Riesz transforms for the Dunkl Ornstein–Uhlenbeck semigroup

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Scheme of the talk

1. Introduction, preliminaries and Main result
   - Dunkl operators
   - Generalized Hermite polynomials
   - $\mathbb{Z}_2^d$ Riesz 'Dunkl' transforms for the O–U semigroup (RDOU)
2. 'Alternative' Laguerre setting and supplementary Riesz transforms
   - 'Alternative' Laguerre polynomials setting
   - Supplementary Riesz transforms
3. Proof of the Main Theorem - the one-dimensional case
   - Preliminaries
   - The even case
   - The odd case
   - Some results needed for the odd case
4. The multidimensional case
   - The multidimensional case: a draft

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Definitions and remarks

- Dunkl operators on $\mathbb{R}^d$ → *differential-difference* associated to some finite reflection group and a nonnegative multiplicity function $k\colon R \longrightarrow [0,\infty)$ on a root system $R \subset \mathbb{R}^d$.
- Consider the finite reflection group generated by σ_j, $j = 1, \dots, d$,

$$\sigma_j(x_1, \dots, x_j, \dots, x_d) = (x_1, \dots, -x_j, \dots, x_d).$$

  - Isomorphic to $\mathbb{Z}_2^d = \{0,1\}^d$.
  - We may think $k = (\alpha_1 + 1/2, \dots, \alpha_d + 1/2)$, $\alpha_j \geq -1/2$.
  - Dunkl operators T_j^α, $j = 1, \dots, d$, given by

$$T_j^\alpha f(x) = \partial_j f(x) + (\alpha_j + 1/2)\frac{f(x) - f(\sigma_j x)}{x_j}.$$

Definitions and remarks (2)

- In Dunkl's theory the operator

  $$\Delta_\alpha = \sum_{j=1}^{d} (T_j^\alpha)^2$$

  plays the role of the Euclidean Laplacian.
- The explicit form of the Dunkl Laplacian is

$$\Delta_\alpha f(x) = \sum_{j=1}^{d} \left(\frac{\partial^2 f}{\partial x_j^2}(x) + \frac{2\alpha_j + 1}{x_j} \frac{\partial f}{\partial x_j}(x) - (\alpha_j + 1/2) \frac{f(x) - f(\sigma_j x)}{x_j^2} \right).$$

Operators analogues of classical second order DE

The operator

$$-\Delta_\alpha + \|x\|^2,$$

which, due to the harmonic confinement $\|x\|^2$, we call the *Dunkl harmonic oscillator*, becomes the classical harmonic oscillator when $\alpha \equiv (-1/2, \dots, -1/2)$.
This operator has a discrete spectrum and the corresponding eigenfunctions are the generalized Hermite functions (Rösler, *Lecture Notes in Math.*, 2003).
The operator

$$L_\alpha = -\Delta_\alpha + 2x \cdot \nabla,$$

becomes the classical infinitesimal generator of an *Ornstein–Uhlenbeck process* when $\alpha \equiv (-1/2, \dots, -1/2)$.
In a similar way L_α has a discrete spectrum and the corresponding eigenfunctions are the generalized Hermite polynomials (Rösler).

Generalized Hermite polynomials

In dimension one, for $\alpha > -1$, $n \in \mathbb{N}^d$,

$$H_{2n}^{\alpha+1/2}(x) = (-1)^n 2^{2n} n! L_n^{\alpha}(x^2),$$
$$H_{2n+1}^{\alpha+1/2}(x) = (-1)^n 2^{2n+1} n! x L_n^{\alpha+1}(x^2).$$

The system is orthogonal in $L^2(\mathbb{R}, \mu_\alpha)$, where μ_α denotes the measure in $\mathbb{R}$ given by

$$d\mu_\alpha(x) = e^{-x^2} w_\alpha(x)\, dx$$

and

$$w_\alpha(x) = |x|^{2\alpha+1}, \qquad x \in \mathbb{R}.$$

The d-dimensional setting,

$$H_n^{\alpha+1/2}(x) = H_{n_1}^{\alpha_1+1/2}(x_1) \cdot \ldots \cdot H_{n_d}^{\alpha_d+1/2}(x_d).$$

