Maximal operators and Riesz transforms for nonsymmetric Ornstein-Uhlenbeck semigroups

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Articles

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O-U process in $\mathbb{R}^d$

Q, B real $d \times d$ matrices,

Q symmetric, positive definite, $\quad \mathrm{Re}\,\lambda < 0$, $\lambda \in \sigma(B)$

W Wiener process on $\mathbb{R}^d$ with covariance matrix Q

Ornstein-Uhlenbeck stochastic diff. eqn.

$$\begin{cases} \mathrm{d}X = BX\,\mathrm{d}t + \mathrm{d}W \\ X(0) = x \end{cases}$$

The solution

$$X(t,x) = \mathrm{e}^{tB}x + \int_0^t \mathrm{e}^{(t-s)B}\,\mathrm{d}W(s)$$

is the O-U process with *drift* B and *covariance* Q.

O-U semigroup

$$H_t f(x) = \mathbb{E}\big[f(X(t,x))\big] \qquad \forall f \in C_b(\mathbb{R}^d).$$

Kolmogorov's formula

$$H_t f(x) = \int_{\mathbb{R}^d} f(\mathrm{e}^{tB}x - y)\,\mathrm{d}\gamma_t(y)$$

γ_t : Gauss measure on $\mathbb{R}^d$ with mean zero and covariance

$$Q_t = \int_0^t \mathrm{e}^{sB} Q \mathrm{e}^{sB^*}\,\mathrm{d}s \qquad \forall t \in (0,\infty].$$

$$\mathrm{d}\gamma_t(x) = \frac{1}{2\pi^{d/2}}(\det Q_t)^{-1/2}\mathrm{e}^{-\frac{1}{2}\langle Q_t^{-1}x,x\rangle}.$$

The semigroup kernel w.r.t. γ_∞

The sgp (H_t) can also be written as a sgp of integral operators w.r.t. the measure γ_∞:

$$H_t f(x) = \int h_t(x,y)\, f(y)\, \mathrm{d}\gamma_\infty(y)$$

where

$$h_t(x,y) = \det(Q_\infty Q_t^{-1})^{1/2} \mathrm{e}^{-\frac{1}{2}\left[\langle Q_t^{-1}(\mathrm{e}^{tB}x - y), \mathrm{e}^{tB}x - y\rangle - \langle Q_\infty^{-1} y, y\rangle\right]}$$

When $Q = -B = I$ this reduces to the classical Mehler kernel

$$h_t(x,y) = \frac{1}{\left(2\pi(1-\mathrm{e}^{-2t})\right)^{d/2}} \mathrm{e}^{-\frac{\left|\mathrm{e}^{-t}x - y\right|^2}{1-\mathrm{e}^{-2t}}}\ \mathrm{e}^{|y|^2/2}$$

Properties of the O-U sgp

- γ_∞ is the unique invariant measure for H_t :

$$\int H_t f \, \mathrm{d}\gamma_\infty = \int f \, \mathrm{d}\gamma_\infty \qquad \forall f \in C_b(\mathbb{R}^d), \forall t > 0.$$

- (H_t) extends to a C^0 -sgp of positive contractions on $L^p(\gamma_\infty)$ for $1 \le p < \infty$.
- it is symmetric iff $QB^* = BQ$ [CM&G '02], [MPRS '02].
- The generator is

$$\mathscr{L} = \frac{1}{2} \operatorname{tr} Q\nabla^2 + \langle Bx, \nabla \rangle.$$

- For $1 < p < \infty$ the spectrum of $\mathscr{L}$ on $L^p(\gamma_\infty)$ is

$$\sigma(\mathscr{L}_p) = \left\{ \sum_i n_j \lambda_j : \; n_j \in \mathbb{N}, \; \lambda_j \in \sigma(B) \right\}$$

