Transference between Laguerre and Hermite settings

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Non negative selfadjoint second order differential operator **L**

- $T_t = e^{-t\mathbf{L}}$ diffusion semigroup associated to L.
- $P_t = e^{-t\sqrt{\mathbf{L}}}$ Poisson subordinated semigroup:

$$P_t f(x) = \frac{1}{\sqrt{\Pi}} \int_0^\infty \frac{t}{s^{3/2}} e^{-t^2/4s} T_s f(x) ds$$

Operators in Harmonic Analysis (Stein)

1. Maximal Operators

$$T_t^* f(x) = sup_{t>0}|T_t f(x)|, \quad P_t^* f(x) = sup_{t>0}|P_t f(x)|$$

2. Riesz transform: If $\mathbf{L} = \partial^* \partial$, ∂ first order differential operator & $\mathbf{L}^{-\frac{1}{2}} = \frac{1}{\sqrt{\Pi}} \int_0^\infty \frac{T_t f(x)}{\sqrt{t}} dt$

$$R^+ = \partial \mathbf{L}^{-1/2}, \; R^- = \partial^* \mathbf{L}^{-1/2}$$

3. Littlewood-Paley Functions $(1 < q < \infty)$

$$g^T(f)(x) = ||\partial_t T_t f(x)||^q_{L^q(\mathbb{R}^+, t^{q-1} dt)}$$

$$g^P(f)(x) = ||\partial_t P_t f(x)||^q_{L^q(\mathbb{R}^+, t^{q-1} dt)}$$

Hermite (Harmonic oscillator)

$$H = -\frac{d^2}{dx^2} + x^2.$$

- H is selfadjoint.
- Eigenfunctions: Hermite functions $\{h_n\}_{n=0}^{\infty}$

$$h_n(x) = (\sqrt{\pi}2^n n!)^{-1/2} H_n(x) e^{-x^2/2}, \; x \in \mathbb{R}$$

  where H_n denotes the n-th Hermite polynomials.
- Eigenvalues: $H(h_n) = (2n+1)h_n$.
- $\{h_n\}_{n=0}^{\infty}$ complete orthonormal system in $L^2(\mathbb{R})$.

Heat Semigroup

Diffusion Integral

$$e^{-tH}f(x) = \int_{-\infty}^{\infty} \mathscr{W}_t(x,y)f(y)dy, \ f \in L^2(\mathbb{R}),$$

Heat Kernel

$$\begin{aligned} \mathscr{W}_t(x,y) &= \sum_{n=0}^{\infty} e^{-(2n+1)t} h_n(x)h_n(y) \\ &= \frac{1}{\sqrt{\pi}}(\sinh\ 2t)^{-1/2} e^{-\frac{1}{2}|x-y|^2 \coth\ 2t - xy \tanh\ t} \end{aligned}$$

for $x, y \in \mathbb{R}, t > 0$.

Laguerre-Hermite Differential Operator

$$\mathbf{L}_\alpha = \frac{1}{2}\left\{ -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\left(\alpha^2 - \frac{1}{4}\right)\right\}, \quad x \in (0,\infty), \quad \alpha > -1.$$

- $\mathbf{L}_\alpha$ is selfadjoint.
- Eigenfunction: Laguerre's functions $\{\varphi_n^\alpha\}_{n=0}^\infty$

$$\varphi_n^\alpha(x) = \left(\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\right)^{1/2} e^{-x^2/2} x^\alpha L_n^\alpha(x^2)(2x)^{1/2},$$

where $\{L_n^\alpha\}_{n=0}^\infty$ are the Laguerre polynomials of order α .

- Eigenvalues: $\mathbf{L}_\alpha(\varphi_n^\alpha) = (2n+\alpha+1)\varphi_n^\alpha$.
- $\{\varphi_n^\alpha\}_{n=0}^\infty$ complete orthonormal system $L^2((0,\infty))$.

Heat semigroup

Diffusion Integral

$$e^{-t\mathbf{L}}f(x) = \int_0^\infty W_t^\alpha(x,y)f(y)dy, \;\; f \in L^2(\mathbb{R}),$$

Heat Kernel

$$\begin{aligned} W_t^\alpha(x,y) &= \sum_{n=0}^\infty e^{-(2n+1+\alpha)t}\varphi_n^\alpha(x)\varphi_n^\alpha(y) \\ &= 2(xy)^{1/2}\frac{e^{-t}}{1-e^{-2t}} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) e^{-\frac{1}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}} \end{aligned}$$

para $x, y \in \mathbb{R}, t > 0$.

Maximal Theorem Hermite (Muckenhoupt, Sjögren)(García Cuerva, Torrea, Mauceri, Meda)

Let the semigroup $\{T_t\}$ associated to H. Then $T^*, \ P^*$ are bounded on L^p, $1 < p \leq \infty$ and weak-L^1.

