)x - x, \qquad x \in X,$$

is a Hilbert family of operators.

Conversely, let $H \equiv (H(s)_{s\in\mathbb{R}}$ be a Hilbert family. Then $E^H \equiv (E^H(s))_{s\in\mathbb{R}}$,

$$E^H(s) := I + \frac{1}{2}\left(H(s) - H^2(s)\right), \qquad s \in \mathbb{R},$$

is a spectral family.

$E = E^{H^E}$ and $H = H^{E^H}$.

2. Hilbert families of operators

X Banach space,

A C_0-group, $(T(t))_{t\in\mathbb{R}} \subset \mathcal{B}(X)$,

$$T(t+s) = T(s)T(t), \qquad t, s \in \mathbb{R},$$

$\lim_{t\to 0} T(t)x = x$.

A cosine function, $(C(t))_{t\geq 0} \subset \mathcal{B}(X)$,

$$C(t+s) + C(t-s) = 2C(t)C(s), \qquad t, s \geq 0$$

$t \mapsto C(t)x$ is continuous on X.

2. Hilbert families of operators

Let X be a UMD Banach space, $T \equiv (T(t))_{t\in\mathbb{R}}$ a C_0-group in X.

Corollary The set $(H^T(s))_{s\in\mathbb{R}}$ is a Hilbert family where

$$H^T(s)x := \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{e^{-its}T(t)x}{t}dt, \qquad x \in X.$$

Corollary Let X be a UMD Banach space and $C \equiv (C(t))_{t\in\mathbb{R}}$ a uniformly bounded cosine function,

$$H_0^C(x) := \frac{2i}{\pi} \lim_{\varepsilon\to 0, N\to\infty} \int_{\varepsilon}^{N} \frac{A^{\frac{1}{2}} \int_0^s C(u)x du}{s} ds,$$

exists for $x \in X$ and $H_0^C \in \mathcal{B}(X)$.

3. Dirichlet families of operators

Definition A Dirichlet family X, $D : \mathbb{R}^+ \to \mathcal{B}(X)$

(i) $\sup\{\|D(\lambda)\| : \lambda \in [0, \infty)\} < \infty$;

(ii) $D(\lambda)D(\mu) = D(\mu)D(\lambda) = D(\mu)$, for $\lambda > \mu \geq 0$;

(iii) $D(\cdot)x$ has a left-hand limit, $D(s^-)x$, and a right-hand limit, $D(s^+)x$, $s \in [0, \infty)$ and $x \in X$;

(iv) $D^2(s)x = \dfrac{D(s)x + D(s^-)x}{2}$ for $s \in [0, \infty)$ and $x \in X$;

(v) $D(0) = 0$ and $D(s)x \to x$ when $s \to \infty$.

3. Dirichlet families of operators

Proposition Let $(H(s))_{s\in\mathbb{R}}$ be a Hilbert family. Then $(D(s))_{s\geq 0}$,

$$D(s) := \frac{1}{2}\left(H(s) - H(-s)\right), \qquad s \in \mathbb{R},$$

is a Dirichlet family.

Let $(E(\lambda))_{\lambda\in\mathbb{R}}$ be a spectral family. Then $(D(s))_{s\geq 0}$,

$$D(s)x := \frac{1}{2}\left(E(s)x + E(s^-)x - E(-s)x - E(-s^-)x\right)$$

is a Dirichlet family.

3. Dirichlet families of operators

$(AC(\mathbb{R}^+), \| \quad \|_{AC})$, $\|f\|_{AC} := \int_0^\infty |f'(s)|ds < \infty$.

Theorem Let $D \equiv (D(s))_{s\geq 0}$ be a Dirichlet family. Define

$$\Phi_D(f)x := -\int_0^\infty f'(s)D(s)xds, \qquad f \in AC(\mathbb{R}^+).$$

Φ_D is an algebra homomorphism of $AC(\mathbb{R}^+) \to \mathcal{B}(X)$

$$\|\Phi_D(f)\| \leq \|f\|_{AC} \sup_{s\geq 0} \|D(s)\|, \qquad f \in AC(\mathbb{R}^+).$$