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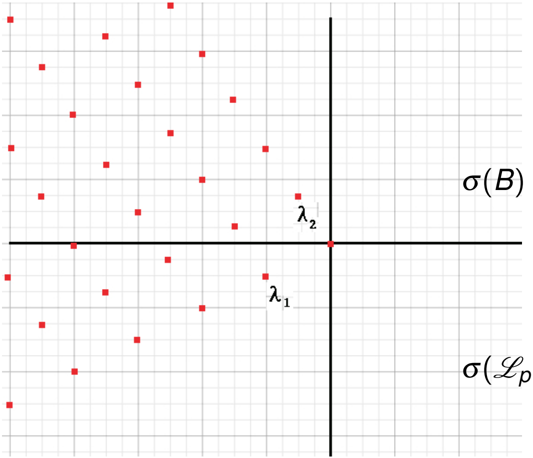
The spectrum of $\mathscr{L}$ on $L^p(\gamma_\infty)$, $1 < p < \infty$

[Metafune-Pallara-Priola '02] for $d < \infty$
[Van Neerven '05] for $d = \infty$



$\sigma(B) = \{\lambda_1, \lambda_2\}$

$\sigma(\mathscr{L}_p) \subset -\mathbf{S}_\phi$

Return

Harmonic analysis of the O-U sgp

Consider the following operators

- The O-U maximal operator

$$H_* f(x) = \sup_{t>0} |H_t f(x)|$$

- Riesz transforms $\partial^\alpha (-\mathscr{L})^{-|\alpha|/2}$
- Imaginary powers $(-\mathscr{L})^{iu}$ or more general spectral multipliers $m(\mathscr{L})$

Are they bounded on $L^p(\gamma_\infty)$ for $1 < p < \infty$ and of weak type $(1,1)$?

Previous results for the maximal operator:

- L^p estimates, $1 < p < \infty$ for (H_t) symmetric follow from Stein's L-P theory for symmetric diffusion sgps.
- weak type $(1,1)$ estimate: only when $Q = -B = I$ [Muckenhoupt] for $d = 1$, [Sjögren] $d < \infty$

  Simpler proofs have been found by

  [Ménarguez-Pérez-Soria]
  [García Cuerva-Mauceri-Meda-Sjögren-Torrea].

  The result extends easily to the case

$$Q = I,\ B = -\lambda I,\ \lambda > 0$$

  but even for the case $B = \mathrm{diag}(\lambda_1, \lambda_2)$ $\lambda_1 \neq \lambda_2$ the answer is unknown.

Riesz transforms: L^p estimates, $1 < p < \infty$

$\partial^\alpha(-\mathscr{L})^{-|\alpha/2|}$ is bounded on $L^p(\gamma_\infty)$, $1 < p < \infty$ for every α

when

- $Q = -B = I$ [Gundy, Meyer, Pisier, Urbina, Dragicevic-Volberg ...]
- $Q = I$, B symmetric, $d < \infty$, [Gutierrez-Segovia-Torrea '96]
- $Q = I$, B symmetric (unbounded) on a separable Hilbert space [Chojnowska M.-Goldys '01]
- general Q, B, $d < \infty$ for $|\alpha| = 2$ [Metafune-Prüss-Rhandi-Schnaubelt '02]

Riesz transforms: weak type $1-1$ estimates

Results are known only for a particular symmetric operator: $Q = -B = I$

$\partial^\alpha(-\mathscr{L})^{-|\alpha/2|}$ is of weak type $1-1$ iff $|\alpha| \leq 2$

[Muckenhoupt, Fabes-Gutierrez-Scotto, Forzani-Scotto, Pérez-Soria, García Cuerva-Mauceri-Sjögren-Torrea ...]

The result extends easily to the case

$$Q = I, \; B = -\lambda I, \; \lambda > 0$$

but even for the case $B = \mathrm{diag}(\lambda_1, \lambda_2)$ $\lambda_1 \neq \lambda_2$ the answer is unknown.

Summary of our results

To what extent the previous results extend to non-symmetric O-U semigroups?

- L^p estimates, $1 < p < \infty$, hold
  1. for the maximal operator H_* if (H_t) is normal
  2. for Riesz transf. of any order for general O-U sgps.

- the weak type $(1,1)$ estimate holds for H_* and 1^{st} order R. T. for a special class of normal O-U sgps.