Maximal Theorem Laguerre (Stempak, Macías, Segovia, Torrea, Chicco Ruiz, Harboure)(Muckenhoupt, Dinger)

- L^*_α and P^*_α are bounded in L^p $(1 < p \leq \infty)$ and L^1 into weak-L^1 $(\alpha > -\frac{1}{2})$
- L^*_α and P^*_α are bounded in L^p $(-1 < \alpha < -\frac{1}{2})$ with an optimal interval of $p's$ depending on α

Riesz-Hermite Transform (Thangavelu, Torrea, Stempak)

Factorization

$$H = \frac{1}{2}(\partial^*\partial + \partial\partial^*)$$

where $\partial^* = \frac{d}{dx} - x$ & $\partial = \frac{d}{dx} + x$

Riesz-Hermite Transform

Integral representation in L^2

- $\mathscr{R}^+ f(x) = \left(\frac{d}{dx} + x\right) H^{-1/2} f(x) = \int_{-\infty}^{\infty} R^+(x,y) f(y) dy$
- $\mathscr{R}^- f(x) = \left(\frac{d}{dx} - x\right) H^{-1/2} f(x) = \int_{-\infty}^{\infty} R^-(x,y) f(y) dy$

Kernels $R^{+}(x,y)$, $R^{-}(x,y)$

- Standard Calderón-Zygmund Kernel. Principal Value.
- Boundedness in L^p and weak-L^1
- Estimates with weights in Muckenhoupt class.
- Banach valued functions. Riesz Transform in Lebesgue-Bochner spaces.

Riesz-Laguerre Transform (Gutiérrez, Incognito, Torrea, Harboure, Viviani, Segovia, Nowak, Stempak...)

- Factorization

$$\mathbf{L}_\alpha = \frac{1}{2}\mathbf{D}_\alpha^* \mathbf{D}_\alpha + \alpha + 1,$$

where $\mathbf{D}_\alpha f = x^{\alpha+1/2}\frac{d}{dx}(x^{-(\alpha+1/2)}f) + xf$.

Riesz-Laguerre Transform

Integral representation in L^2

- $\mathbf{R}_\alpha f(x) = \mathbf{D}_\alpha \mathbf{L}^{-1/2} f(x) = \int_0^\infty R^\alpha(x,y) f(y)\, dy$

- Kernel $R^\alpha(x,y)$: Standard Calderón-Zygmund. Principal Value.

Other Laguerre expansion

- Different families of Laguerre functions (Differential Operators Laguerre Type).
- The infinitesimal generators are conjugated by "isometries"in L^2 (Macías, Segovia, Torrea).
- Estimations for a family are transferred to other family via conjugation.

Hermite 'versus' Laguerre

Classical relation: Hermite d-dimensional $\Rightarrow$ Laguerre 1-dimensional (Gutiérrez, Incognito, Torrea, ...)

For $\alpha = d/2 - 1$, $x \in \mathbb{R}^d$

$$L_k^\alpha(|x|^2) = \sum_{|r|=k} a_r H_{2r}(x), \; r = (r_1, \ldots, r_n) \in \mathbb{N}^d$$

where

$$H_r(x) = \prod_{i=1}^{d} H_{r_i}(x_i)$$

- $H_r(x)$: Eigenfunctions of Hermite $\Delta + |x|^2$ (d-dimensional) with $d = r_1 + \cdots + r_d$.
- Laguerre $\equiv$ Hermite d-dimensional & Transplantation

Hermite 1-dimensional ‘$\Leftrightarrow$’ Laguerre 1-dimensional

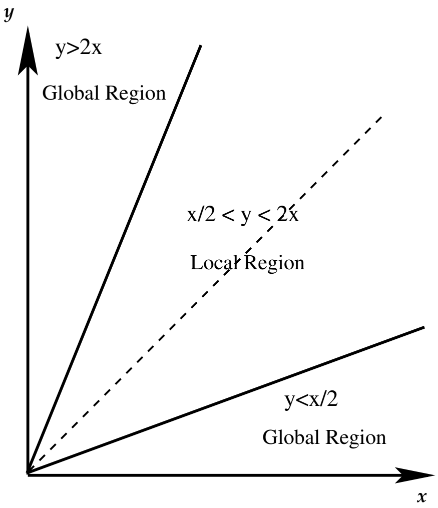
Pointwise Identity between heat kernels

$$W_t^\alpha(x,y) - \mathscr{W}_{t/2}(x,y) = h(t,x,y)\mathscr{W}_{t/2}(x,y)$$

where

$$h(t,x,y) = \left\{ \sqrt{2\pi}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) e^{-\frac{2xye^{-t}}{1-e^{-2t}}} - 1 \right\}.$$

Outline Proof



- Estimation in global Region.
- Comparison in local Region.

