An overview on boundedness properties of operators related to Laguerre functions expansions.
(A tribute to the memory of Carlos Segovia)

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Laguerre Polynomials

For a given $\alpha > -1$, the Laguerre polynomials L_n^α are

$$e^{-x} x^\alpha L_n^\alpha(x) = \frac{1}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^{n+\alpha} \right)$$

for $x \in (0, \infty)$.
They are also eigenvectors with eigenvalues n for

$$L^\alpha = -x \frac{d^2}{dx^2} - (\alpha + 1 - x) \frac{d}{dx}, \quad x > 0.$$

- L^α is self-adjoint on C_c^∞ with respect to the measure $e^{-x} x^\alpha dx$.
- L^α is positive.

After normalization, they constitute a basis for $L^2(\mathbb{R}^+, e^{-x} x^\alpha dx)$.
More precisely

$$\int_0^\infty L_k^\alpha(x) L_j^\alpha(x) e^{-x} x^\alpha \, dx = \frac{\Gamma(k + \alpha + 1)}{\Gamma(k+1)} \delta_{kj}.$$

Systems of Laguerre functions, $\alpha > -1$

$$\mathcal{L}_n^\alpha(x) = c_n^\alpha L_n^\alpha(x) e^{-x/2} x^{\alpha/2}$$

$$\varphi_n^\alpha(x) = c_n^\alpha L_n^\alpha(x^2) e^{-x^2/2} x^{\alpha+1/2} = \mathcal{L}_n^\alpha(x^2)(2x)^{1/2}$$

$$\ell_n^\alpha(x) = c_n^\alpha L_n^\alpha(x) e^{-x/2} = \mathcal{L}_n^\alpha(x) x^{-\alpha/2}$$

They are respectively the eigenfunctions of

$$L_{\mathcal{L}}^\alpha = -x\frac{d^2}{dx^2} - \frac{d}{dx} + \frac{x}{4} + \frac{\alpha^2}{4x}$$

$$L_\varphi^\alpha = \frac{1}{4}\left\{-\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\left(\alpha^2 - \frac{1}{4}\right)\right\}$$

$$L_\ell^\alpha = -x\frac{d^2}{dx^2} - (\alpha+1)\frac{d}{dx} + \frac{x}{4}$$

In all cases the sequences of eigenvalues are $\left\{n + \frac{\alpha+1}{2}\right\}_{n=0}^\infty$

L with a discrete spectrum

For L self-adjoint with respect to μ and with positive eigenvalues λ_k and eigenfunctions ϕ_k the heat semigroup is

$$T_t f = e^{-tL} f = \sum_{k=1}^{\infty} e^{-t\lambda_k} \langle f, \phi_k \rangle \phi_k$$

$u(x,t) = T_t f(x)$ solves the problem

$$\begin{cases} \frac{\partial u}{\partial t} = -Lu, & t > 0, \\ u(x,0) = f(x). \end{cases}$$

It may be seen as an integral operator

$$T_t f(x) = \int_X W(t,x,y) f(y) \mu(y) dy,$$

where $W(t,x,y) = \sum_{k=1}^{\infty} e^{-t\lambda_k} \phi_k(x) \phi_k(y)$

Some operators derived from T_t

- The heat maximal operator

$$T^*f = \sup_{t>0} |T_t f|.$$

- The Fractional integration

$$L^{-\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty T_t f(x) t^{\sigma-1} dt, \; \sigma > 0$$

- Riesz Transforms: If $L = D_L^* \cdot D_L$ we may introduce

$$D_L \circ L^{-1/2} \qquad \text{and} \qquad D_L^* \circ L^{-1/2}$$

- The Litlewood-Paley square function

$$g(f)(x) = \left(\int_0^\infty |\frac{\partial}{\partial t} T_t f(x)|^2 t \, dt \right)^{1/2}$$

Heat kernels for Laguerre systems

$$K_{\mathcal{L}^\alpha}(t,x,y) = \sum_{n=0}^{\infty} e^{-t\lambda_n}\mathcal{L}_n^\alpha(x)\mathcal{L}_n^\alpha(y)$$

$$K_{\varphi^\alpha}(t,x,y) = \sum_{n=0}^{\infty} e^{-t\lambda_n}\varphi_n^\alpha(x)\varphi_n^\alpha(y) = 2(xy)^{1/2}K_{\mathcal{L}^\alpha}(t,x^2,y^2)$$

$$K_{\ell^\alpha}(t,x,y) = \sum_{n=0}^{\infty} e^{-t\lambda_n}\ell_n^\alpha(x)\ell_n^\alpha(y) = (xy)^{-\alpha/2}K_{\mathcal{L}^\alpha}(t,x,y).$$