Generalized Hermite polynomials (2)

We now define

$$\mathcal{H}^{\alpha_j}_{2n_j}(x_j) = d_{2n_j,\alpha_j} L^{\alpha_j}_{n_j}(x_j^2), \qquad \mathcal{H}^{\alpha_j}_{2n_j+1}(x_j) = d_{2n_j+1,\alpha_j} x_j L^{\alpha_j+1}_{n_j}(x_j^2),$$

and

$$\mathcal{H}^{\alpha}_{n}(x) = \mathcal{H}^{\alpha_1}_{n_1}(x_1) \cdot \ldots \cdot \mathcal{H}^{\alpha_d}_{n_d}(x_d),$$

where d_{2n_j,α_j} and d_{2n_j+1,α_j} are the normalizing constants.
$\{\mathcal{H}^{\alpha}_n : n \in \mathbb{N}^d\}$ orthonormal basis in $L^2(\mathbb{R}^d, \mu_\alpha)$,
$d\mu_\alpha(x) = e^{-\|x\|^2} \prod_{j=1}^d |x_j|^{2\alpha_j+1}\, dx$,

$$L_\alpha \mathcal{H}^{\alpha}_n = 2|n|\mathcal{H}^{\alpha}_n.$$

Spectral decomposition of $\mathcal{L}_\alpha$

$$\mathcal{L}_\alpha f = \sum_{m=0}^{\infty} 2m \sum_{|n|=m} \langle f, \mathcal{H}^{\alpha}_n \rangle_\alpha \mathcal{H}^{\alpha}_n, \qquad f \in \mathrm{Dom}(\mathcal{L}_\alpha).$$

$\mathbb{Z}_2^d$ RDOU: formal definition

A direct computation shows that $L_\alpha = -\Delta_\alpha + 2x \cdot \nabla$ can be written in the form

$$L_\alpha = \frac{1}{2}\sum_{j=1}^{d}\left(\delta_j^*\delta_j + \delta_j\delta_j^*\right) - A, \qquad A = 2\left(1 + \sum_{j=1}^{d}(\alpha_j + 1/2)\right),$$

where

$$\delta_j = T_j^\alpha,$$

and

$$\delta_j^* = -T_j^\alpha + 2x_j$$

is the adjoint of δ_j in $L^2(\mathbb{R}^d, \mu_\alpha)$.

$\mathbb{Z}_2^d$ RDOU: formal definition (2)

It is reasonable to define, at least formally, the Riesz transform $\mathcal{R}^\alpha = (R_1^\alpha, \dots, R_d^\alpha)$ associated with L_α as

$$R_j^\alpha = \delta_j (\mathcal{L}_\alpha)^{-1/2} \Pi_0;$$

here $\mathcal{L}_\alpha$ is a suitable self-adjoint extension in $L^2(\mathbb{R}^d, \mu_\alpha)$ of L_α and Π_0 denotes the orthogonal projection onto the orthogonal complement of the eigenspace corresponding to the eigenvalue 0.

$\mathbb{Z}_2^d$ RDOU: motivation

Dunkl operators, origin:

- Dunkl (*Trans. AMS*, 1989).

Generalized Hermite polynomials and Dunkl operators:

- Rösler (*Commun. Math. Phys.*, 1998, *Lecture Notes in Math.*, 2003).

Riesz transforms related O–U semigroup

- One dimension: Muckenhoupt (*Trans. Amer. Math. Soc.*).
- Any dimension, any order: P.A. Meyer (*Lecture Notes in Math.*, 1984).
- Other proofs: Gundy, Pisier, Gutiérrez, Segovia, Torrea, Fabes, Scotto, Sjögren...

