Riesz transforms for the Dunkl harmonic oscillator - the $\mathbb{Z}_2^d$ group case

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Introduction

- Recent development on Riesz transforms related to orthogonal expansions
- Unified approach to the L^2-theory of Riesz transforms for orthogonal expansions (differential "Laplacians")
- Motivation: extension to non-differential "Laplacians"
- Choice of Dunkl's framework
- Interest in L^2 (primarily) and in L^p theory (whenever possible)

Setting (emerging from reflection symmetries in $\mathbb{R}^d$)

- Finite reflection group $G \subset \mathcal{O}(\mathbb{R}^d)$
- Root system $\mathcal{R} \subset \mathbb{R}^d$
- G-invariant multiplicity function $k\colon \mathcal{R} \mapsto [0,\infty)$

Dunkl's difference-differential operators

$$T_j^{G,k} f(x) = \partial_j f(x) + \sum_{\beta\in\mathcal{R}_+} k(\beta)\beta_j \frac{f(x) - f(\sigma_\beta x)}{\langle \beta, x\rangle}$$

$\mathcal{R}_+$ - fixed positive subsystem of $\mathcal{R}$
σ_β - reflection in the hyperplane orthogonal to β

The system $\{T_j^{G,k} : j = 1,\dots,d\}$ commutes

Dunkl's Laplacian

Dunkl's Laplacian

$$\Delta_{G,k} = \sum_{j=1}^{d} \left(T_j^{G,k} \right)^2$$

This (difference-differential) operator is symmetric in

$$L^2\left(\mathbb{R}^d, w_{G,k}\right)$$

with the (G-invariant) measure given by

$$w_{G,k}(x) = \prod_{\beta \in \mathcal{R}_+} |\langle \beta, x \rangle|^{2k(\beta)}$$

Dunkl's harmonic oscillator (DHO)

Harmonic oscillator related to $\Delta_{G,k}$

$$\mathbb{L}_{G,k} = -\Delta_{G,k} + \|x\|^2$$

- self-adjoint positive extension
- discrete spectrum
- eigenfunctions: multivariable generalized Hermite functions (defined and investigated by M. Rösler)
- choice of eigenfunctions **is not** unique

Riesz transforms for DHO

- Partial derivatives associated with $\mathbb{L}_{G,k}$

$$\delta_j^{G,k} = T_j^{G,k} + x_j$$

- Motivation:

$$\mathbb{L}_{G,k} = \frac{1}{2}\sum_{j=1}^{d} \left(\delta_j^{G,k}\right)^* \delta_j^{G,k} + \delta_j^{G,k} \left(\delta_j^{G,k}\right)^*$$

Riesz transforms

$$R_j^{G,k} = \delta_j^{G,k} \left(\mathbb{L}_{G,k}\right)^{-1/2}, \qquad j = 1, \ldots, d$$

- These are L^2-**bounded operators** (mild assumptions)
- **Aim:** study L^p behavior of $R_j^{G,k}$; **restrict to** $\mathbf{G} = \mathbb{Z}_2^d$

Setting ($\mathbb{Z}_2^d$ group case)

- $G \simeq \mathbb{Z}_2^d$ generated by reflections

$$\sigma_j(x_1, \dots, x_j, \dots, x_d) = (x_1, \dots, -x_j, \dots, x_d)$$

- root system $\mathcal{R} = \{\pm\sqrt{2}e_j : j = 1, \dots, d\}$
- positive subsystem $\mathcal{R}_+ = \{\sqrt{2}e_j : j = 1, \dots, d\}$
- multiplicity function $k \colon \mathcal{R} \mapsto [0, \infty)$ det. by values on $\mathcal{R}_+$
- we may think

$$k = (\alpha_1 + 1/2, \dots, \alpha_d + 1/2), \qquad \alpha_j \geq -1/2$$

Setting (cont.)