A formula for the $\mathcal{L}_k^\alpha$ case

After performing a change in the parameter

$$K_{\mathcal{L}^\alpha}(t,x,y) = W_{\mathcal{L}^\alpha}\left(\frac{1-e^{-t/2}}{1+e^{-t/2}}, x, y\right)$$

$$W_{\mathcal{L}^\alpha}(s,x,y) = \frac{1}{2}\frac{1-s^2}{2s} e^{-\frac{1}{4}(s+\frac{1}{s})|x^{1/2}-y^{1/2}|^2} e^{-\frac{1}{2}(s+\frac{1}{s})(xy)^{1/2}} I_\alpha\left(\frac{1-s^2}{2s}(xy)^{1/2}\right)$$

for $0 < s < 1$, and $I_\alpha(z) = e^{-i\alpha\pi} J_\alpha(iz)$, with J_α the usual Bessel function.

Searching for Riesz Transforms for $\mathcal{L}_k^\alpha$

Question: What are the RT in this setting?
A path to follow: Thangavelu definition for Hermite functions + Formula relating Hermite d-dimensional with Laguerre on $(0, \infty)$ for $2\alpha = d - 2$

$$\mathcal{L}_\alpha - (\frac{\alpha+1}{2}) = (\delta^\alpha)^* \delta^\alpha = \partial^{\alpha+1}\delta^\alpha.$$

As first order differential operators

$$\delta^\alpha = \sqrt{x}\frac{d}{dx} + \frac{1}{2}(\sqrt{x} - \frac{\alpha}{\sqrt{x}}) \quad \text{and} \quad \partial^{\alpha+1} = -\sqrt{x}\frac{d}{dx} + \frac{1}{2}(\sqrt{x} - \frac{\alpha+1}{\sqrt{x}}).$$

Their action on the corresponding basis is

$$\delta^\alpha(\mathcal{L}_k^\alpha) = -\sqrt{k}\ \mathcal{L}_{k-1}^{\alpha+1}, \qquad \text{and} \qquad \partial^{\alpha+1}(\mathcal{L}_k^{\alpha+1}) = -\sqrt{k+1}\ \mathcal{L}_{k+1}^\alpha.$$

Riesz Transforms for $\mathcal{L}_k^\alpha$

We define

$$R_+^\alpha = \delta^\alpha (\mathcal{L}_\alpha)^{-1/2}, \quad \text{and} \quad R_-^{\alpha+1} = \partial^{\alpha+1} (\mathcal{L}_{\alpha+1})^{-1/2}.$$

Action on the corresponding basis

$$R_+^\alpha(\mathcal{L}_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k + \frac{\alpha+1}{2}}}\, \mathcal{L}_{k-1}^{\alpha+1}$$

$$R_-^{\alpha+1}(\mathcal{L}_k^{\alpha+1}) = -\frac{\sqrt{k+1}}{\sqrt{k + \frac{\alpha+2}{2}}}\, \mathcal{L}_{k+1}^{\alpha}.$$

From Hermite to Laguerre

Relation between Hermite d-dimensional and Laguerre functions:

$$\mathcal{L}_k^\alpha(|x|^2) = c_k^\alpha \sum_{|r|=k} \frac{a_r}{b_{2r}} \Phi_{2r}(x)|x|^\alpha, \quad x \in \mathbb{R}^d, \quad \alpha = \frac{d}{2} - 1.$$

Trough this relation, according to our definition it holds

$$|R_+^\alpha(f) \circ \rho(x)| = \left(\sum_{j=1}^{d} |R_j^+ (\frac{f \circ \rho}{|\cdot|^\alpha})(x)|^2 \right)^{1/2} |x|^\alpha,$$

where $\alpha = (d-2)/2$ and $\rho(x) = |x|^2$, with a similar equation for the others RT.

First power weighted inequalities for special α

Let $\alpha > -1$ and $\alpha = (d-2)/2$.
We (HTV) take advantage of Stempak-Torrea results for Hermite trough the following result:
For a pair as above, when the Hermite version T_d is bounded from

$$L^p(\mathbb{R}^d, |x|^\gamma dx) \mapsto L^p_F(\mathbb{R}^d, |x|^\gamma dx),$$

the Laguerre version T^α is bounded from

$$L^p((0,\infty), y^\delta dy) \mapsto L^p_E((0,\infty), y^\delta dy),$$

where δ satisfies

$$\gamma = \alpha(p-2) + 2\delta.$$

$$-d < \gamma < d(p-1) \ \mapsto \ -1 - \frac{\alpha}{2}p < \delta < \frac{\alpha}{2}p + p - 1$$