Riesz transforms in the Dunkl framework:

- Thangavelu and Xu, *Riesz transform and Riesz potentials for Dunkl transform*, J. Comput. Appl. Math, 2007.
- Nowak and Stempak, *Riesz transforms for the Dunkl harmonic oscillator*, preprint .

$\mathbb{Z}_2^d$ RDOU: rigorous definition

We have

$$\delta_j \mathcal{H}_n^\alpha = m_j(n,\alpha)\mathcal{H}_{n-e_j}^\alpha,$$

where

$$m_j(n,\alpha) = \begin{cases} \sqrt{2n_j}, & \text{if } n_j \text{ is even,} \\ \sqrt{2n_j + 4\alpha_j + 2}, & \text{if } n_j \text{ is odd.} \end{cases}$$

Also

(1) the system $\{\delta_j \mathcal{H}_n^\alpha : n_j \geq 1\}$, is an orthogonal system in $L^2(\mathbb{R}^d, \mu_\alpha)$;

(2) $\|\delta_j \mathcal{H}_n^\alpha\| = \mathcal{O}(|n|^{1/2})$,

then, rigorous definition of R_j^α, bounded on $L^2(\mathbb{R}^d, \mu_\alpha)$

$$R_j^\alpha f = \sum_{n \in \mathbb{N}_*^d} \frac{m_j(n,\alpha)}{\sqrt{2|n|}} \langle f, \mathcal{H}_n^\alpha \rangle_\alpha \, \mathcal{H}_{n-e_j}^\alpha.$$

RDOU: Main result

Main Theorem

Assume that $\alpha = (\alpha_1, \ldots, \alpha_d)$ *is a multi-index such that* $\alpha_i \geq -1/2$, $i = 1, \ldots, d$, *and* $1 < p < \infty$. *Then the Riesz operators* R_j^α, $j = 1, \ldots, d$, *defined on* $L^2(\mathbb{R}^d, \mu_\alpha)$ *extend to bounded linear operators on* $L^p(\mathbb{R}^d, \mu_\alpha)$.

'Alternative' Laguerre polynomials setting

- New setting to discuss → from Nowak, *J. Funct. Anal.*, 2004 by the change of variables $(x_1, \dots, x_d) \mapsto (x_1^2, \dots, x_d^2)$ on $\mathbb{R}_+^d$.
- Measure $\hat{\mu}_\alpha$ in $\mathbb{R}_+^d$ given by $d\hat{\mu}_\alpha(x) = e^{-\|x\|^2} \big(\prod_{j=1}^d x_j^{2\alpha_j+1} \big)\, dx$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_j > -1$.
- The operator

$$\widehat{\mathcal{L}}^\alpha = -\sum_{j=1}^d \left[\partial_j^2 + \frac{2\alpha_j + 1 - 2x_j^2}{x_j} \partial_j \right].$$

- We have

$$\widehat{\mathcal{L}}^\alpha L_n^\alpha(x^2) = 4|n| L_n^\alpha(x^2).$$

‘Alternative’ Laguerre polynomials setting (2)

A direct computation shows that

$$\widehat{\mathcal{L}}^{\alpha} = \sum_{j=1}^{d} \hat{\delta}_j^* \hat{\delta}_j$$

where

$$\hat{\delta}_j = \partial_j$$

and the formal adjoint is

$$\hat{\delta}_j^* = -\partial_j - \frac{2\alpha_j + 1 - 2x_j^2}{x_j}$$

In this setting, the Riesz transform $\widehat{\mathcal{R}}^{\alpha} = (\widehat{R}_1^{\alpha}, \dots, \widehat{R}_d^{\alpha})$ associated with $\widehat{\mathcal{L}}^{\alpha}$ is then formally defined by

$$\widehat{R}_j^{\alpha} = \hat{\delta}_j (\widehat{\mathcal{L}}^{\alpha})^{-1/2} \Pi_0 .$$

