An uncertainty inequality for Fourier–Dunkl expansions

and more...

Óscar Ciaurri and Juan Luis Varona

Departamento de Matemáticas y Computación
Universidad de La Rioja
Logroño, Spain

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Scheme of the talk

1. Introduction
    - The classical uncertainty inequality
    - An uncertainty inequality for the Dunkl transform
    - An uncertainty inequality for the Fourier series
    - Other uncertainty inequalities
2. An uncertainty inequality for the Fourier–Dunkl expansions
    - The Fourier–Dunkl orthonormal system
    - The uncertainty inequality
    - Some consequences
3. More about uncertainty inequalities and orthonormal expansions
    - Work in progress
    - Some questions

The classical uncertainty inequality

A nonzero function and its Fourier transform cannot be sharply localized

This assert, with its physics meaning, was given by Heisenberg in 1927.
The quantitative expression was formulated by Weyl in 1928 (crediting the result to Pauli).
Being the variance defined as

$$\mathrm{V}(f) = \int_{\mathbb{R}} (x - \langle xf, f \rangle_{L^2(\mathbb{R})})^2 |f(x)|^2 \, dx,$$

for functions $f \in L^2(\mathbb{R})$, such that $\|f\|_{L^2(\mathbb{R})} = 1$,

$$\mathrm{V}(f)\,\mathrm{V}(\hat{f}) \geq \frac{1}{4}.$$

Moreover, the equality holds for a proper modification of the gaussian function $e^{-\gamma x^2}$, with $\gamma > 0$.

In the *n*-dimensional case the uncertainty inequality (UI) becomes

$$\mathrm{V}(f)\,\mathrm{V}(\hat{f}) \geq \frac{n^2}{4}$$

where, now,

$$\mathrm{V}(f) = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \left| x_i - \langle x_i f, f \rangle_{L^2(\mathbb{R}^n)} \right|^2 |f(x)|^2 \, dx.$$

Note that

$$\mathrm{V}(\hat{f}) = \left\| \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} - \left\langle \frac{\partial f}{\partial x_i}, f \right\rangle_{L^2(\mathbb{R}^n)} \right) f \right\|_{L^2(\mathbb{R}^n)}^2 .$$

“The uncertainty principle: a mathematical survey” by G. B. Folland and A. Sitaram (*J. Fourier Anal. Appl.*, 1997).

An uncertainty inequality for the Dunkl transform

We define the Hankel transform of order α, with $\alpha \geq -1/2$, as

$$\mathcal{H}_\alpha g(s) = \int_0^\infty g(r) j_\alpha(sr)\, d\omega_\alpha(r), \qquad s > 0,$$

with $d\omega_\alpha(r) = (2^\alpha \Gamma(\alpha+1))^{-1} r^{2\alpha+1}\, dr$ and $j_\alpha(z) = \Gamma(\alpha+1)(z/2)^{-\alpha} J_\alpha(z)$, being J_α the Bessel function of order α. For a radial function $f(x) = g(\|x\|_2)$ on $\mathbb{R}^n$, we have

$$\hat{f}(\xi) = \mathcal{H}_{n/2-1} g(\|\xi\|_2).$$

In this setting we have the following UI: for $f \in L^2((0,\infty), d\omega_\alpha)$ such that $\|f\|_{L^2((0,\infty),d\omega_\alpha)} = 1$

$$\int_0^\infty (rf(r))^2\, d\omega_\alpha(r) \int_0^\infty (s\mathcal{H}_\alpha f(s))^2\, d\omega_\alpha(s) \geq (\alpha+1)^2$$

Moreover, the equality holds if and only if $f(r) = de^{-cr^2/2}$ ($d \in \mathbb{C}$ and $c > 0$).

Different proofs of this UI are known:

- Corollary of the n-dimensional situation for the Fourier transform when $\alpha = n/2 - 1$.
- Proved, in the general case, by Boyle in 1971 (*SIAM J. Math. Anal.*).
- Generalized for the Dunkl transform by Rösler–Voit in 1999 (*Proc. Amer. Math. Soc.*).
- An unpublished proof was given by Roosenraad in his Ph. D. Thesis in 1969.

