Characterizations of Hankel multipliers

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Consider: Convolution with radial kernels in $\mathbb{R}^d$ acting on radial functions in L^p_{rad}.

$K(x) = K_0(|x|)$, $f(x) = f_0(|x|)$. Characterize

- $\|K * f\|_{L^p_{\text{rad}}} \lesssim \|f\|_{L^p_{\text{rad}}}$.

Analogous formulation (which makes sense for all $d \geq 1$)

$$\mathcal{L} = D_r^2 + \frac{d-1}{r} D_r$$

acting on functions on $\mathbb{R}^+$, with measure

$d\mu_d = r^{d-1} dr$.

Then characterize m so that

$$\|m(\sqrt{-\mathcal{L}})f\|_{L^p(\mu_d)} \lesssim \|f\|_{L^p(\mu_d)}.$$

Def.: $m \in \mathfrak{M}^p_d$.

Connection: $K = \mathcal{F}^{-1}_{\mathbb{R}^d}[m(|\cdot|)\widehat{f}]$.

Some history:

I. Carl Herz (PNAS 1954): $\frac{2d}{d+1} < p \leq 2$:

$\chi_{[0,t]}(\sqrt{-\mathcal{L}})$ bounded on $L^p(\mu_d)$.

(many subsequent extensions to other expansions of orthogonal polynomials and other transforms).

II. Now $1 < p < 2d/(d+1)$.

Schoenberg (1938): continuity is necessary condition for $m \in \mathfrak{M}_d^p$, and some differentiability property is also needed:

Essentially sharp result (Gasper-Trebels, 1982): For

$b = (d-1)(1/p - 1/2)$, $1 < p < 2d/(d+1)$:

$$\sup_{R>0} \left(\int_R^{2R} \left| s^b m^{(b)}(s) \right|^{p'} \frac{ds}{s} \right)^{1/p'} \lesssim \|m\|_{\mathfrak{M}_d^p}$$

Sufficient conditions (Gasper-Trebels, 1979): If $\alpha > d(1/p - 1/2)$, $1 < p < 2d/(d+1)$ then

$$\|m\|_{\mathfrak{M}^p_d} \lesssim \sup_{R>0} \Big(\int_R^{2R} \Big| s^\alpha m^{(\alpha)}(s) \Big|^2 \frac{ds}{s} \Big)^{1/2}.$$

One can almost close the gap in differentiability if one uses conditions involving $\mathcal{F}^{-1}[\phi m(t\cdot)]$:

Müller-S. (02, for $L^p_{\text{rad}}(L^2_{\text{sph}})$ multipliers),

N. Arai (Hankel, different scales).

Wolff's local smoothing result yields results for Fourier multipliers (in a smaller range).

No characterizations, though.

III. Transplantation theorems for Hankel transforms:

Guy (1960), Stempak-Trebels (1997), Stempak (2002), Nowak-Stempak (2006), ...

Trivial necessary conditions for $m \in \mathfrak{M}^p_d$

- If $\widehat{K}$ has compact support then $K \in L^p(\mathbb{R}^d)$.

- For general K: If Φ is *any* radial Schwartz function then $m \in \mathfrak{M}^p_d$ implies that

$$\sup_{t>0} \left\| \Phi * t^{-d} K(t^{-1} \cdot) \right\|_p < \infty$$

(An equivalent necessary condition is that the $\mathbb{R}^d$ - Fourier transforms of $\phi m(t|\cdot|)$ have uniform L^p norms.)

A characterization for $m \in \mathfrak{M}^p_d$

Theorem: If $1 < p < \frac{2d}{d+1}$ then

$$\|m\|_{\mathfrak{M}^p_d} \approx \sup_{t>0} \left\| \Phi * t^{-d} K(t^{-1} \cdot) \right\|_{L^p_{\mathrm{rad}}(\mathbb{R}^d)}$$

Corollaries

Assumption $1 < p < 2d/(d+1)$.