Outline Proof

Main tools

- asymptotic behavior of modified Bessel function I_α.
- Singularity localizable in $\dfrac{2xye^{-t}}{1-e^{-2t}} > 1$.

Kernel's estimation: global region

- $W_t^\alpha(x,y) \leq Cy^{\alpha+1/2}x^{-\alpha-3/2}$, $t>0$, $0<y<x/2$.
- $W_t^\alpha(x,y) \leq Cx^{\alpha+1/2}y^{-\alpha-3/2}$, $t>0$, $0<y<x/2$
- $|\mathbf{R}_\alpha(x,y)| \leq Cy^{\alpha+1/2}x^{-\alpha-3/2}$, $0<y<x/2$.
- $|\mathbf{R}_\alpha(x,y)| \leq Cx^{\alpha+3/2}y^{-\alpha-5/2}$, $2x<y$.

Comparison: Local Region: $0 < x/2 < y < 2x$.

- $|W_t^\alpha(x,y) - \mathscr{W}_{t/2}(x,y)| \leq \frac{C}{y}$, $t > 0$.
- $\left|\mathbf{R}_\alpha(x,y) - \mathscr{R}^+(x,y)\right| \leq \frac{C}{y}\left(1 + \frac{(xy)^{1/4}}{|x-y|^{1/2}}\right)$

Riesz-Laguerre Transform

- Global Region
  - Positives Operators of Hardy Type (potential weights)
- Local Region
  - Positive Operator with integrable singularity.
  - Local Riesz-Hermite transform (Calderón-Zygmund local -Nowak, Stempak, 2005-; potential weights)

Subordinated Poisson semigroup

- $P_t^H f = \dfrac{t}{2\sqrt{\pi}} \displaystyle\int_0^\infty \dfrac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} W_u^H f \, du, \qquad t > 0$(Hermite)
- $P_t^\alpha f = \dfrac{t}{2\sqrt{\pi}} \displaystyle\int_0^\infty \dfrac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} W_u^\alpha f \, du, \qquad t > 0$ (Laguerre)

g-functions

- $g_q^H(f)(x) = \left\{ \displaystyle\int_0^\infty \left\| t \frac{\partial}{\partial t} P_t^H(f)(x) \right\|_B^q \frac{dt}{t} \right\}^{1/q}$
- $g_q^\alpha(f)(x) = \left\{ \displaystyle\int_0^\infty \left\| t \frac{\partial}{\partial t} P_t^\alpha(f)(x) \right\|_B^q \frac{dt}{t} \right\}^{1/q}.$

Boundedness g-function

Boundedness g-function Laguerre

- $g^S_{q,\mathrm{loc}}(f)(x) = \left\{ \int_0^\infty \left\| t \int_{x/2}^{2x} \frac{\partial}{\partial t} P^S_t(x,y) f(y) dy \right\|^q_B \frac{dt}{t} \right\}^{1/q}$, $x \in (0,\infty)$ ($S = \alpha$ or $S = H$ Laguerre or Hermite respectively)(local version)
- $g^\alpha_{q,\mathrm{glob},+}(f)(x) = \left\{ \int_0^\infty \left\| t \int_{2x}^{\infty} \frac{\partial}{\partial t} P^\alpha_t(x,y) f(y) dy \right\|^q_B \frac{dt}{t} \right\}^{1/q}$ (global version)
- $g^\alpha_{q,\mathrm{glob},-}(f)(x) = \left\{ \int_0^\infty \left\| t \int_{0}^{x/2} \frac{\partial}{\partial t} P^\alpha_t(x,y) f(y) dy \right\|^q_B \frac{dt}{t} \right\}^{1/q}$ (global version)

Estimation Global Operator

- $g^{\alpha}_{q,\mathrm{glob},+}(f)(x) \leq Cx^{\alpha+\frac{1}{2}} \int_{2x}^{\infty} \frac{\|f(y)\|_B}{y^{\alpha+\frac{3}{2}}} dy$
- $g^{\alpha}_{q,\mathrm{glob},-}(f)(x) \leq \frac{C}{x^{\alpha+\frac{3}{2}}} \int_{0}^{x/2} \|f(y)\|_B y^{\alpha+\frac{1}{2}} dy$
- $|g^{\alpha}_{q,\mathrm{loc}}(f)(x) - \sqrt{2} g^{H}_{q,\mathrm{loc}}(f)(x)| \leq$

$$C \int_{x/2}^{2x} \|f(y)\|_B \int_0^{\infty} \frac{1}{s} |W^{\alpha}_s(x,y) - \sqrt{2} W^{H}_s(x,y)| ds dy \leq$$

$$C \int_{x/2}^{2x} \|f(y)\|_B \frac{1}{x} \left(1 + \left(\frac{x}{|x-y|} \right)^{1/2} \right) dy$$

Applications. Laguerre Riesz Transform

Banach Spaces B with UMD property (Unconditional Martingale Difference)

- Extension of Riesz inequality (conjugate function - Hilbert transform) to Lebesgue-Bochner spaces. $L^p_B((0,1),dx)$, $1 < p < \infty$.
- UMD $\Leftrightarrow$ Boundedness of Hilbert transform for (all) p, $1 < p < \infty$ (Burkholder-Bourgain).
- Independent Property on p, $1 < p < \infty$.