Dunkl operators on $\mathbb{R}^n$ are differential-difference operators associated with some finite reflection groups. With the group $\mathbb{Z}_2$ on $\mathbb{R}$ and for $\alpha \geq -1/2$, the Dunkl operator Λ_α is given by

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha + 1}{x} \left(\frac{f(x) - f(-x)}{2} \right).$$

The Dunkl kernel E_α is, for $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the unique solution of the initial value problem

$$\begin{cases} \Lambda_\alpha f(x) = \lambda f(x), \quad x \in \mathbb{R}, \\ f(0) = 1. \end{cases}$$

and can be written as

$$E_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz).$$

Taking $d\mu_\alpha(x) = \frac{1}{2} d\omega_\alpha(|x|)$, the Dunkl transform of order α, is defined by

$$\mathcal{F}_\alpha f(y) = \int_{\mathbb{R}} f(x) E_\alpha(-iyx)\, d\mu_\alpha(x).$$

Using that $E_{-1/2}(z) = e^z$, we obtain that $\mathcal{F}_{-1/2}(f) = \hat{f}$.

Moreover, for even functions

$$\mathcal{F}_\alpha(f(y)) = \mathcal{H}_\alpha f(|y|).$$

Being

$$\operatorname{var}(f) = \|(x - \langle xf, f\rangle_{L^2(\mathbb{R},d\mu_\alpha)})f\|^2_{L^2(\mathbb{R},d\mu_\alpha)}$$

and

$$\operatorname{var}(\mathcal{F}_\alpha f) = \|(\Lambda_\alpha - \langle \Lambda_\alpha f, f\rangle_{L^2(\mathbb{R},d\mu_\alpha)})f\|^2_{L^2(\mathbb{R},d\mu_\alpha)},$$

the following result holds: for $f \in L^2(\mathbb{R}, d\mu_\alpha)$ such that $\|f\|_{L^2(\mathbb{R},d\mu_\alpha)} = 1$

$$\operatorname{var}(f)\cdot\operatorname{var}(\mathcal{F}_\alpha f) \geq \left(\left(\alpha + \frac{1}{2}\right)(\|f_e\|_{L^2(\mathbb{R},d\mu_\alpha)} - \|f_o\|_{L^2(\mathbb{R},d\mu_\alpha)}) + \frac{1}{2}\right)^2$$

where f_o and f_e are the odd and the even part of f respectively.

An uncertainty inequality for Fourier series

Breitenberger in 1985 (*Found. Phys.*) proposed a UI for the Fourier series. For a function $f \in L^2(\mathbb{T}, dm)$ such that $\|f\|_{L^2(\mathbb{T},dm)} = 1$ and $f(z) = \sum_{k\in\mathbb{Z}} a_k z^k$, the frequency variance is given by

$$\operatorname{var}_F(f) = \sum_{k\in\mathbb{Z}} k^2 |a_k|^2 - \left(\sum_{k\in\mathbb{Z}} \operatorname{sgn}(k) k |a_k|^2 \right)^2 .$$

Defining the mean localization of the function as

$$\tau(f) = \int_{\mathbb{T}} z |f(z)|^2 \, dm,$$

the angular variance of f is given by

$$\operatorname{var}_A(f) = 1 - |\tau(f)|^2.$$

With this notation the UI for Fourier series establishes that

$$\operatorname{var}_A(f) \cdot \operatorname{var}_F(f) \geq \frac{1}{4} |\tau(f)|^2. \qquad (1)$$

Note that

$$\operatorname{var}_A(f) = \|(z - \langle zf, f\rangle_{L^2(\mathbb{T},dm)})f\|^2_{L^2(\mathbb{T},dm)}$$

and

$$\operatorname{var}_F(f) = \left\| \left(\frac{d}{dz} - \left\langle \frac{df}{dz}, f \right\rangle_{L^2(\mathbb{T},dm)} \right) f \right\|^2_{L^2(\mathbb{T},dm)}.$$