- Dunkl's d-d operators

$$T_j^{\alpha} f(x) = \partial_j f(x) + (\alpha_j + 1/2)\frac{f(x) - f(\sigma_j x)}{x_j}$$

- Dunkl's Laplacian

$$\Delta_{\alpha} f(x) = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}(x) + \frac{2\alpha_j + 1}{x_j}\frac{\partial f}{\partial x_j}(x) - (\alpha_j + 1/2)\frac{f(x) - f(\sigma_j x)}{x_j^2}$$

- Associated measure

$$w_{\alpha}(x) \simeq \prod_{j=1}^{d} |x_j|^{2\alpha_j + 1}$$

Setting (cont.)

- Dunkl's harmonic oscillator $\mathbb{L}_\alpha = -\Delta_\alpha + \|x\|^2$
- generalized Hermite functions (1-dim)

$$h_n^\alpha(x) = c_{n,\alpha} \exp(-x^2/2) H_n^{\alpha+1/2}(x)$$

here $H_n^{\alpha+1/2}$ are **genuine** generalized Hermite polynomials

$$H_{2n}^{\alpha+1/2}(x) = (-1)^n 2^{2n} n! L_n^\alpha(x^2)$$
$$H_{2n+1}^{\alpha+1/2}(x) = (-1)^n 2^{2n+1} n!\, x L_n^{\alpha+1}(x^2)$$

- $\{h_n^\alpha : n \in \mathbb{N}^d\}$ forms ONB in $L^2(\mathbb{R}^d, w_\alpha)$ and

$$\mathbb{L}_\alpha h_n^\alpha = (2|n| + 2|\alpha| + 2d) h_n^\alpha$$

Semigroup generated by $\mathbb{L}_\alpha$

$$\exp(-t\mathbb{L}_\alpha)f = \int_{\mathbb{R}^d} G_t^\alpha(x,y)f(y)\,dw_\alpha(y)$$

Heat kernel $G_t^\alpha(x,y)$ in 1-dim

$$\frac{\exp\left(-\frac{1}{2}\coth(2t)(x^2+y^2)\right)}{2\sinh 2t}\left[\frac{I_\alpha\left(\frac{xy}{\sinh 2t}\right)}{(xy)^\alpha} + xy\frac{I_{\alpha+1}\left(\frac{xy}{\sinh 2t}\right)}{(xy)^{\alpha+1}}\right]$$

here I_ν is the modified Bessel function

$$I_\nu(z) = \sum_{k=0}^{\infty}\frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$$

Heat kernel

In dimension *d*

$$G_t^{\alpha}(x,y) = \sum_{\varepsilon\in\{0,1\}^d} G_t^{\alpha,\varepsilon}(x,y)$$

with the component kernels

$$G_t^{\alpha,\varepsilon}(x,y) =$$

$$\frac{\exp\left(-\frac{1}{2}\coth(2t)(\|x\|^2+\|y\|^2)\right)}{(2\sinh 2t)^d} \prod_{j=1}^{d} (x_j y_j)^{\varepsilon_j} \frac{I_{\alpha_j+\varepsilon_j}\left(\frac{x_j y_j}{\sinh 2t}\right)}{(x_j y_j)^{\alpha_j+\varepsilon_j}}$$

Riesz transforms

$$R_j^\alpha = \delta_j^\alpha (\mathbb{L}_\alpha)^{-1/2}$$

with *partial derivatives*

$$\delta_j^\alpha f(x) = T_j^\alpha f(x) + x_j = \partial_j f(x) + (\alpha_j + 1/2)\frac{f(x) - f(\sigma_j x)}{x_j} + x_j$$

For $f \in L^2(\mathbb{R}^d, w_\alpha)$

$$R_j^\alpha f = \sum_{n \in \mathbb{N}^d} \frac{m(n_j, \alpha_j)}{\sqrt{2|n| + 2|\alpha| + 2d}} \langle f, h_n^\alpha \rangle_\alpha h_{n-e_j}^\alpha$$

$$m(n_j, \alpha_j) = \begin{cases} \sqrt{2n_j} & \text{if } n_j \text{ is even} \\ \sqrt{2n_j + 4\alpha_j + 2} & \text{if } n_j \text{ is odd} \end{cases}$$

