Eigenfuncions of the Hardy-Littlewood maximal operator

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Outline

Definition

Definition (Uncentered Hardy-Littlewood maximal operator on the real line)

Let I be an interval $I \subset \mathbb{R}$. Let f be a locally integrable function on I. Then, for any $x \in I$,

$$Mf(x) := \sup_{y \in I, y \neq x} \frac{1}{x-y} \int_y^x |f(z)| dz.$$

Theorem (Hardy-Littlewood-Wiener)

$$\|Mf\|_p \leq c\|f\|_p \qquad p > 1$$

Extremal problem

$$(p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$$

For each $p > 1$ there exists $\lambda \equiv \lambda_p > 1$ that satisfies the equation and viceversa

Theorem (Grafakos, Montgomery-Smith (1997))

For each $p > 1$, $\|Mf\|_p \leq \lambda_p \|f\|_p$ and the constant λ_p can't be smaller.

In other words: $\|M\|_{L^p(\mathbb{R})} = \lambda_p$.

Question

There exists an extremal or on the contrary the norm is only attained asymptotically?
In other words: $\exists\, f$ s.t. $\|Mf\|_p = \lambda_p \|f\|_p$ or $\|Mf\|_p < \lambda_p \|f\|_p$ for all f.

If *f* is an extremal:

1. $f \in L^p(\mathbb{R})$.
2. from Blackwell-Dubins and Colzani-Laeng-Morpugo

$$\|Mf\|_p \leq \|M(Sf)\|_p$$

then, if *f* is an extremal

$$\lambda_p\|f\|_p = \|Mf\|_p \leq \|M(Sf)\|_p \leq \lambda_p\|Sf\|_p = \lambda_p\|f\|_p.$$

In consequence *Sf* is an extremal.

3. from Grafakos-Montgomery-Smith

$$\int_{-\infty}^{\infty} |f(x)||Mf(x)|^{p-1}dx \leq \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |Mf(x)|^p dx\right)^{\frac{p-1}{p}}$$

If f is an extremal:

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   In consequence Sf is an extremal.
3. from Grafakos-Montgomery-Smith

$$\int_{-\infty}^{\infty} |f(x)||Mf(x)|^{p-1} dx = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |Mf(x)|^p dx\right)^{\frac{p-1}{p}}$$

This implies $Mf = \lambda f$, with λ the operator norm λ_p.

Fixed points for maximal operators

$Mf = f$
constant functions are trivial fixed points

- Korry: M_c over balls has non zero fixed points in L^p if and only if the space dimension is $d \geq 3$ and $d/(d-2) < p \leq \infty$. Such fixed points are positive super harmonic functions.
- Korry: Strong centered maximal operator over parallelograms with sides parallel to the axes has no fixed points in L^p for every d and $1 \leq p < \infty$.
- Martín-Soria: extend to rearrangement invariant spaces

A natural continuation of the study of fixed points, $Mf = 1 \cdot f$, is the study of the eigenfunctions.

Outline

Objective

Study the eigenfunctions of M on the real line.

Example

The homogeneus functions $|x|^{-\alpha}$ $(0 < \alpha < 1)$ are eigenfunctions. Also translations and dilations.

unimodal:= with the shape: increases if $x < c$ and decreases if $x > c$,

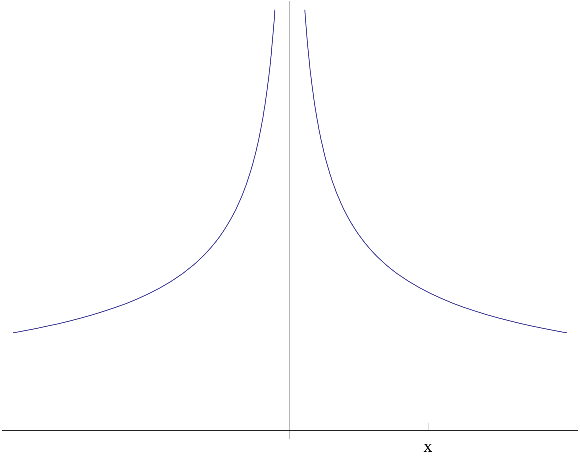
Theorem

$$\left.\begin{array}{l} f \text{ unimodal on } \mathbb{R} \\ Mf = \lambda f \quad \lambda > 1 \end{array}\right\} \Rightarrow f(x) = d|x-c|^{-1/p} \text{ with } \\ (p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$$

Steps of the proof:

$$\exists\, y \in \mathbb{R} \text{ s.t.} \qquad Mf(x) = \frac{1}{y-x}\int_x^y |f(u)|du.$$

For such y, $f(y) = Mf(x)$.