The constant $1/4$ appearing in the IU for the Fourier series is optimal in the following sense: it is not attained by any function but there exists a function f_t, with $t > 0$, verifying the equality

$$\lim_{t\to 0} \frac{\operatorname{var}_A(f_t)}{t} \lim_{t\to 0} t \operatorname{var}_F(f_t) = \frac{1}{4} \lim_{t\to 0} |\tau(f_t)|^2.$$

The function f_t is given by

$$f_t(x) = \sum_{k\in\mathbb{Z}} e^{-tk^2} e^{ikx}.$$

This result was established by Prestin and Quak (1999, *Proc. Edinb. Math. Soc.*).

Other uncertainty inequalities

Some UI's have been proved in other settings:

- For ultraspherical expansions (Rösler–Voit, 1997, *J. Math. Anal. Appl.*).
- On the sphere $\mathbb{S}^n$ with the quantum angular moment $\Omega = ix \times \nabla$ and the Laplace–Beltrami operator $\Delta_{\mathbb{S}}^n = -\Omega \cdot \Omega$. The case $n = 2$ is due to Narcowich–Ward (1996, *Appl. Comput. Harmon. Anal.*).
- For Jacobi expansions (Li–Liu, 2003, *J. Math. Anal. Appl.*).
- For expansions related to eigenfunctions of Sturm–Liouville operators (Li–Liu, 2005, *Constr. Approx.*). As particular cases, UI's for Laguerre polynomials and for generalized Hermite polynomials are deduced.
- For the discrete Fourier expansions (Grünbaum, 2005, *Appl. Comput. Harmon. Anal.*).

Note. In the previous cases the sharpness of the constant involved in the UI has been analyzed following the method developed by Prestin–Quad.

- For self-adjoint operators L defined on spaces measures (X, μ) equipped with a metric such that
  - a) the measure of the balls are controlled by powers of the radius,
  - b) the heat semigroup satisfies estimates with polynomial growth.

  The result in this case is a weighted UI (Ciatti–Ricci–Sundari, 2007, *Advances in Math.*)
- For continuous Jacobi transforms (Ma, 2007, *J. Math. Anal. Appl.*).

A result of other kind:

$$\text{UI for the Fourier series} \Longrightarrow \text{UI for the Fourier transform on } \mathbb{R}$$

(Prestin–Quak–Rauhut–Selig, 2003, *J. Fourier Anal. Appl.*).

The Fourier–Dunkl orthonormal system

We consider the functions

$$e_{\alpha,j}(x) = \frac{2^{\alpha/2}(\Gamma(\alpha+1))^{1/2}}{|j_\alpha(s_j)|} E_\alpha(is_j x), \quad j \in \mathbb{Z} \setminus \{0\}, \quad x \in (-1,1),$$

and $e_{\alpha,0}(x) = 2^{(\alpha+1)/2}(\Gamma(\alpha+2))^{1/2}$, where $\{s_j\}_{j\in\mathbb{Z}}$ are the zeros of the function $\operatorname{Im}(E_\alpha(ix)) = \frac{x}{2(\alpha+1)} j_{\alpha+1}(x)$ (with $s_{-j} = -s_j$ and $s_0 = 0$).

Theorem (C–V, 2007, *Proc. Amer. Math. Soc.*)

For $\alpha \geq -1/2$, the sequence of functions $\{e_{\alpha,j}\}_{j\in\mathbb{Z}}$ is a complete and orthornormal system in $L^2((-1,1), d\mu_\alpha)$.

The Fourier–Dunkl series are given by

$$f \sim \sum_{j\in\mathbb{Z}} a_j(f) e_{\alpha,j}, \qquad a_j(f) = \int_{-1}^{1} f(y)\overline{e_{\alpha,j}(y)}\, d\mu_\alpha.$$

The sequence of functions $\{e_{-1/2,j}\}_{j\in\mathbb{Z}}$ corresponds with the classical exponential system.