- Putting dyadic pieces together

$$\|m\|_{\mathfrak{M}^p_d} \lesssim \sup_{t>0} \|\phi m(t\cdot)\|_{\mathfrak{M}^p_d}.$$

- Interpolation (using one of Calderón's complex methods)

$$[\mathfrak{M}^{p_0}_{d_0}, \mathfrak{M}^{p_1}_{d_1}]^{\vartheta} = \mathfrak{M}^p_d;$$

$$(1-\vartheta)(\tfrac{1}{p_0}, d_0) + \vartheta(\tfrac{1}{p_1}, d_1) = (\tfrac{1}{p}, d).$$

In contrast $M^p(\mathbb{R})$ is no interpolation space of $M^{p_0}(\mathbb{R})$ and $M^{p_1}(\mathbb{R})$. (Zafran, ...)

- Sharp Hörmander multiplier theorem. Take a nontrivial $\phi \in C_c^\infty(1,2)$:

  Then

  $$\|m\|_{\mathfrak{M}^p_d} \lesssim \sup_{t>0} \|\phi m(t\cdot)\|_{B^2_{d(\frac{1}{p}-\frac{1}{2}),p}}.$$

  Compare with the necessary condition

  $$\sup_{t>0} \|\phi m(t\cdot)\|_{B^{p'}_{(d-1)(\frac{1}{p}-\frac{1}{2})}} \lesssim \|m\|_{\mathfrak{M}^d_p}$$

  One proves these by proving the corresponding inequalities for

  $$\sup_{t>0} \|\mathcal{F}^{-1}[\phi(|\cdot|)m(t|\cdot|)]\|_p$$

  which is now equivalent to $\|m\|_{\mathfrak{M}^d_p}$.

Versions for multipliers of Fourier-Bessel transforms (aka modified Hankel transforms), i.e. Extension to $m(\sqrt{-\mathcal{L}})$ for all real $d > 1$.

Define

$$B_d(\rho) = \rho^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(\rho)$$

(corresponds to Fourier transform of surface measure on unit sphere in $\mathbb{R}^d$).

For functions on $(0, \infty)$ define

$$\mathcal{B}_d g(r) = \int_0^\infty B_d(\rho r) g(\rho) \rho^{d-1} d\rho$$

and then

$$T_m f \equiv m(\sqrt{-\mathcal{L}}) f = \mathcal{B}_d[m \mathcal{B}_d f].$$

Note that $\mathcal{B}_d = \mathcal{B}_d^{-1}$.

Let ϕ be a nontrivial $C_c^\infty(\mathbb{R}^+)$ function.

Theorem: Let $1 < p < \frac{2d}{d+1}$. **TFAE:**

(i) T_m is bounded on $L^p(\mu_d)$.

(ii) T_m maps $L^{p,1}(\mu_d)$ to $L^p(\mu_d)$.

(iii) $\sup_{t>0} \|\mathcal{B}_d[\phi m(t\cdot)]\|_{L^p(\mu_d)} < \infty$

(iv) For $\kappa_t := \mathcal{F}_{\mathbb{R}}^{-1}[\phi m(t\cdot)]$ we have

$$\sup_{t>0} \Big(\int |\kappa_t(r)|^p (1+|r|)^{(d-1)(1-p/2)} dr \Big)^{1/p} < \infty.$$

Structure of the proof:

(i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) : trivial

(iii) $\Longrightarrow$ (iv): easy version of transplantation.

(iv) $\Longrightarrow$ (i): needs to be proved.

Proof that (iii) $\Longrightarrow$ (iv), i.e.

$$\sup_{t>0} \|\mathcal{B}_d[\phi m(t\cdot)]\|_{L^p(\mu_d)} < \infty$$

implies

$$\sup_{t>0} \Big(\int |\kappa_t(r)|^p (1+|r|)^{(d-1)(1-p/2)} dr \Big)^{1/p} < \infty.$$

Note that $\mathcal{B}_1$ is the cosine transform.