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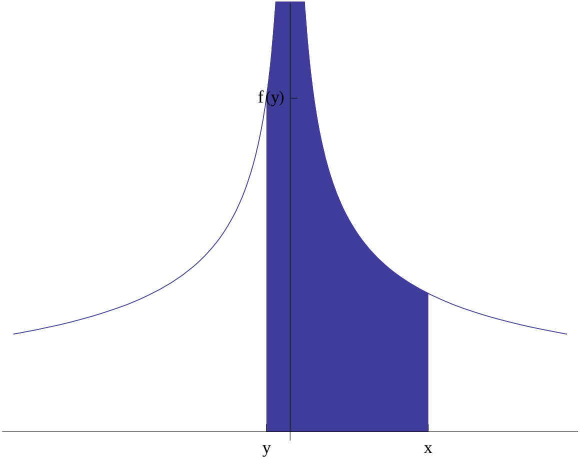

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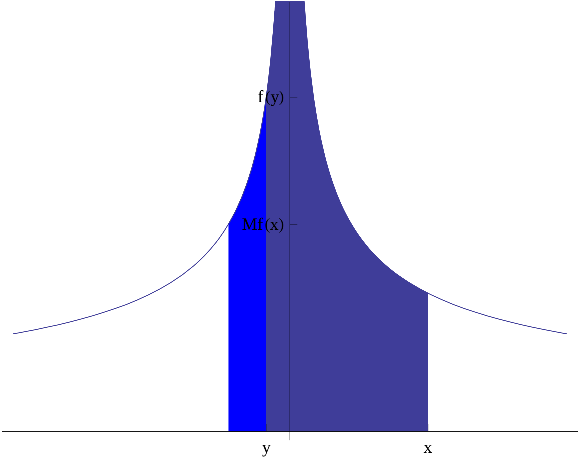

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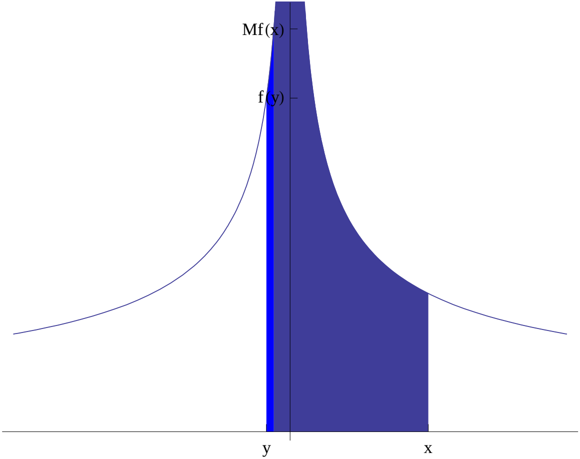

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Steps of the proof:

$$\exists\, y \in \mathbb{R} \text{ s.t.} \qquad Mf(x) = \frac{1}{y-x}\int_x^y |f(u)|du.$$

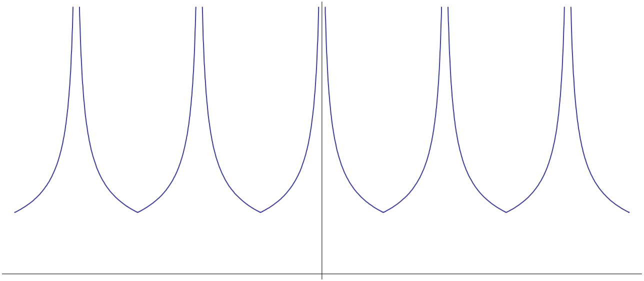
- For such y, $f(y) = Mf(x)$.
- Integral equation system
- Retarded differential equation system
- Changes of variable and take the Fourier transform

Theorem

$$\left.\begin{array}{l} f \text{ unimodal on } \mathbb{R} \\ Mf = \lambda f \quad \lambda > 1 \end{array}\right\} \Rightarrow f(x) = d|x-c|^{-1/p} \text{ with } (p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$$

Can we delete the assumption of unimodality?