The uncertainty inequality

To enounce the uncertainty inequality related to the Fourier–Dunkl series, we consider the variances defined by

$$\mathbf{var}_A^\alpha(f) = \|(e_{\alpha,1} - \tau_\alpha(f))f\|^2_{L^2((-1,1),d\mu_\alpha)},$$

where

$$\tau_\alpha(f) = \langle e_{\alpha,1} f, f\rangle_{L^2((-1,1),d\mu_\alpha)},$$

and

$$\mathbf{var}_F^\alpha(f) = \|(\Lambda_\alpha - \langle \Lambda_\alpha f, f\rangle_{L^2((-1,1),d\mu_\alpha)})f\|^2_{L^2((-1,1),d\mu_\alpha)}.$$

Theorem

For $f \in C_c^1(-1,1)$, with $\|f\|_{L^2((-1,1),d\mu_\alpha)} = 1$, and $\alpha \geq -1/2$, there holds

$$\mathbf{var}_A^\alpha(f)\,\mathbf{var}_F^\alpha(f) \geq \left| \frac{s_1}{2}\tau_\alpha(f) - \int_{-1}^{1} \frac{2\alpha+1}{x} \operatorname{Im}(e_{\alpha,1}(x))|f_o(x)|^2 \, d\mu_\alpha(x) \right|^2,$$

where f_o denotes the odd part of the function f.

Some consequences

The UI for Fourier–Dunkl series can be written in a more usual way. It is clear that

$$\mathbf{var}_A^\alpha(f) = \int_{-1}^{1} |e_{\alpha,1}(x) f(x)|^2 \, d\mu_\alpha(x) - \tau_\alpha(f)^2.$$

Moreover, if each function $f \in C_c^2(-1,1)$ has the Fourier–Dunkl expansion

$$f(x) = \sum_{j\in\mathbb{Z}} a_j(f) e_{\alpha,j}(x),$$

we have

$$\begin{aligned} \mathbf{var}_F^\alpha(f) &= |\langle \Lambda_\alpha^2 f, f\rangle| - |\langle \Lambda_\alpha f, f\rangle|^2 \\ &= \sum_{j\in\mathbb{Z}} s_j^2 |a_j(f)|^2 - \left(\sum_{j\in\mathbb{Z}} (\operatorname{sgn} j) s_j |a_j(f)|^2 \right)^2. \end{aligned}$$

Corollary

For $\alpha \geq -1/2$ *and* $f \in L^2((-1,1), d\mu_\alpha)$ *verifying that* $\|f\|_{L^2((-1,1),d\mu_\alpha)} = 1$, *the inequality*

$$\left(\int_{-1}^{1} |e_{\alpha,1}(x) f(x)|^2 \, d\mu_\alpha(x) - \tau_\alpha(f)^2 \right) \times \left(\sum_{j\in\mathbb{Z}} s_j^2 |a_j(f)|^2 - \left(\sum_{j\in\mathbb{Z}} (\operatorname{sgn} j) s_j |a_j(f)|^2 \right)^2 \right) \geq \left| \frac{s_1}{2} \tau_\alpha(f) - \int_{-1}^{1} \frac{2\alpha+1}{x} \operatorname{Im}(e_{\alpha,1}(x)) |f_o(x)|^2 \, d\mu_\alpha(x) \right|^2$$

holds

The extension to $L^2((-1,1), d\mu_\alpha)$ is obtained with a standard density argument.

- The Breitenberger UI for Fourier series is a consequence of our main result.

  This follows from the identity

$$\{e_{-1/2,j}(t)\}_{j\in\mathbb{Z}} = \left\{ \left(\frac{2}{\pi}\right)^{1/4} e^{ij\pi t} \right\}_{j\in\mathbb{Z}} .$$

  and the relation between the variances var_A, var_F and $\mathbf{var}_A^{-1/2}$, $\mathbf{var}_F^{-1/2}$

- For even and odd functions the Fourier–Dunkl series become Fourier–Dini and Fourier–Bessel series respectively. With the appropriate modifications, we can deduce UI for these orthonormal systems.