Take $\chi \in C_c^\infty$ so that $\chi\phi = \phi$, then

$$\mathcal{B}_1[\phi m(t\cdot)] = \mathcal{B}_1 \chi \mathcal{B}_d \mathcal{B}_d[\phi m(t\cdot)].$$

Use pointwise bound

$$|\mathcal{B}_1 \chi \mathcal{B}_d g(r)| \lesssim \int_0^\infty \frac{|g(s)| s^{d-1}}{(1+|r-s|)^M} (1+s)^{-\frac{d-1}{2}} \, ds.$$

Main implication (iv) $\Longrightarrow$ (i): Condition $\sup_{t>0}\Big(\int|\kappa_t(r)|^p(1+|r|)^{(d-1)(1-p/2)}dr\Big)^{1/p}<\infty$ implies $L^p(\mu_d)$ boundedness.

We present here the special case of m compactly supported in $(1,2)$.

Observe that $T_m f(r)=\int K(r,s)f(s)s^{d-1}ds$ where

$K(r,s)=\int_0^\infty B_d(r\rho)B_d(s\rho)m(\rho)\rho^{d-1}d\rho.$

Use Asymptotics for large x:

$B_d(x)=(c_1e^{i|x|}+c_2e^{-i|x|})|x|^{-(d-1)/2}+\ldots$

This leads to

$$|K(r,s)|\lesssim\sum_{(\pm,\pm)}\int\frac{|\kappa(\pm r\pm s\pm u)|(1+|u|)^{-N}}{(1+r)^{(d-1)/2}(1+s)^{(d-1)/2}}\,du.$$

Here $\kappa=\mathcal{F}_{\mathbb{R}}^{-1}[m]$.

Ignoring the u-integral, one needs to verify the inequality

$$\Big(\int_0^\infty \Big|\int_0^\infty \frac{\kappa(\pm r \pm s) r^{\frac{d-1}{p}}}{(1+r)^{\frac{d-1}{2}}} \frac{|f(s)| s^{\frac{d-1}{p}}}{(1+s)^{\frac{d-1}{2}-\frac{d-1}{p'}}}\, ds\Big|^p dr\Big)^{\frac{1}{p}}$$

$$\lesssim \Big(\int_0^\infty |f(s)|^p s^{d-1} ds\Big)^{\frac{1}{p}}.$$

Use:

$(d-1)(\frac{1}{2}-\frac{1}{p'}) > \frac{1}{p'} \iff p < \frac{2d}{d+1}$.

Analyze the extreme case:

$$s \approx 1 \text{ and } r \gg 1.$$

For *general* $m = \sum_j m_j(2^{-j}\cdot)$ (support of m_j in $(1,2)$) use Littlewood-Paley theory, singular integrals, and vector-valued versions of Hardy's inequality.

Let

$$T_j f(r) = \int 2^{jd} K_j(2^j r, 2^j s) f(s) s^{d-1} ds$$

and we need to prove

$$\Big\| \Big(\sum_j |T_j f_j|^2 \Big)^{1/2} \Big\|_p \lesssim \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_p$$

- Extreme case for T_j: localize where

  $$r \gg s \approx 2^{-j}.$$

  Define corresponding operator H_j and because of *disjointness on the function side* it suffices to prove

  $$\Big\| \Big(\sum_j |H_j f_j|^p \Big)^{1/p} \Big\|_p \lesssim \Big(\sum_j \|f_j\|_p^p \Big)^{1/p}.$$

- Similar analysis for the cases $r \ll s \approx 2^{n-j}$, $n > 0$. Gain $2^{-n\epsilon}$ (typical for Hardy operator estimates).

- Calderón-Zygmund analysis needed for the localization $r \approx s$. An ϵ loss occurs when putting dyadic cases together.

  However: the conditions needed for boundedness of this part are weaker than our assumptions (essentially as in 1 D, without weights). Thus the ϵ loss does not matter.

- The case $r \gg s$ is a more standard Hardy type operator - works with much weaker assumptions.

Weak type (p,p) bounds.