‘Alternative’ Laguerre polynomials setting (3)

With respect to the orthonormalized system $\{\widetilde{L}_n^\alpha(x^2) : n \in \mathbb{N}^d\}$

$$\widehat{R}_j^\alpha f(x) = -\sum_{n\in\mathbb{N}_*^d} \left(\frac{n_j}{|n|}\right)^{1/2} \langle f, \widetilde{L}_n^\alpha((\cdot)^2)\rangle_{L^2(\mathbb{R}_+^d,\,\hat{\mu}_\alpha)}\, x_j \widetilde{L}_{n-e_j}^{\alpha+e_j}(x^2), \quad x \in \mathbb{R}_+^d.$$

$\widehat{R}_j^\alpha$ is a bounded operator on $L^2(\mathbb{R}_+^d, \hat{\mu}_\alpha)$. We have

$$\widehat{R}_j^\alpha L_n^\alpha(x^2) = -|n|^{-1/2} x_j L_{n-e_j}^{\alpha+e_j}(x^2), \qquad x \in \mathbb{R}_+^d. \tag{1}$$

Theorem 1 (Nowak, Theorem 13)

Assume that $\alpha = (\alpha_1, \ldots, \alpha_d)$ *is a multi-index such that* $\alpha_i \geq -1/2$, $i = 1, \ldots, d$, *and* $1 < p < \infty$. *Then, for all polynomials* $f \in \mathbb{R}_+^d$

$$\|\widehat{R}_j^\alpha f\|_{L^p(\mathbb{R}_+^d,\,\hat{\mu}_\alpha)} \leq C_p \|f\|_{L^p(\mathbb{R}_+^d,\,\hat{\mu}_\alpha)}.$$

Supplementary Riesz transforms

The operator $(\widehat{R}_j^\alpha)^*$, adjoint to $\widehat{R}_j^\alpha$ in $L^2(\mathbb{R}^d_+, \hat{\mu}_\alpha)$, is given by

$$(\widehat{R}_j^\alpha)^* g(x) = -\sum_{n\in\mathbb{N}^d} \left(\frac{n_j+1}{|n|+1}\right)^{1/2} \langle g, x_j \widetilde{L}_n^{\alpha+e_j}((\cdot)^2)\rangle_{L^2(\mathbb{R}^d_+,\hat{\mu}_\alpha)} \, \widetilde{L}_{n+e_j}^\alpha(x^2),$$

In particular we have

$$(\widehat{R}_j^\alpha)^*(x_j L_n^{\alpha+e_j}(x^2)) = -\frac{n_j+1}{\sqrt{|n|+1}} L_{n+e_j}^\alpha(x^2), \quad x\in\mathbb{R}^d_+, \; n\in\mathbb{N}^d. \quad (2)$$

As a consequence of Theorem 1 one has

$$\|(\widehat{R}_j^\alpha)^* f\|_{L^p(\mathbb{R}^d_+,\, \hat{\mu}_\alpha)} \le C_p \|f\|_{L^p(\mathbb{R}^d_+,\, \hat{\mu}_\alpha)}, \quad 1 < p < \infty.$$

Preliminaries

We write R^α, $\widehat{R}^\alpha$, δ, ∂ in place of R_1^α, $\widehat{R}_1^\alpha$, δ_1, ∂_1. Given $f \in L^2 \cap L^p(\mathbb{R}, \mu_\alpha)$, we decompose it into even and odd parts,

$$f = f_e + f_o.$$

Thus, to prove the Main Theorem for R^α it is sufficient to show the inequalities

$$\|R^\alpha f_e\|_{L^p(\mathbb{R},\mu_\alpha)} \le C\|f_e\|_{L^p(\mathbb{R},\mu_\alpha)}, \qquad \|R^\alpha f_o\|_{L^p(\mathbb{R},\mu_\alpha)} \le C\|f_o\|_{L^p(\mathbb{R},\mu_\alpha)}.$$

The even case

- The associated derivative δ acting on even functions coincides with ∂. Thus,

$$R^\alpha : L_n^\alpha(x^2) \longmapsto -n^{-1/2} x L_{n-1}^{\alpha+1}(x^2), \qquad n \in \mathbb{N} \quad \text{even}.$$

- Comparing with (1) $\longrightarrow$ R^α on the class of even functions coincides with $\widehat{R}^\alpha$.