Riesz Transforms with any α

Our starting point is

$$R_+^\alpha = T_{\alpha+1}^{\beta+1} \circ T_m^{\beta+1} \circ R_+^\beta \circ T_\beta^\alpha,$$

where T_β^α is the transplantation operator

$$T_\beta^\alpha(\sum a_k \mathcal{L}_k^\alpha) = \sum a_k \mathcal{L}_k^\beta,$$

and T_m^β is a multiplier operator for the system $\mathcal{L}_k^\beta$.
Reminding that

$$R_+^\alpha(\mathcal{L}_k^\alpha) = c_k^\alpha \, \mathcal{L}_{k-1}^{\alpha+1}$$

Clearly the above chain is

$$\mathcal{L}_k^\alpha \mapsto \mathcal{L}_k^\beta \mapsto c_k^\beta \mathcal{L}_{k-1}^{\beta+1} \mapsto c_k^\alpha \mathcal{L}_{k-1}^{\beta+1} \mapsto c_k^\alpha \mathcal{L}_{k-1}^{\alpha+1}$$

Riesz Transforms with any α

Reminding

$$R_+^\alpha = T_{\alpha+1}^{\beta+1} \circ T_m^{\beta+1} \circ R_+^\beta \circ T_\beta^\alpha,$$

Needed tools:

- $L^p(x^\delta)$ inequalities for R_+^α, $\alpha = (d-2)/2$.
- $L^p(x^\delta)$ inequalities for multipliers with $\alpha = (d-2)/2$. It can be derived from Hermite or from results of Stempak and Trebels, when $\alpha > 0$.
- A power weighted version of the Transplantation Theorem of Kanjin and Sato due to Stempak and Trebels.

Some power inequalities for Riesz Transforms

Theorem

Let $\alpha > -1$ and $1 < p < \infty$. Then the Laguerre Riesz Transforms R_+^{α} and $R_-^{\alpha+1}$ are bounded on $L^p((0,\infty), x^{\delta}dx)$ whenever

$$-1 - \alpha p/2 < \delta < (p-1) + \alpha p/2, \qquad \textit{and} \qquad \alpha \leq 0$$

or

$$-1 < \delta < p - 1, \qquad \textit{and} \qquad \alpha > 0.$$

- **Remark**: The range for $\alpha \leq 0$ was the expected one, but one might expect the same range also for $\alpha > 0$. In fact, for the special values of α we do not recover what was proven.
- **Moral**: Multiplier and Transplantation Theorems should be improved.(To be continued).
- **Alternative**: Face the Riesz kernels. (To be continued).

Meanwhile....

Macias-Segovia-Torrea attacked the Maximal heat kernel.
Stempak (1994) For $\alpha \geq 0$

$$W_t^\alpha f(y) \leq C_\alpha e^{-t/4} M_0 f(y).$$

MST (2005) Same bound for $\alpha \geq 0$ and...
If $-1 < \alpha \leq 0$, we denote $\beta = -\alpha$. Then

$$W_t^\alpha f(y) \leq C\left(e^{-t/4} M_0 f(y) + e^{-t(1-\beta)/4} y^{-\beta/2} M_\beta\left(z^{-\beta/2} f(z)\right)(y)\right).$$

Theorem

For $-1 < \alpha < 0$, let $N_\alpha = \left(2/\left(2+\alpha\right), 2/\left(-\alpha\right)\right)$. Then, W_α^* is

(a) *strong type (p, p) if p belongs to N_α*

(b) *weak type on the right end.*

(c) *restricted weak type on the left end.*

Moreover, the results are sharp

Also, some weighted results for the maximal

[MST] refined their estimates to get for the interval

$$N_\alpha^\delta = \begin{cases} \left(\frac{2(1+\delta)}{2+\alpha}, \frac{2(1+\delta)}{-\alpha}\right) \cap (1,\infty) & \text{if } -1<\alpha<0, \\ \left(\frac{2(1+\delta)}{2+\alpha}, \infty\right] \cap (1,\infty] & \text{if } \alpha \geq 0. \end{cases}$$

Theorem

Let $-1<\alpha<\infty$ and $-1<\delta<\infty$. Then, W_α^* is of strong type (p,p) with respect to the measure $x^\delta dx$ whenever p is in N_α^δ. Moreover for the left end is of restricted weak type and for the right is of weak type, and all the results are sharp.