Theorem WT: Let $1 < p < \frac{2d}{d+1}$. **TFAE:**

(i) T_m maps $L^p(\mu_d)$ to $L^{p,\infty}(\mu_d)$.

(ii) T_m maps $L^{p,1}(\mu_d)$ to $L^{p,\infty}(\mu_d)$.

(iii) $\sup_{t>0} \|\mathcal{B}_d[\phi m(t\cdot)]\|_{L^{p,\infty}(\mu_d)} < \infty$

(iv) For $\kappa_t := \mathcal{F}_{\mathbb{R}}^{-1}[\phi m(t\cdot)]$ we have

$$\sup_{t>0} \Big\|(1+|\cdot|)^{-\frac{d-1}{2}}\kappa_t\Big\|_{L^{p,\infty}((1+r)^{d-1}dr)} < \infty.$$

Remark: If m is compactly supported in $(1,2)$ the weak type (p,p) (i.e. $L^p(\mu_d) \to L^{p,\infty}(\mu_d)$) can be upgraded to boundedness on $L^{p,\infty}(\mu_d)$. This effect has been observed before by Colzani, Travaglini and Vignati.

Question: Are there versions of these results for Fourier multipliers?

T. Wolff's $\ell^q(L^q) \to L^q$ inequality for decompositions of cone multipliers ($q = p' \gg 2$) yields a smoothing result (local in time) for the wave equation

$$\Big(\int_{-R}^{R} \big\| e^{it\sqrt{-\Delta}} f \big\|_q^q dt \Big)^{1/q} \le C_R \|f\|_{L^p_\alpha},$$

for

$$\begin{aligned} \alpha &> d(1/2 - 1/q) - 1/2 \\ &= (d-1)(1/2 - 1/q) - 1/q. \end{aligned}$$

"Smoothing" means gain of $1/q - \varepsilon$ derivatives when compared to the fixed time $L^p(\mathbb{R}^d)$ bounds for $e^{it\sqrt{-\Delta}}$.

Connection to radial multipiers: Use

$$m(\sqrt{-\Delta}) = \frac{1}{2\pi} \int \widehat{m}(t) e^{it\sqrt{-\Delta}} dt.$$

Setting $p = q'$ this implies that

$$\|\mathcal{F}^{-1}[m(|\cdot|)\widehat{f}]\Big\|_p \lesssim \|f\|_p$$

holds under the condition

$$\sup_{t>0} \Big\|\mathcal{F}^{-1}[\phi m(t|\cdot|)]\Big\|_{L^p((1+|x|)^\epsilon)} < \infty.$$

"ε loss of smoothness".

Current range of validity (Garrigós, Schlag, S.):

$$1 < p < \frac{2d(d+3)}{d^2+7d+2}, \text{ if } d \geq 3.$$

$$1 < p < \frac{20}{19}, \text{ if } d = 2.$$

ε loss because of method of proof (induction on scales)

A new development: Work in progress on characterizations of *radial Fourier multipliers*.

Theorem: (**F. Nazarov**, A.S.). Let $1 < p < 2$. Then there is $d(p)$ so that for dimensions $d > d(p)$ the following characterization holds.

- $m(|\cdot|) \in M^p$ if and only if

$$\sup_{t>0} \left\| \mathcal{F}^{-1}[\phi m(t|\cdot|)] \right\|_p < \infty.$$

Hence, *for large dimensions* $d > d(p)$, $m(\sqrt{-\Delta})$ is bounded on $L^p(\mathbb{R}^d)$ if and only if it is bounded on $L^p_{\text{rad}}(\mathbb{R}^d)$. In fact it suffices to test the operator on the one-parameter family $t^{d/p}\Phi(t\cdot)$, for *one* nontrivial radial $\Phi \in \mathcal{S}(\mathbb{R}^d)$.

Currently this gives a nontrivial range of $p's$ in dimensions $d \geq 6$, with $1 < p < \frac{2(d-3)}{d-1}$ (i.e. $d > d(p) = \frac{6-p}{2-p}$). Analogous result for $p = 1$ involving Hardy space H^1.