The canonical form

[Metafune Prüss Rhandi Schnaubelt]

After a linear change of coordinates in $\mathbb{R}^d$ we can reduce to the case

$$Q = I, \qquad B = -\frac{1}{2} D_{\lambda^{-1}} + R,$$

where

$$D_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d), \qquad RD_\lambda = -D_\lambda R^t.$$

Thus

$$\begin{aligned} \mathscr{L} &= \frac{1}{2}\Delta - \frac{1}{2}\langle D_\lambda^{-1} x, \nabla\rangle + \langle Rx, \nabla\rangle \\ &= \mathscr{L}^0 + \mathscr{R} \end{aligned}$$

where $\mathscr{L}^0$ is symmetric and $\mathscr{R}$ is antisymmetric on $L^2(\gamma_\infty)$.

Normal O-U sgps

Theorem

Let

$$\mathscr{L} = \mathscr{L}^0 + \mathscr{R}$$

be the symmetric+antisymmetric decomposition of $\mathscr{L}$ *, where*

$$\mathscr{R} = \langle Rx, \nabla \rangle.$$

Then the following are equivalent

- (H_t) *is normal on* $L^2(\gamma_\infty)$
- $[\mathscr{L}^0, \mathscr{R}] = 0$
- $R \in \mathfrak{so}(d, \mathbb{R})$ *, i.e.* $R + R^t = 0$
- $[D_\lambda, R] = 0$ *.*

Building blocks of normal O-U operators

The operators

$$\mathscr{L}_{\alpha,R} = \frac{1}{2}\Delta - \alpha\langle (I+R)x, \nabla\rangle$$

with $Q = I$ and drift $B = -\alpha(I+R)$, $\alpha > 0$, $R \in \mathfrak{so}(d,\mathbb{R})$ are the basic building blocks of normal O-U operators:

Theorem

If (H_t) is normal on $L^2(\gamma_\infty)$ after a linear change of coordinates we may write

$$\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k}, \qquad \mathscr{L} = \sum_j \mathscr{L}_{\alpha_j,R_j}$$

where $R_j \in \mathfrak{so}(d_j,\mathbb{R})$ and each $\mathscr{L}_{\alpha_j,R_j}$ acts on $\mathbb{R}^{d_j}$.
The $\mathscr{L}_j$ commute.

The main result

Theorem

Let (H_t) be the O-U sgp with covariance $Q = I$ and drift $B = -\alpha(I + R)$, where $\alpha > 0$ and $R \in \mathfrak{so}(d, \mathbb{R})$. If the one-parameter group of rotations of $\mathbb{R}^d$ generated by R is periodic then the maximal operator

$$H_* f(x) = \sup_{t>0} |H_t f(x)|$$

and the first order Riesz transforms

$$\partial_j \big(-\mathscr{L}\big)^{-1/2} \qquad j = 1, \ldots, d$$

are of weak type $(1,1)$ with respect to the invariant measure γ_∞.

Remark

By a rescaling argument we may assume that $\alpha = 1$.

The $\mathbb{R}^2$ case

$$\mathfrak{so}(2,\mathbb{R}) = \left\{ R_\theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} : \theta \in [0,2\pi) \right\}$$

The operator with drift $-(I+R_\theta)$ is

$$\begin{aligned} \mathscr{L}_\theta &= \frac{1}{2}\Delta - \langle x,\nabla\rangle - \langle R_\theta x,\nabla\rangle \\ &= \mathscr{L}^{(s)} - \langle R_\theta x,\nabla\rangle. \end{aligned}$$

The kernel of the semigroup $(\mathrm{e}^{t\mathscr{L}_\theta})$ is

$$h_t(x,y) = h_t^{(s)}(\mathrm{e}^{tR_\theta}x,y) = h_t^{(s)}(x,y)\, k_{t\theta}(x,y)$$

where $h_t^{(s)}$ is the Mehler kernel and

$$k_{t\theta}(x,y) = \exp\left\{ -\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-2t}} \left[\big(1-\cos(t\theta)\big)\langle x,y\rangle + \sin(t\theta)\, x\wedge y \right] \right\}$$

Here $x \wedge y = x_1\, y_2 - x_2\, y_1$.

The $\mathbb{R}^2$ case

$$h_t(x,y) = h_t^{(s)}(x,y)\; k_{t\theta}(x,y)$$

$$k_{t\theta}(x,y) = \exp\left\{-\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-2t}}\left[\big(1-\cos(t\theta)\big)\langle x,y\rangle + \sin(t\theta)\, x\wedge y\right]\right\}$$

Remark

Note that $k_{t\theta}(x,y) = 1$ for $t\theta = 2n\pi$, $n \in \mathbb{N}$.