New and better bounds

Lemma

Let $-1 < \alpha$. We have the following estimates for the heat diffusion integral $W_t^\alpha f(y)$:

a) If $-1 < \alpha \leq 0$, we denote $\beta = -\alpha$. Then, we get

$$W_t^\alpha f(y) \leq C_\alpha \left\{ e^{-t/4} M_0 f\,(y) + e^{-t(1-\beta)/2} y^{-\beta/2} M_\beta \left(z^{-\beta/2} f\,(z) \right) (y) \right\}.$$

b) If $\alpha \geq 0$, we have

$$W_t^\alpha f(y) \leq C_\alpha e^{-t/4} \left\{ M^R f\,(y) + M^+ f\,(y) + y^{-\alpha/2} \frac{1}{y} \int_0^y z^{\alpha/2} f\,(z)\, dz \right\}.$$

Riesz Transforms revisited

With some new weapons, we take courage to face the Riesz kernels

- The estimates and technics developed to handle the maximal operator.
- The theory of local Calderón-Zygmund singular integrals introduced by Nowak-Stempak when studying the Hankel transform Transplantation operator.

The plan:

- Show that when $x \approx y$ the restriction of R_+^α and $R_-^{\alpha+1}$ are local Calderón-Zygmund operators
- Outside of that region try to bound in terms of the global part of the maximal operators or else, Hardy type operators.

It turned out that the boundedness properties for the pair of RT were a little bit worse than those obtained for the maximal operator.

Maximal revisited

Some new tasks for the maximal operator:

- Get rid of the restriction $\delta > -1$.
- Obtain the same kind of bounds for all values of $\alpha > -1$.
- Enlarge the family of weights
- Extend the results to the other systems.

Achievements (with A. Chicco-Ruiz)

- For all values of $\alpha > -1$ we get

$$\mathrm{W}^*_{L^\alpha} \lesssim H_0^{\alpha/2} + M_{loc}^{16} + T^{\alpha/2},$$

  where the last operator is bounded by $H_\infty^{\alpha/2}$
- For power weights the same results of MST but for any δ
- Boundedness on $L^p(\omega)$ as long as $\omega(x)x^{p\eta-\eta-\beta} \in A_p(x^{\eta+\beta}dx)$, with $\eta = \beta = \alpha/2$ for the system $\mathcal{L}_k^\alpha$

Precise bounds for the maximal operators

$$H_0^\beta f(x) = x^{-\beta-1} \int_0^x f(y) y^\beta dy$$

$$H_\infty^\eta f(x) = x^\eta \int_x^\infty f(y) y^{-\eta-1} dy,$$

$$T^\eta f(x) = \sup_{0<s<1} |x^\eta \int_x^\infty \varphi(y/s) f(y) y^{-\eta-1} dy|$$

$$M_{loc}^\kappa f(x) = \sup_{0<a<x<b<\kappa a} \frac{1}{b-a} \int_a^b |f(y)| dy$$

Strong type: H_0^β: $\delta < \beta p + p - 1$. H_∞^η: $\delta > -\eta p - 1$. M_{loc}^κ: any δ. For the system $\mathcal{L}_k^\alpha$ we get

$$\beta = \eta = \alpha/2 \qquad \text{and} \qquad \varphi(u) = u^\epsilon e^{-cu}$$

The Maximal for the other systems

From the equalities

$$K_{\varphi^\alpha}(t,x,y) = 2(xy)^{1/2} K_{\mathcal{L}^\alpha}(t,x^2,y^2)$$

$$K_{\ell^\alpha}(t,x,y) = (xy)^{-\alpha/2} K_{\mathcal{L}^\alpha}(t,x,y),$$

we immediately obtain

$$W^*_{\varphi^\alpha} \lesssim H_0^{\alpha+\frac{1}{2}} + M^4_{loc} + T^{\alpha+\frac{1}{2}},$$

and

$$W^*_{\ell^\alpha} \lesssim H_0^{\alpha} + M^{16}_{loc} + T^0.$$

Similarly we get bounds from below.

RT final version

Here is the result we obtained with Segovia, Torrea and Viviani

$$\|\mathcal{R}^\alpha\|(f) = \left(|R_+^\alpha f|^2 + |R_-^{\alpha+1} f|^2\right)^{1/2}.$$

Theorem

Let $\alpha > -1$ and δ a real number. Then the operator $\|\mathcal{R}^\alpha\|$ satisfies:

(a) *Strong type (p,p) with respect to $x^\delta\, dx$ when $-\frac{\alpha}{2}p - 1 < \delta < \frac{\alpha}{2}p + p - 1$ and $1 < p < \infty$.*