Main Theorem

Define

$$R_j^\alpha(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \delta_{j,x}^\alpha G_t^\alpha(x,y) \frac{dt}{\sqrt{t}}$$

Theorem

The Riesz operators R_j^α, defined on L^2 by ..., are CZ operators, in the sense of $(\mathbb{R}^d, w_\alpha, \|\cdot\|)$, with the associated kernels $R_j^\alpha(x,y)$.

It follows that R_j^α extend to bounded operators

- on $L^p(\mathbb{R}^d, W dw_\alpha)$ for $1 < p < \infty$ and $W \in A_p^\alpha = A_p^\alpha(\mathbb{R}^d, w_\alpha)$
- from $L^1(\mathbb{R}^d, W dw_\alpha)$ to $L^{1,\infty}(\mathbb{R}^d, W dw_\alpha)$ for $W \in A_1^\alpha$

Typical application

Let $1 < p < \infty$ and $i, j \in \{1, \ldots, d\}$. Then

A priori bounds

$$\|(\delta_i^\alpha)^* \delta_j^\alpha f\|_{L^p(\mathbb{R}^d, w_\alpha)} \lesssim \|\mathbb{L}_\alpha f\|_{L^p(\mathbb{R}^d, w_\alpha)}, \qquad f \in C_c^\infty$$

Proof: check that

$$(\delta_i^\alpha)^* \delta_j^\alpha f = (R_i^\alpha)^* R_j^\alpha \mathbb{L}_\alpha f$$

(reduce this task to $f = h_n^\alpha$) and apply the Theorem.

Proof of the Theorem

standard Kernel estimates

$$|R_j^\alpha(x,y)| \lesssim \frac{1}{w_\alpha(B(x,\|y-x\|))}, \qquad x \neq y$$

$$\|\nabla_{x,y} R_j^\alpha(x,y)\| \lesssim \frac{1}{\|x-y\|}\,\frac{1}{w_\alpha(B(x,\|y-x\|))}, \qquad x \neq y$$

Recall that

$$R_j^\alpha(x,y) = \sum_{\varepsilon\in\{0,1\}^d} R_j^{\alpha,\varepsilon}(x,y)$$

with the component kernels

$$R_j^{\alpha,\varepsilon}(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \delta_{j,x}^\alpha G_t^{\alpha,\varepsilon}(x,y)\frac{dt}{\sqrt{t}}$$

Reduction to $\mathbb{R}^d_+$

The following symmetries hold for all $\eta, \xi \in \{-1,1\}^d$

$$|R_j^{\alpha,\varepsilon}(x,y)| = |R_j^{\alpha,\varepsilon}(\eta x, \xi y)|$$
$$\|\nabla_{x,y} R_j^{\alpha,\varepsilon}(x,y)\| = \|\nabla_{x,y} R_j^{\alpha,\varepsilon}(\eta x, \xi y)\|$$
$$w_\alpha(x) = w_\alpha(\xi x)$$

Consequently, it suffices to prove

standard estimates in $(\mathbb{R}^d_+, w_\alpha^+, \|\cdot\|)$

$$|R_j^{\alpha,\varepsilon}(x,y)| \lesssim \frac{1}{w_\alpha^+(B^+(x, \|y-x\|))}, \qquad x,y \in \mathbb{R}^d_+, \quad x \neq y$$
$$\|\nabla_{x,y} R_j^{\alpha,\varepsilon}(x,y)\| \lesssim \frac{1}{\|x-y\|} \frac{1}{w_\alpha^+(B^+(x, \|y-x\|))}$$