Thus

$$h_t(x,y) \sim h_t^{(s)}(x,y) \qquad \text{when} \quad t \sim 2n\pi/\theta, n \in \mathbb{N}$$

Reduction to $\mathbb{R}^2$

If $R \in \mathfrak{so}(d,\mathbb{R})$, $d = 2m$, there exists $O \in SO(d,\mathbb{R})$ such that

$$ORO^t = \begin{pmatrix} R_{\theta_1} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & R_{\theta_m} \end{pmatrix} \qquad R_{\theta_j} = \begin{pmatrix} 0 & \theta_j \\ -\theta_j & 0 \end{pmatrix}$$

Thus (e^{tR}) is periodic iff the θ_j's satisfy the rationality condition

$$\theta_j = r_j\,\theta, \qquad r_j \in \mathbb{Q}, \quad j = 1, \ldots, m.$$

Reduction to $\mathbb{R}^2$

Theorem

Suppose that $d = 2m$. *Then, after an orthogonal change of coordinates in* $\mathbb{R}^{2d}$, *we can write*

$$\mathbb{R}^d = \mathbb{R}^2_1 \times \cdots \times \mathbb{R}^2_m, \qquad \mathscr{L}_{1,R} = \sum_{j=1}^{m} \mathscr{L}_{\theta_j},$$

where each $\mathscr{L}_{\theta_j}$ *acts on* $\mathbb{R}^2_j$. *Moreover*

$$h_t(x,y) = h_t^{(s)}(x,y) \prod_{j=1}^{m} k_{t\theta_j}(x_j, y_j),$$

where $h_t^{(s)}(x,y)$ *is the Mehler kernel.*

Idea of the proof 1

If (e^{tR}) is periodic of period T then

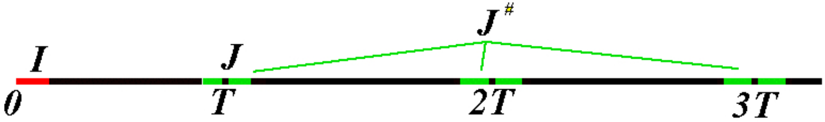
$$h_{nT}(x,y) = h^{(s)}_{nT}(x,y) \qquad \forall n \in \mathbb{N}.$$

Exploiting this fact we obtain good estimates of

$$\left|\nabla^j_{x,y} h_t(x,y)\right|, \qquad j = 0,1,2$$

when

- $t \in I = (0, t_0]$ for some small $t_0 > 0$;
- $t \in J = (T-\varepsilon, T+\varepsilon]$ for some small $\varepsilon > 0$;
- $t \in J^\sharp = \cup_n (J + nT)$.



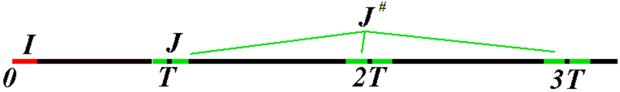
Idea of the proof: the maximal operator

For $E \subset \mathbb{R}_+$ define the maximal operator

$$H_*^E f(x) = \sup_{t \in E} |H_t f(x)|$$

- H_*^I is of weak type $(1,1)$.
- If (e^{tR}) is periodic then $H_*^{J^\sharp}$ is of weak type $(1,1)$.

$$\mathbb{R}_+ = \left(\cup_{\ell=0}^{M} (I + \ell t_0)\right) \bigcup \left(\cup_{m=0}^{N} (J^\sharp + m t_1)\right)$$



$$H_* f(x) \le \max_{\ell, m} \left\{ H_*^{I+\ell t_0} f(x), H_*^{J^\sharp + m t_1} f(x) \right\}$$

Now $\quad H_*^{E+s} f = H_*^E \, H_s f \quad$ because $\quad H_{t+s} = H_t H_s$

Idea of the proof: Riesz transforms

Split

$$\partial_j(-\mathscr{L})^{-1/2} = \frac{1}{\Gamma(1/2)}\partial_j \int_0^\infty t^{-1/2} H_t \,\mathrm{d}t$$

as a finite sum of integrals

$$\partial_j \int_{I+mt_0} t^{-1/2} H_t \,\mathrm{d}t \qquad \text{and} \qquad \partial_j \int_{J^\sharp+mt_1} t^{-1/2} H_t \,\mathrm{d}t$$

All these integrals are of the form

$$\partial_j \int_{E+s} t^{-1/2} H_t \,\mathrm{d}t$$

for some $E \subset \mathbb{R}^+$ and $s > 0$.