(b) *Weak type $(1,1)$ with respect to $x^\delta\, dx$ when $-\frac{\alpha}{2} - 1 \leq \delta \leq \frac{\alpha}{2}$ if $\alpha \neq 0$ and $-1 < \delta \leq 0$ when $\alpha = 0$.*

(c) *Restricted weak type (p,p) with respect to $x^\delta\, dx$, if either $\alpha \neq 0$ and $-\frac{\alpha}{2}p = \delta + 1$ for some $1 < p < \infty$, or if $\frac{\alpha+2}{2}p = \delta + 1$ for some $1 < p < \infty$.*

Multipliers and Transplantation

Some similarities

$$T^{\alpha}_{\beta} : \mathcal{L}^{\alpha}_{k} \to \mathcal{L}^{\beta}_{k}$$

$$R^{\alpha}_{+} : \mathcal{L}^{\alpha}_{k} \to c_k \mathcal{L}^{\alpha+1}_{k-1}$$

Our hope: to get a similar range of powers.
Result: (with Garrigós, Signes, Torrea, Viviani)

Theorem

Let $-1 < \alpha < \beta$ and $1 < p < \infty$. Then the operators T^{α}_{β} and T^{β}_{α} admit a bounded extension to $L^p(x^{\delta}, dx)$ if and only if

$$-1 - \frac{\alpha}{2}p < \delta < \frac{\alpha}{2}p + p - 1$$

Previously known range:

$$\frac{|\gamma|}{2}p - 1 < \delta < p - 1 - \frac{|\gamma|}{2}p, \quad \text{where} \quad \gamma := \min\{\alpha, 0\}.$$

Fractional integration ...under construction

For $\alpha > -1$ and $0 < \sigma < \min\{\alpha + 1, 1\}$ set

$$I_\alpha^\sigma f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty T_t f(x) t^\sigma \frac{dt}{t}.$$

We get the estimates

$$I_\alpha^\sigma \lesssim T_0^{\alpha,\sigma} + T_{loc}^{\sigma,\kappa} + T_\infty^{\alpha,\sigma},$$

where

$$T_0^{\alpha,\sigma} f(x) = e^{-x} x^\sigma H_0^{\alpha/2} f(x),$$

$T_\infty^{\alpha,\sigma} f(x)$ is the adjoint of the above operator, and

$$T_{loc}^{\sigma,\kappa} f(x) = \min_{0 \le \epsilon \le \sigma} \frac{x^{\sigma-\epsilon}}{1 + x^{2\sigma-\epsilon}} \ M_{loc}^{\epsilon,\kappa} f(x),$$

with

$$M_{loc}^{\epsilon,\kappa} f(x) = \sup_{0<a<x<b<\kappa a} \frac{1}{(b-a)^{1-\epsilon}} \int_a^b |f(y)| dy.$$

Some power weighted results for I_α^σ

Boundedness on L^p_δ, $L^{p,1}_\delta$, $L^{p,\infty}_\delta$, where $\|f\|_{L^{p,q}_\delta} = \|f\|_{L^{p,q}(x^{\delta p})}$.

Theorem

Let $\alpha > -1$, $0 < \sigma < \min\{1, \alpha+1\}$ and $1 \le p, q \le \infty$ such that $\frac{1}{q} = \frac{1}{p} - \sigma$. Then I_α^σ satisfies the following sharp results:

- *For $1 < p, q \le \infty$, I_α^σ continuously maps L^p_δ in L^q_δ if* $-\frac{1}{p} + \sigma - \frac{\alpha}{2} < \delta < -\frac{1}{p} + \frac{\alpha}{2} + 1.$
- *For $1 < p, q < \infty$, I_α^σ continuously maps $L^{p,1}_\delta$ in $L^{q,\infty}_\delta$ if* $\delta = -\frac{1}{p} + \frac{\alpha}{2} + 1$ *or* $\delta = -\frac{1}{p} + \sigma - \frac{\alpha}{2}$ *with $\alpha \ne 0$ and $\alpha \ne 2\sigma$.*
- *For $p = 1$ and $q = \frac{1}{1-\sigma}$, I_α^σ continuously maps L^p_δ in $L^{q,\infty}_\delta$ if* $-1 + \sigma - \frac{\alpha}{2} \le \delta \le \frac{\alpha}{2}$*, excluding the left end when $\alpha = 0$.*
- *For $p = \frac{1}{\sigma}$ and $q = \infty$, I_α^σ continuously maps $L^{p,1}_\delta$ in L^q_δ if* $\delta = -\frac{\alpha}{2}$ *or* $\delta = \frac{\alpha}{2} + 1 - \sigma$ *and $\alpha \ne 2\sigma$.*