Theorem

Let B Banach space, $\alpha > -1$, $\sigma \in \mathbb{R}$, $1 < p < \infty$. TFAE:

i) B has the UMD property.

ii) $\mathbf{R}_\alpha$ has a bounded extension on $L^p_B((0,\infty), x^\sigma dx)$, for σ verifying: $-p(\alpha+3/2)-1 < \sigma < p(\alpha+3/2)-1$

Outline of proof

- B is UMD $\Leftrightarrow$ Boundedness of Hilbert transform $\Leftrightarrow$ Boundedness of Riesz-Hermite transform.
- Boundedness of Riesz-Hermite transform $\Leftrightarrow$ Boundedness of Riesz-Laguerre transform.

Applications. Laguerre g-functions

Theorem

Let B a Banach space, $q \geq 2$ and $\alpha > -1$. TFAE:

- *B has q-martingale cotype.*
- *We denote $\Omega_\alpha = (1, \infty)$, when $\alpha > -\frac{1}{2}$, and $\Omega_\alpha = \left(\frac{2}{2\alpha+3}, \frac{-2}{2\alpha+1}\right)$, when $-1 < \alpha \leq -\frac{1}{2}$. For every (or, equivalently, for some) $p \in \Omega_\alpha$ there exists $C_p > 0$ such that*

$$||g_q^\alpha(f)||_{L^p(0,\infty)} \leq C_p ||f||_{L_B^p(0,\infty)}, \qquad f \in L_B^p(0,\infty).$$

- *For every (or, equivalently, for some) $1 < p < \infty$ there exists $C_p > 0$ such that*

$$||g_q^H(f)||_{L^p(\mathbb{R})} \leq C_p ||f||_{L_B^p(\mathbb{R})}, \qquad f \in L_B^p(\mathbb{R}).$$

Outline of the proof of Theorem cotype

Lemma 1

Let B be a Banach space and $1 < p, q < \infty$. The following assertions are equivalent.

a) $g_q^{H-I/2}$ is bounded from $L_B^p(\mathbb{R})$ into $L^p(\mathbb{R})$.
b) $g_{q,\mathrm{loc}}^{H-I/2}$ is bounded from $L_B^p(0,\infty)$ into $L^p(0,\infty)$.
c) g_q^H is bounded from $L_B^p(\mathbb{R})$ into $L^p(\mathbb{R})$.
d) $g_{q,\mathrm{loc}}^H$ is bounded from $L_B^p(0,\infty)$ into $L^p(0,\infty)$.

Lemma 2

Let B be a Banach space and $1 < q < \infty$. Then, if $g^H_{q,\mathrm{loc}}$ defines a bounded operator from $L^p_B(0,\infty)$ into $L^p(0,\infty)$, for some $1 < p < \infty$, then it is true for every $1 < p < \infty$.

Lemma 3

Let B be a Banach space, $\alpha > -1$, $1 < q < \infty$, and $p \in \Omega_\alpha$. The following assertions are equivalent.

1. g^α_q is bounded from $L^p_B(0,\infty)$ into $L^p(0,\infty)$.
2. $g^H_{q,\mathrm{loc}}$ is bounded from $L^p_B(0,\infty)$ into $L^p(0,\infty)$.

Applications. Laguerre Riesz Transform

Outline of proof

- $g_q^{2H-I} = g_q^{H-I/2}$, $1 < q < \infty$.
- $g_q^{\mathbb{L}}(f) = U^{-1} g_q^{2H-I}(Uf)$, $f = \sum_k c_k h_k$ ($\mathbb{L}$ Ornstein-Uhlenbeck operator) ($Uf(x) = e^{-x^2/2} f(x)$)
- B has martingale cotype $\Leftrightarrow$ $g_q^{\mathbb{L}}$ bounded on $L_B^2(\mathbb{R}, e^{-x^2}dx)$ (Martínez, Torrea, Xu).

Work in progress

- L^p-boundedness of higher order Riesz Transform for Laguerre expansion (preprint).
- L^p-boundedness of higher order g_q^α-function.