Idea of the proof: Riesz transforms

Next observe that

$$\begin{aligned}\partial_j \int_{E+s} t^{-1/2} H_t \, \mathrm{d}t &= \partial_j \int_E (t+s)^{-1/2} H_{t+s} \, \mathrm{d}t \\ &= \partial_j \int_E (t+s)^{-1/2} H_t \, \mathrm{d}t \; H_s \\ &= \mathscr{R}_j^{E,s} \; H_s\end{aligned}$$

Since H_s is bounded on $L^1(\gamma_\infty)$ it suffices to prove

Proposition

For every $s > 0$ if $E = I$ or $E = J^\sharp$ then the operator

$$\mathscr{R}_j^{E,s} = \partial_j \int_E (t+s)^{-1/2} H_t \, \mathrm{d}t$$

is of weak type $(1,1)$.

Sketch of the proof

Set $\mathscr{R} = \mathscr{R}_j^{E,s}$ for brevity.

Step 1 $\mathscr{R}$ is bounded on $L^2(\gamma_\infty)$

Step 2 The kernel of $\mathscr{R}$

$$K_j^{E,s}(x,y) = \int_E (t+s)^{-1/2} \partial_{x_j} h_t(x,y)\, \mathrm{d}t$$

satisfies the standard Calderòn-Zygmund estimates in the local region

$$L = \left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \ |x-y| < \min\left(1, \frac{1}{|x+y|}\right) \right\}$$

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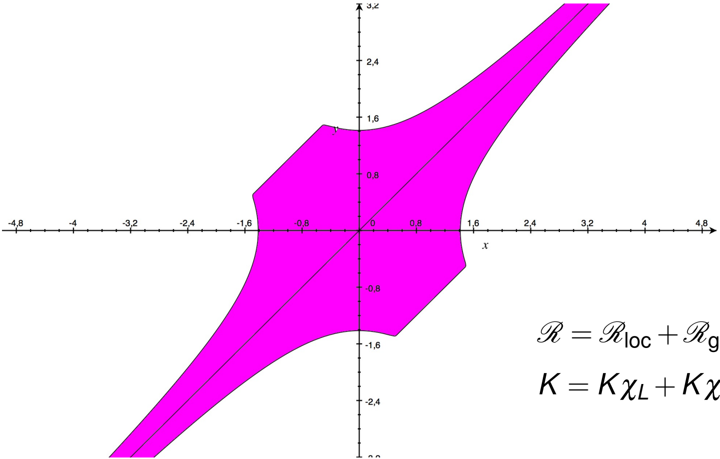
Thus $\mathscr{R}_{\mathrm{loc}}$ is of weak type $(1,1)$.

The local region L



$$\mathscr{R} = \mathscr{R}_{\text{loc}} + \mathscr{R}_{\text{glob}}$$

$$K = K\chi_L + K\chi_{L^c}$$

Return

Step 3: Estimate of $\mathscr{R}_{\text{glob}}$

If $E = I, J^\sharp$ there exists C such that $\forall s > 0$

$$\left| K_j^{E,s}(x,y) \right| \leq C\, \mathrm{e}^{|x|^2} \min\left\{ (1+|x|)^d, (|x| \sin\vartheta)^{-d} \right\} \qquad \forall (x,y) \in L^c.$$

where $\vartheta = \vartheta(x,y)$ is the angle $\widehat{xy}$.

The conclusion follows by the following

Proposition [GC-M-M-S-T]

The operator

$$Tf(x) = \int \mathrm{e}^{|x|^2} \min\left\{ (1+|x|)^d, (|x| \sin\vartheta)^{-d} \right\} f(y)\, \mathrm{d}\gamma_\infty(y)$$

is of weak type $(1,1)$.