Moreover $(A, D(A))$, (generator) $\sigma(A) \subset [0, \infty)$ such that

$$(\lambda + A)^{-1}x = \int_0^\infty \frac{D(s)x}{(\lambda + s)^2} ds, \qquad x \in X, \qquad \lambda \in \mathbb{C}\backslash(-\infty, 0].$$

3. Dirichlet families of operators

Corollary Let X be a UMD Banach space and $T \equiv (T(t))_{t\in\mathbb{R}}$ a uniformly bounded C_0-group. Then $(D^T(s))_{s>0}$

$$D^T(s)x := \lim_{N\to\infty} \int_{-N}^{N} \frac{\sin(sr)}{r} T(r)x \frac{dr}{\pi}, \qquad x \in X, s \geq 0,$$

is a Dirichlet family.

Corollary Let X be a UMD Banach space and $C \equiv (C(t))_{t\in\mathbb{R}}$ a cosine function. Then $(D^C(s))_{s>0}$ is a Dirichlet family where

$$D^C(s)x := \lim_{N\to\infty} 2\int_{0}^{N} \frac{\sin(sr)}{r} C(r)x \frac{dr}{\pi}, \qquad x \in X, s \geq 0,$$

4. Fejér families of operators

Definition A Fejér family of operators, $F : [0, \infty) \to \mathcal{B}(X)$

(i) the map $t \mapsto F(t)x$, is continuous.

(ii) $F(t)F(s)x = 2\int_0^s F(u)xdu + (t-s)F(s)x$, $t \geq s \geq 0, x \in X$.

(iii) There exists $C > 0$ such that $\|F(t)\| \leq Ct$ for any $t \geq 0$.

(iv) $F(0) = 0$ and $\lim_{t\to\infty} \frac{1}{t}F(t)x = x$ for $x \in X$.

Proposition Let $(D(s))_{s\geq 0}$ be a Dirichlet family. Then $(F(s))_{s\geq 0}$,

$$F(s)x := \int_0^s D(u)xdu, \qquad s \geq 0,$$

is a Fejér family.

4. Fejér families of operators

$(AC^{(1)}(\mathbb{R}^+), \| \quad \|_{AC^{(1)}})$, $\|f\|_{AC^{(1)}} := \int_0^\infty |f''(t)| t dt.$

Theorem Let $F \equiv (F(s))_{s \geq 0}$ be a Fejér family. Define

$$\Phi_F(f)x := \int_0^\infty f''(s)F(s)x ds, \qquad f \in AC^{(1)}(\mathbb{R}^+).$$

Then Φ_F is an algebra homomorphism of $AC^{(1)}(\mathbb{R}^+)$ into $\mathcal{B}(X)$,

$$\|\Phi_F(f)\| \leq \|f\|_{AC^{(1)}} \sup_{s \geq 0} (\frac{1}{s} \|F(s)\|), \qquad f \in AC^{(1)}(\mathbb{R}^+).$$

Moreover A (generator) with $\sigma(A) \subset [0, \infty)$ such that

$$(\lambda + A)^{-1} x = -2 \int_0^\infty \frac{F(s)x}{(\lambda + s)^3} ds, \qquad x \in X, \qquad \lambda \in \mathbb{C} \backslash (-\infty, 0].$$

4. Fejér families of operators

(1)[GP] $(T(z))_{z\in\mathbb{C}^+}$, $\|T(z)\| \le C\left(\frac{|z|}{\Re z}\right)^{\alpha}$ with $0 \le \alpha < 1$,

$$F^T(s)x := \frac{1}{2\pi i}\int_{\Re z=1}\frac{T(z)x}{z^2}e^{sz}\,dz, \qquad x \in X.$$

(2) $(T(t))_{t\in\mathbb{R}}$ such that $\|T(t)\| \le C$, then

$$F^T(s)x := \int_{-\infty}^{\infty}\frac{1-\cos(st)}{t^2}T(t)x\frac{dt}{\pi}, \qquad x \in X.$$

(3) $(C(t))_{t>0}$ such that $\|C(t)\| \le C$, then

$$F^C(s)x := 2\int_0^{\infty}\frac{1-\cos(st)}{t^2}C(t)x\frac{dt}{\pi}, \qquad x \in X.$$

5. Comments and examples

Open question: given $(D(s))_{s\geq 0}$ a Dirichlet family in a UMD space, does there exist $(H(s))_{s\in\mathbb{R}}$ a Hilbert family such that

$$D(s) = \frac{1}{2}(H(s) - H(-s))?$$

In the case of C_0-group, YES.

