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A REARRANGEMENT INEQUALITY AND THE NORM OF THE HARDY LITTLEWOOD MAXIMAL OPERATOR ON LORENTZ AND MARCINKIEWICZ SPACES

The distribution function of the uncentered Hardy Littlewood maximal function on the real line increases when a function is symmetrically rearranged. This leads to the exact norm of the maximal operator acting on Lorentz and Marcinkiewicz spaces.

DISTRIBUTION FUNCTIONS AND REARRANGEMENTS

Let $f(x)$ be a measurable function on a measure space. The distribution function $\mu\,(f,t)$ is the measure of the level sets of a function:

$$\mu\,(f,t) = \left|\{x \in \mathbb{R},\ |f(x)| > t\}\right|.$$

The non increasing rearrangement $f^*(s)$ is a non negative non increasing function on $\mathbb{R}_+$ with the same distribution of the function:

$$f^*(s) = \inf\left\{t \in \mathbb{R}_+,\ \mu\,(f,t) \leq s\right\}.$$

The symmetric rearrangement $\mathcal{S}f(x)$ is a non negative even function on $\mathbb{R}$, non increasing on $\mathbb{R}_+$, with the same distribution of the function:

$$\mathcal{S}f(x) = f^*(|2x|).$$

LORENTZ AND MARCINKIEWICZ SPACES

A function space is rearrangement invariant if functions with the same distributions have equal norms.

Let the weight $W(t)$ be continuous concave increasing in $0 \leq t < +\infty$, with $W(0) = 0$ and $W(+\infty) = +\infty$. The Lorentz space $\mathbb{L}(W)$ is the smallest rearrangement invariant function spaces where characteristic functions $\chi_A(x)$ have norms $W(|A|)$.

$$\|f\|_{\mathbb{L}(W)} = \int_0^{+\infty} f^*(s) \frac{d}{ds} W(s) ds.$$

The Marcinkiewicz space $\mathbb{M}(W)$ is the largest rearrangement invariant function spaces where characteristic functions $\chi_A(x)$ have norms $W(|A|)$.

$$\|f\|_{\mathbb{M}(W)} = \sup \left\{ W(s) f^*(s),\ s > 0 \right\}.$$

LORENTZ AND MARCINKIEWICZ SPACES

The Lorentz space $\mathbb{L}(W)$ is generated by characteristic functions. Any negative simple functions has an atomic decompositions with

$$\|f\|_{\mathbb{L}(w)} =$$

$$\inf\left\{\sum_j |\alpha_j|\,,\ f(x) = \sum_j \alpha_j W\left(|A_j|\right)^{-1} \chi_{A_j}(x)\right\}.$$

A function is in the Marcinkiewicz space $\mathbb{M}(W)$ if and only if its symmetric rearrangement is dominated by $W(|2x|)^{-1}$,

$$\mathcal{S}f(x) \leq \|f\|_{\mathbb{M}(W)} W(|2x|)^{-1}.$$

If $W(t) = t^{1/p}$, $1 < p < +\infty$, the Lorentz space is $\mathbb{L}(p,1)$ and the Marcinkiewicz space is $\mathbb{L}(p,\infty)$.

HARDY LITTLEWOOD MAXIMAL FUNCTION

The left and right Hardy Littlewood maximal functions of locally integrable functions on the real line:

$$\mathcal{M}_{-}f(x)=\sup_{y<x}\left\{\frac{1}{x-y}\int_{y}^{x}|f(z)|\,dz\right\},$$
$$\mathcal{M}_{+}f(x)=\sup_{y>x}\left\{\frac{1}{y-x}\int_{x}^{y}|f(z)|\,dz\right\}.$$

The uncentered maximal function:

$$\mathcal{M}f(x)=\sup_{a<x<b}\left\{\frac{1}{b-a}\int_{a}^{b}|f(y)|\,dy\right\}.$$

Hardy Littlewood maximal theorem (1930):

$$\left(\mathcal{M}_{\pm}f\right)^{*}(s)\leq\frac{1}{s}\int_{0}^{s}f^{*}(u)du.$$

The rearrangement of the maximal function is smaller than the maximal function of the rearrangement,

$$\left(\mathcal{M}_{\pm}f\right)^{*}(s)\leq\mathcal{M}_{-}\left(f^{*}\right)(s).$$

MAXIMAL FUNCTION AND REARRANGEMENTS

THEOREM: The distribution function of the uncentered maximal function increases if the function is symmetrically rearranged,

$$\mu\left(\mathcal{M}f,t\right)\leq\mu\left(\mathcal{M}\mathcal{S}f,t\right).$$

Equivalently,

$$\mathcal{S}\mathcal{M}f(x)\leq\mathcal{M}\mathcal{S}f(x).$$

Conversely, if $g(x)$ is non negative and $\mu\left(\mathcal{M}g,t\right)=\mu\left(\mathcal{M}\mathcal{S}g,t\right)$ for every t, then, up to a translation, $g(x)=\mathcal{S}g(x)$ for almost every x.

The direct part of the theorem is due to Blackwell and Dubins. The inequality $\mathcal{S}\mathcal{M}f(x)\leq c\mathcal{M}\mathcal{S}f(x)$ with a suitable constant holds also for the centered maximal function and in any dimension.

NORM OF THE MAXIMAL OPERATOR

THEOREM: The norm of the Hardy-Littlewood maximal operator on the Lorentz space $\mathbb{L}(W)$ is attained at characteristic functions of intervals.

THEOREM: The norm of the Hardy-Littlewood maximal operator on the Marcinkiewicz space $\mathbb{M}\,(W)$ is attained at the function $W(|2x|)^{-1}$.