Work in progress

The UI for the Fourier–Bessel series obtained as consequence of our result is not completely satisfactory.
For $j \geq 1$, $k \geq 0$ and $\alpha > -1$, we consider the functions

$$\phi_j^{\alpha,k}(r) = d(j,\alpha,k) r^{-\alpha} J_{\alpha+k}(\lambda_{j,\alpha+k} r)$$

where $\{\lambda_{j,\beta}\}_{j\geq 1}$ are the zeros of the equation $J_\beta(z) = 0$. This family of functions is orthonormal and complete in $L^2((0,1), r^{2\alpha+1}\,dr)$.
For the case $k = 0$, we have proved the UI

$$\int_0^1 r^2 |f(r)|^2 r^{2\alpha+1}\,dr \cdot \sum_{j=1}^{\infty} \lambda_{j,\alpha}^2 |c_{j,\alpha,0}(f)|^2 \geq (\alpha+1)^2,$$

where $\|f\|_{L^2((0,1),r^{2\alpha+1}\,dr)} = 1$ and $c_{j,\alpha,0}(f) = \int_0^1 f(t)\phi_j^{\alpha,k}(t) t^{2\alpha+1}\,dt$.

We are working in the sharpness of the constant. The main point is the evaluating of the integral

$$\int_0^1 r^2 \phi_j^{\alpha,0}(r)\phi_k^{\alpha,0}(r) r^{2\alpha+1}\, dr.$$

The following step is the proof of a vector valued UI for the multidimensional Fourier–Bessel expansions.

Being $\{\mathcal{Y}_{\ell,k}\}_{1\le\ell\le D(k)}$, a basis of spherical harmonics of degree k orthonormal in $L^2(\mathbb{S}^{n-1}, d\sigma)$, we consider the functions

$$\Phi_j^{n/2-1,k,\ell}(x) = \phi_j^{n/2-1,k}(|x|)\mathcal{Y}_{\ell,k}\left(\frac{x}{|x|}\right).$$

This system is orthonormal and complete in

$$L^2(B^n) = L^2((0,1)\times\mathbb{S}^{n-1}, r^{n-1}\, dr\, d\sigma).$$

In this setting, our target is the proof of the UI

$$\int_0^1 \sum_{k=0}^{\infty} \sum_{\ell=1}^{D(k)} r^2 |f_{\ell,k}(r)|^2 \, r^{n-1} \, dr \\ \times \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{D(k)} \lambda_{j,n/2-1+k}^2 |c_{j,n/2-1,k}(f_{\ell,k})|^2 \geq \frac{n^2}{4},$$

with

$$\|f\|_{L^2(B^n)}^2 = \int_0^1 \sum_{k=0}^{\infty} \sum_{\ell=1}^{D(k)} |f_{\ell,k}(r)|^2 r^{n-1} \, dr = 1,$$

where

$$f_{\ell,k}(r) = \int_{\mathbb{S}^{n-1}} f(r,\theta) \mathcal{Y}_{\ell,k}(\theta) \, d\sigma(\theta)$$

and $c_{j,n/2-1,k}(f_{\ell,k}) = \int_0^1 f_{\ell,k}(t) \phi_j^{n/2-1,k}(t) t^{n-1} \, dt$.

The vector value UI can be interpreted in terms of the classical laplacian $-\Delta$.

Indeed, for a function such that $\|f\|_{L^2(B^n)} = 1$, the inequality

$$\|xf(x)\|^2_{L^2(B^n)} \cdot \|(-\Delta)^{1/2} f(x)\|^2_{L^2(B^n)} \geq \frac{n^2}{4}$$

is equivalent to the UI given for the multidimensional Fourier–Bessel.

Some questions

- In the same way that the UI for the Fourier series implies the UI for Fourier transform, we conjecture that

  $$\text{UI for Fourier–Dunkl series} \Longrightarrow \text{UI for the Dunkl transform.}$$

  Problem: a Poisson summation formula is not available for the Dunkl transform.
- Study of weighted UI for series using the ideas developed in the paper by Ciatti–Ricci–Sundari.
- Study of inequalities with other information measures (Shannon entropy, Fisher information,...) for series.