No. The following functions are still eigenfunctions:



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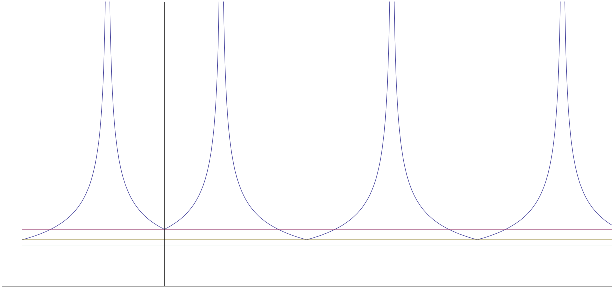

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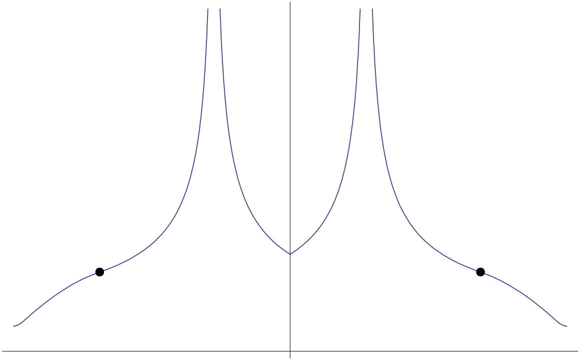

Corollary

There are not extremals for the operator M on $L^p(\mathbb{R})$

Remember that if f is an extremal

1. $f \in L^p(\mathbb{R})$.
2. f is symmetrically decreasing.
3. $Mf = \lambda f$, with λ the operator norm λ_p.

Corollary

Let I be an interval.

$$\left.\begin{array}{l} f \text{ unimodal on } I, \\ \displaystyle\lim_{x\to\ell(I)^+} f(x) = \lim_{x\to r(I)^-} f(x) \\ Mf = \lambda f \quad \lambda > 1 \end{array}\right\} \Rightarrow f(x) = d|x-c|^{-1/p} \text{ with}$$

$$(p-1)\lambda^p - p\lambda^{p-1} - 1 = 0$$

Outline

1. Motivation
   - Extremal problem
   - Fixed points
2. Results for the uncentered H-L maximal operator
3. Further results and conjectures

Definition (Left hand sided H-L maximal operator)

$$M_{-}f(x) := \sup_{y\in I, y<x} \frac{1}{x-y}\int_{y}^{x} |f(z)|dz.$$

$\|M_{-}\|_{L^p(\mathbb{R})} = \frac{p}{p-1}$

Theorem

The unimodal eigenfunctions of M_{-} on $\mathbb{R}$ (with eigenvalue $\frac{p}{p-1} > 1$) are $f(x) = b(x-a)_{+}^{-1/p}$

Of course, analogous results holds for the right hand sided H-L maximal operator M_{+}.

Conjectures

Definition (Centered H-L maximal operator)

$$M_c f(x) := \sup_{r>0: x-r, x+r \in I} \frac{1}{2r} \int_{x-r}^{x+r} |f(z)| dz.$$

Conjectures

It can be seen that the homogeneus functions are eigenfunctions. That is, $M_c(|\cdot|^{-1/p})(x) = c_p|x|^{-1/p}$ $(p > 1)$.

Definition (peak-shaped functions)

positive functions on $\mathbb{R}$, convex except one point, where it can be discontinuous.

The set of peak-shaped functions on $\mathbb{R}$ will be denoted by $\mathcal{P}$

Theorem (Grafakos-Montgomery-Smith-Motrunich)

For peak-shaped functions, the best constant for the inequality

$$\|M_c f\|_p \leq c(p)\|f\|_p \qquad f \in L^p(\mathbb{R}) \cap \mathcal{P}$$

is the eigenvalue of the harmonic functions of above.

Conjecture

$$\|M_c\|_{L^p(\mathbb{R})} = \text{“eigenvalue of } |\cdot|^{-1/p}\text{”}?$$

Conjecture

The unique peak-shaped eigenfunctions for M_c are, up to translations and dilations, the homogeneus?