Schläfli's formula of Poisson's type

$$\mathbf{I}_\nu(\mathbf{z}) = \mathbf{z}^\nu \int_{-1}^{1} \exp(-\mathbf{z}s)\, \Pi_\nu(d\mathbf{s}), \qquad \nu \geq -1/2;$$

here the measure Π_ν is given by

$$\Pi_\nu(ds) = \frac{(1-s^2)^{\nu-1/2}ds}{\sqrt{\pi}2^\nu\Gamma(\nu+1/2)}, \qquad s\in(-1,1)$$

for $\nu > -1/2$, and in the limiting case $\nu = -1/2$

$$\Pi_{-1/2} = \frac{1}{\sqrt{2\pi}}(\eta_{-1}+\eta_1)$$

Symmetric form of heat kernel

Using Schläfli's formula and transforming

$$t = t(\zeta) = \frac{1}{2}\log\frac{1+\zeta}{1-\zeta}, \qquad \zeta \in (0,1)$$

Convenient heat (component) kernel formula

$$G_t^{\alpha,\varepsilon}(x,y) =$$

$$\frac{1}{2^d}\left(\frac{1-\zeta^2}{2\zeta}\right)^{d+|\alpha|+|\varepsilon|}(xy)^\varepsilon \int_{[-1,1]^d} \exp\left(-\frac{1}{4\zeta}q_+ - \frac{\zeta}{4}q_-\right)\Pi_{\alpha+\varepsilon}(ds)$$

$$q_\pm(x,y,s) = \|x\|^2 + \|y\|^2 \pm 2\sum_{i=1}^d x_i y_i s_i, \qquad \Pi_{\alpha+\varepsilon} = \bigotimes_{i=1}^d \Pi_{\alpha_i+\varepsilon_i}$$

Riesz kernels

$$\beta_{d,\alpha}(\zeta)=\frac{\sqrt{2}}{2^d\sqrt{\pi}}\left(\frac{1-\zeta^2}{2\zeta}\right)^{d+|\alpha|}\frac{1}{1-\zeta^2}\left(\log\frac{1+\zeta}{1-\zeta}\right)^{-1/2}$$

Riesz (component) kernels

$$R_j^{\alpha,\varepsilon}(x,y)=$$

$$\int_{[-1,1]^d}\int_0^1\beta_{d,\alpha+\varepsilon}(\zeta)\,\delta_j\left[(xy)^{\varepsilon}\exp\left(-\frac{1}{4\zeta}q_+-\frac{\zeta}{4}q_-\right)\right]d\zeta\,\Pi_{\alpha+\varepsilon}(ds)$$

$$\delta_j=\frac{\partial}{\partial x_j}+x_j\ \text{ if }\ \varepsilon_j=0,\qquad \delta_j=\frac{\partial}{\partial x_j}+x_j+\frac{2\alpha_j+1}{x_j}\ \text{ if }\ \varepsilon_j=1$$

Transition Lemma

Let $\gamma, \kappa \in [0,\infty)^d$ be fixed. Then

$$|K(x,y)| \lesssim (x+y)^{2\gamma} \int_{[-1,1]^d} \big(q_+(x,y,s)\big)^{-d-|\alpha|-|\gamma|} \, \Pi_{\alpha+\gamma+\kappa}(ds),$$

implies

$$|K(x,y)| \lesssim \frac{1}{w_\alpha^+(B^+(x,\|y-x\|))}, \qquad x \neq y.$$

Similarly,

$$\|\nabla_{x,y} K(x,y)\| \lesssim (x+y)^{2\gamma} \int_{[-1,1]^d} \big(q_+\big)^{-d-|\alpha|-|\gamma|-1/2} \, \Pi_{\alpha+\gamma+\kappa}(ds)$$

implies

$$\|\nabla_{x,y} K(x,y)\| \lesssim \frac{1}{\|x-y\|} \, \frac{1}{w_\alpha^+(B^+(x,\|y-x\|))}, \qquad x \neq y.$$

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Thank you for your attention.