We obtain

$$\int_{\mathbb{R}} |R^\alpha f_e(x)|^p \, d\mu_\alpha(x) = 2 \int_0^\infty \left|\widehat{R}^\alpha(f_e)(x)\right|^p d\hat{\mu}_\alpha(x).$$

The odd case

- The associated derivative δ acting on odd functions coincides with $\partial + \frac{2\alpha+1}{x}$,

$$R^\alpha \colon xL_n^{\alpha+1}(x^2) \longmapsto -\frac{n+\alpha+1}{\sqrt{n+1/2}} L_n^\alpha(x^2), \qquad n \in \mathbb{N} \quad \text{odd}.$$

- Comparing with (2) → R^α on the class of odd functions coincides with $(\widehat{R}^\alpha)^*$, up to multiplier and shift operators.

Some results needed for the odd case

Proposition 1 (GLLNU, *J. Math. Pures Appl., 2005*)

Let h be analytic in a neighborhood of the origin, $\{\psi(n)\}_{n\in\mathbb{N}}$ *a sequence of real numbers such that* $\psi(n)=h(n^{-\beta})$, $n\geq n_0>0$ *and* $\beta\in(0,1]$. *Then*

$$\Big\|\sum_{n\in\mathbb{N}^d}\psi(|n|)a_n\widetilde{L}_n^\alpha(x^2)\Big\|_{L^p(\mathbb{R}^d_+,\hat{\mu}_\alpha)}\leq A_p\Big\|\sum_{n\in\mathbb{N}^d}a_n\widetilde{L}_n^\alpha(x^2)\Big\|_{L^p(\mathbb{R}^d_+,d\hat{\mu}_\alpha)}.$$

$f=\sum_{n\in\mathbb{N}^d}a_nL_n^\alpha(x^2)$, $f\in L^2\cap L^p(\mathbb{R}^d_+,\hat{\mu}_\alpha)$.

Proposition 2

Let $\alpha>-1$, $1<p<\infty$. *Then for* $f\in L^2\cap L^p(\mathbb{R}_+,\hat{\mu}_\alpha)$

$$\Big\|\sum_{n=0}^\infty a_n\widetilde{L}_n^\alpha(x^2)\Big\|_{L^p(\mathbb{R}_+,\hat{\mu}_\alpha)}\leq B_p\Big\|\sum_{n=0}^\infty a_n\widetilde{L}_{n+1}^\alpha(x^2)\Big\|_{L^p(\mathbb{R}_+,\hat{\mu}_\alpha)}.$$

Proof of the Proposition 2

We show the dual form

$$\int_0^\infty \Big| \sum_{n=0}^\infty a_n L_{n+1}^\alpha(x) \Big|^q x^\alpha e^{-x}\, dx \le C \int_0^\infty \Big| \sum_{n=0}^\infty a_n L_n^\alpha(x) \Big|^q x^\alpha e^{-x}\, dx.$$

This is a consequence of Proposition 1 and

$$\begin{aligned} \int_0^\infty \Big| \sum_{n=0}^\infty \frac{n+1}{n+\alpha+1} d_n L_{n+1}^\alpha(x) \Big|^q x^\alpha e^{-x}\, dx & \\ \le D_q \int_0^\infty \Big| \sum_{n=0}^\infty d_n L_n^\alpha(x) \Big|^q x^\alpha e^{-x}\, dx, \quad 1 < q < \infty. & \qquad (3) \end{aligned}$$

Proof of the Proposition 2 (3)