THEOREM: The norm of the Hardy-Littlewood maximal operator from the Lorentz space $\mathbb{L}\,(w)$ into the Marcinkiewicz space $\mathbb{M}\,(W)$ is attained at characteristic functions of intervals.

MAXIMAL OPERATOR ON $\mathbb{L}(p,1)$

THEOREM: The norm of the Hardy-Littlewood maximal operator on the Lorentz space $\mathbb{L}(p,1)$, $1 < p < +\infty$, is $2^{1/p}\int_0^1 (1/t - 1/2)^{1/p}\,dt$. This norm is attained on characteristic functions of intervals.

Proof: $\mathbb{L}(p,1)$ has an atomic decomposition:

$$f(x) = \sum a(n)\left[|A|^{-1/p}\chi_A(x)\right],$$

$$\|\mathcal{M}f\|_{\mathbb{L}(p,1)} \le \left\{\sum |a(n)|\right\}\left\{\sup_A \left[|A|^{-1/p}\|\mathcal{M}\chi_A\|_{\mathbb{L}(p,1)}\right]\right\}.$$

Maximal functions get larger when characteristic functions are symmetrically rearranged and, for an interval,

$$\mathcal{M}\chi_{[-\alpha,+\alpha]}(x) = \begin{cases} 1 \text{ if } |x| \le \alpha, \\ 2\alpha/\left(\alpha + |x|\right) \text{ if } |x| \ge \alpha, \end{cases}$$

$$\left\|\mathcal{M}\chi_{[-\alpha,+\alpha]}\right\|_{\mathbb{L}(p,1)} = (4\alpha)^{1/p}\int_0^1 (1/t - 1/2)^{1/p}\,dt.$$

MAXIMAL OPERATOR ON $\mathbb{L}(p,\infty)$

THEOREM: The norm of the Hardy-Littlewood maximal operator on the Marcinkiewicz space $\mathbb{L}(p,\infty)$, $1 < p < +\infty$, is the positive solution to the equation $(p-1)x^p - px^{p-1} - 1 = 0$. This norm is attained at homogeneous functions.

Proof: Maximal functions get larger when functions are symmetrically rearranged:

$$\|\mathcal{M}f\|_{\mathbb{L}(p,\infty)} \leq \|\mathcal{M}\mathcal{S}f\|_{\mathbb{L}(p,\infty)} .$$

By the definition of norm on $\mathbb{L}(p,\infty)$:

$$\mathcal{S}f(x) \leq \|f\|_{\mathbb{L}(p,\infty)} |2x|^{-1/p} .$$

Hence

$$\|\mathcal{M}f\|_{\mathbb{L}(p,\infty)} \leq \|f\|_{\mathbb{L}(p,\infty)} \left\|\mathcal{M}\,|2x|^{-1/p}\right\|_{\mathbb{L}(p,\infty)} .$$

$g(x) = |x|^{-1/p}$ is an eigenfunction of maximal operator, $\mathcal{M}g(x) = \lambda g(x)$, with eigenvalue $\lambda = \mathcal{M}g(1)$ which is the positive solution to $(p-1)x^p - px^{p-1} - 1 = 0$.

TRANSLATION INVARIANT OPERATORS ON $\mathbb{L}(p,\infty)$

THEOREM: A sublinear translation invariant operator bounded on $\mathbb{L}(p,\infty)$ is also bounded on $\mathbb{L}(p)$ and the norm on $\mathbb{L}(p)$ is smaller or equal than the norm on $\mathbb{L}(p,\infty)$.

THEOREM: The norm of the Hardy-Littlewood maximal operator on the Lebesgue space $\mathbb{L}(p)$, $1 < p < +\infty$, is the same as the norm of the maximal operator on $\mathbb{L}(p,\infty)$, but the norm is not attained.

The norm of the maximal operator on $\mathbb{L}(p)$ was first determined by L.Grafakos and S.Montgomery Smith. The non existence of extremals has been proved by J.Pérez Lázaro.

OPERATORS ON $\mathbb{L}(p,\infty)$

Assume that the operator maps functions with bounded supports into functions with bounded. Let $g(x)$ and $\mathcal{M}g(x)$ have support in $-a<x<a$ and let

$$f(x)=\sum_{n=1}^{+\infty}n^{-1/p}g\left(x-2an\right).$$

Then

$$\begin{gathered}t^p\left|\{x\in\mathbb{R},\ |f(x)|>t\}\right|\\ =\sum_{n=1}^{+\infty}t^p\left|\{x\in\mathbb{R},\ |g(x)|^p>nt^p\}\right|\leq\int_{-\infty}^{+\infty}|g(x)|^p\,dx,\\ \lim_{t\to0+}t^p\left|\{x\in\mathbb{R},\ |f(x)|>t\}\right|=\int_{-\infty}^{+\infty}|g(x)|^p\,dx.\end{gathered}$$

Hence $\|g\|_{\mathbb{L}(p)}=\|f\|_{\mathbb{L}(p,\infty)}$ and similarly $\|\mathcal{M}g\|_{\mathbb{L}(p)}=\|\mathcal{M}f\|_{\mathbb{L}(p,\infty)}$. Then, if c is the norm of $\mathcal{M}$ on $\mathbb{L}(p,\infty)$,

$$\|\mathcal{M}g\|_{\mathbb{L}(p)}=\|\mathcal{M}f\|_{\mathbb{L}(p,\infty)}\leq c\,\|f\|_{\mathbb{L}(p,\infty)}=c\,\|g\|_{\mathbb{L}(p)}$$