$(C(t))_{t\geq 0}$ cosine functions in a UMD, there exists $(T(t))_{t\in\mathbb{R}}$

$$C(t) = \frac{T(t) + T(-t)}{2}, \qquad t \geq 0.$$

Some results may be proved using "transference principle"

5. Comments and examples

(1) Take $X = L^1(\mathbb{R})$, and $T \equiv (T(s))_{s\in\mathbb{R}}$ $T(s)f(t) := e^{its}f(t)$.

$$H^T(s)f(t) = \frac{i}{\pi}\int_{\mathbb{R}} \frac{e^{-isr}e^{itr}}{r} dr f(t) = -\mathrm{sgn}(t-s)f(t).$$

For $s \geq 0$, we may compute $(D^T(s))_{s\geq 0}$ by

$$D^T(s)f(t) = \int_{-\infty}^{\infty} \frac{\sin(sr)}{r} e^{irt}\frac{dr}{\pi} f(t) \equiv \chi_{(-s,s)}(t)f(t).$$

Then $(D^T(s))_{s\geq 0}, (H^T(s))_{s\in\mathbb{R}} \subset \mathcal{B}(X)$.

5. Comments and examples

(2) Take $X = C([\pi, 2\pi])$, $T \equiv (T(s))_{s\in\mathbb{R}}$, $T(s)f(t) := e^{its}f(t)$.

$$H^T(s)f(t) = -\mathrm{sgn}(t-s)f(t),$$

for $t \in [\pi, 2\pi]$. Note that $H^T(s)f \in X \Leftrightarrow s > 2\pi$ or $s < \pi$.

$$D^T(s)f(t) = \chi_{(-s,s)}(t)f(t) + \frac{1}{2}\chi_{\{s\}}(t)f(t), \qquad t \in [\pi, 2\pi].$$

Then $D^T(s)f \in X \Leftrightarrow 0 \le s < \pi$, or $s > 2\pi$.

5. Comments and examples

(4) $\mathbb{T}$, $X = C(\mathbb{T})$, $T \equiv (T(t))_{t\in\mathbb{R}}$, $T(t)f(z) := f(e^{-it}z)$

Then $H^T(s) \notin \mathcal{B}(X)$ for $s \in \mathbb{R}$ and $(D^T(s))_{s\geq 0} \subset \mathcal{B}(X)$.

Take $P(z) = \sum_{j=-m}^{m} \hat{P}(j)z^j$

$$(H^T(0)P)(z) = \sum_{j=-m}^{m} \operatorname{sgn}(j)\hat{P}(j)z^j, \qquad z \in \mathbb{T}.$$

Suppose that $H^T(0) \in \mathcal{B}(X)$,

$$\| \sum_{j=-m}^{m} \operatorname{sgn}(j)\hat{P}(j)z^j \|_\infty \leq \|H^T(0)\| \, \| \sum_{j=-m}^{m} \hat{P}(j)z^j \|$$

$f \in C(\mathbb{T})$, the conjugate Fourier series for f is the

Fourier series of a function $\tilde{f} \in C(\mathbb{T})$. False: $H^T(0) \notin \mathcal{B}(X)$.

5. Comments and examples

Similar ideas $H^T(s)$ for $s \in \mathbb{R}$.

Take $Q = \sum_{j=-m}^{m} \hat{Q}(j) z^j$, $s = n + r$. Then

$$(D^T(s)Q)(z) = (D_n(Q))(z) - \frac{1}{2}\chi_{\{0\}}(r)(\hat{Q}(-n)z^{-n} + \hat{Q}(n)z^n),$$

where $D_n(Q) := \sum_{j=-n}^{j=n} \hat{Q}(j) z^j$. $D^T(s) \in \mathcal{B}(X) \Leftrightarrow D_n \in \mathcal{B}(X)$.

Since $\|D_n\| \sim \log(n)$, $(D^T(s))_{s>0} \subset \mathcal{B}(X)$, $\sup_{s>0} \|D^T(s)\| = \infty$.