To prove (3) we use

- the formula (Szegö)

$$\frac{n+1}{n+\alpha+1} L_{n+1}^{\alpha}(x) = L_n^{\alpha}(x) - \frac{x}{n+\alpha+1} L_n^{\alpha+1}(x)$$

- *Koshlyakov's formula*

$$\frac{x}{n+\alpha+1} L_n^{\alpha+1}(x) = \frac{1}{x^{\alpha}} \int_0^x y^{\alpha} L_n^{\alpha}(y)\, dy$$

- Hardy's inequality with weights

$$\int_0^{\infty} \left| \int_0^x g(y)\, dy \right|^q x^{\alpha(1-q)} e^{-x}\, dx \le C \int_0^{\infty} |g(x)|^q x^{\alpha(1-q)} e^{-x}\, dx.$$

.

Coming back to the proof of the Main Theorem

Since

$$\langle f_o, \mathcal{H}^\alpha_{2n+1}\rangle_\alpha = (-1)^n\sqrt{2}\langle f_o, x\widetilde{L}^{\alpha+1}_n(x^2)\rangle_{L^2(\mathbb{R}_+,\hat{\mu}_\alpha)},$$

then, using Proposition 1 for $\psi(n) = \sqrt{\frac{n+\alpha+1}{n+1/2}}$ and Proposition 2:

$$\begin{aligned}
&\int_{\mathbb{R}} |R^\alpha f_o(x)|^p d\mu_\alpha(x)\,dx \\
&\qquad = \int_{-\infty}^{\infty}\Big|\sum_{n=0}^{\infty}\frac{\sqrt{n+\alpha+1}}{\sqrt{n+1/2}}\langle f_o, \mathcal{H}^\alpha_{2n+1}\rangle_\alpha\, \mathcal{H}^\alpha_{2n}(x)\Big|^p d\mu_\alpha(x) \\
&\qquad \le 2A_p^p \int_0^\infty \Big|\sum_{n=0}^{\infty}\langle f_o, x\widetilde{L}^{\alpha+1}_n(x^2)\rangle_{L^2(\mathbb{R}_+,\hat{\mu}_\alpha)}\widetilde{L}^\alpha_n(x^2)\Big|^p d\hat{\mu}_\alpha(x) \\
&\qquad \le 2A_p^p B_p^p \int_0^\infty \Big|\sum_{n=0}^{\infty}\langle f_o, x\widetilde{L}^{\alpha+1}_n(x^2)\rangle_{L^2(\mathbb{R}_+,\hat{\mu}_\alpha)}\widetilde{L}^\alpha_{n+1}(x^2)\Big|^p d\hat{\mu}_\alpha(x) \\
&\qquad = 2A_p^p B_p^p \int_0^\infty \Big|(\widehat{R}^\alpha)^* f_o(x)\Big|^p d\hat{\mu}_\alpha(x).
\end{aligned}$$

The multidimensional case

The situation can be reduced to considering tensor products of the $L_n^\alpha(x^2)$ and $xL_n^{\alpha+1}(x^2)$ systems (2^d orthogonal systems emerge).

- Tensor products of $L_n^\alpha(x^2) \rightarrow$ use directly the known results for 'Riesz Laguerre transforms'.
- Tensor product of $(d-1)$ $L_n^\alpha(x^2)$ and only one $xL_n^{\alpha+1}(x^2)$ $\rightarrow$ component Riesz transform related to this coordinate
    - relate to the adjoint of 'Riesz Laguerre transforms'
    - composition of two multipliers (dimension d and one dimension) and a shift (one dimension).

The multidimensional case (2)

Remaining cases: Tensor products with at least one 'coordinate' of the form $L_n^{\alpha}(x^2)$

- Component Riesz transform related to this coordinate → square functions?

Component Riesz transform related to an odd coordinate

- adjoint of the previous case
- general multiplier theorem (Cowling, Meda) and a shift.