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Representing types in Orlicz and Lorentz sequence spaces

By JESÚS BASTERO
Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

and YVES RAYNAUD
Equipe d'Analyse, Université Paris VI, 4 Place Jussieu, 75252 Paris, Cedex 05, France

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Abstract

The aim of this paper is to give a representation for the types in some stable Banach sequence spaces, namely in the Orlicz, Lorentz and dual of Lorentz sequence spaces. We also find a characterization for the Lorentz sequence spaces whose class of weakly-null types is locally compact for the topology of uniform convergence on bounded subsets.

1. *Introduction and notation*

The class of stable Banach spaces was introduced by Krivine and Maurey (see [7]), in order to extend to this class of Banach spaces a previous result by Aldous concerning the closed infinite-dimensional subspaces of L^1.

A Banach space E is said to be stable if for any pair of bounded sequences $(x_n)_n$, $(y_k)_k$ in E and for any pair of non-trivial ultrafilters $\mathscr{U}, \mathscr{V}$ on $\mathbb{N}$ we have

$$\lim_{n\,\mathscr{U}} \lim_{k\,\mathscr{V}} \|x_n + y_k\| = \lim_{k\,\mathscr{V}} \lim_{n\,\mathscr{U}} \|x_n + y_k\|.$$

We also recall that a *type* in a (separable) Banach space E is a map τ from E into $\mathbb{R}$, defined by $\tau(x) = \lim_{n\,\mathscr{U}} \|x + x_n\|$, where $(x_n)_n$ is a bounded sequence in E and $\mathscr{U}$ is a non-trivial ultrafilter in $\mathbb{N}$. Note that there exists a subsequence $(x'_n)_n$ of $(x_n)_n$ such that $\tau(x) = \lim_{n\to\infty} \|x + x'_n\|$ for all $x \in E$. We therefore say that τ is a weakly-null type if τ can be defined by a weakly-null sequence in E. Let $\mathscr{T}(E)$ (respectively $\mathscr{T}_0(E)$) represent the class of types (respectively weakly-null types) defined on E. The space $\mathscr{T}(E)$ can be equipped with two natural topologies: the topology of pointwise convergence (P.C.T.) and the topology of uniform convergence on bounded sets (T.U.C.B.). Both of them are metrizable and $\mathscr{T}(E)$ is locally compact with respect to P.C.T.

Here we make explicit the types of Orlicz and Lorentz sequence spaces and of their duals. In Section 2 the class $\mathscr{T}_0(l^M)$ appears as a subset of the compact set $C_{M,1}$ naturally associated to the Orlicz function M (see [9], I·4·a·6). This implies that P.C.T. and T.U.C.B. coincide on $\mathscr{T}_0(l^M)$ (as already stated in [3] with shortened proof). In Section 3 we characterize the Lorentz sequence spaces $d(\omega, p)$ for which P.C.T. and

T.U.C.B. coincide on $\mathscr{T}_0(d(\omega,p))$. This class is the Lorentz space analogue of Orlicz spaces with regularly varying Orlicz function studied in [**4**], and the ultrapowers of such spaces can be very simply described. In Section 4 we state a duality result for types and, as a corollary, we obtain the stability of duals of reflexive Lorentz sequence spaces.

If the space E is stable we can define the convolution of types as a separately continuous operation in $\mathscr{T}(E)$. In fact if $\tau = \lim_{n\to\infty} x_n$ and $\sigma = \lim_{k\to\infty} y_k$ the type $\tau * \sigma$ is well defined by

$$\tau * \sigma(x) = \lim_{n\to\infty}\lim_{k\to\infty} \|x + x_n + y_k\|.$$

If λ is a real number, we also may define the type $\lambda\tau$ by

$$\lambda\tau(x) = \lim_{n\to\infty} \|x + \lambda x_n\|$$

$(= \lambda(\tau(x/\lambda))$ if $\lambda \neq 0)$.

We say that a type τ is symmetric if $\tau(x) = \tau(-x)$ for all x in E and a symmetric type τ is an l^p-type (where $1 \leqslant p < \infty$) if $\lambda\tau * \mu\tau = (\lambda^p + \mu^p)^{1/p}\tau$, for all $\lambda, \mu \geqslant 0$.

Recall that (see [**1**] and [**12**]) if E is a Banach space with a 1-symmetric basis, we may represent a type in the following way:

$$\tau(x) = \lim_{n\to\infty} \|x + a \oplus x_n\|,$$

where a is a fixed vector in E, $(x_n)_n$ is a block basic sequence with $\|x_n\| = C$ for all n, and $\oplus$ denotes disjoint sum. The vector a and the scalar C are unique. Indeed, suppose that also we could represent $\tau(x) = \lim_{n\to\infty} \|x + b \oplus y_k\|$ with $\|y_k\| = D$. Then

$$\tau(0) = \tau(-2a) = \tau(2a - 2b) = \ldots = \tau(2na - 2nb) \geqslant \|2na - 2nb\|$$

for all $n \in \mathbb{N}$. Thus $a = b$ and so $C = D$.

Recall that a is the coordinate-wise limit for all bounded sequences defining the same type. Moreover for reflexive symmetric spaces, a type is symmetric if and only if $a = 0$.

In the sequel we will use standard Banach space notation such as may be found in [**9**].

An Orlicz function is a continuous non-decreasing convex function M, defined for $t \geqslant 0$, such that $M(0) = 0$, $M(1) = 1$ and $\lim_{t\to\infty} M(t) = \infty$. We say M satisfies the Δ_2-condition at 0 if $M(2t) \leqslant KM(t)$, for some constant K and for $0 \leqslant t \leqslant t_0$. In the sequel we always require this condition for the function M. The space

$$l^M = \left\{ x = (x(i))_i ; \sum_{i=1}^{\infty} M(|x(i)|) < \infty \right\}$$

equipped with the norm

$$\|x\| = \inf\left\{ \rho > 0 ; \sum_{i=1}^{\infty} M\left(\frac{|x(i)|}{\rho}\right) \leqslant 1 \right\}$$

is an Orlicz sequence space. Note that if $x \neq 0$ then $\|x\|$ is the unique number such that $\sum_{i=1}^{\infty} M(|x(i)|/\|x\|) = 1$. By $M_x(\cdot)$ we denote the modular defined by $M_x(t) = \sum_{i=1}^{\infty} M(|x(i)|\,t)$ for x in l^M and $t \geqslant 0$. The unit vectors form a 1-symmetric basis of l^M.

We represent by $d(\omega, p)$ the Lorentz sequence space

$$d(\omega,p) = \left\{x = (x(i))_i : \sum_{i=1}^{\infty} x(i)^{*p}\,\omega_i < \infty\right\},$$

where $1 \leqslant p < \infty$, $\omega = (\omega_i)_i$ is a non-increasing sequence of positive numbers such that $\lim_{i\to\infty}\omega_i = 0$, $\omega_1 = 1$ and $\sum_{i=1}^{\infty}\omega_i = \infty$, (where $(x(i)^*)_i$ denotes the non-increasing rearrangement of the sequence $(|x(i)|)_i$). In this space we consider the norm $\|x\| = (\sum_{i=1}^{\infty} x(i)^{*p}\,\omega_i)^{1/p}$. The canonical basis is a symmetric basis in this space. Recall that $d(\omega,p)$ does not contain c_0 and it is reflexive when $p > 1$ (see [**9**], proposition I·4·e·3 and I·c·13).

2. *Types in Orlicz sequence spaces*

The stability of these spaces was proved in [**5**] and [**11**]. First, using the representation of types described above, we verify this fact in a simple way. Indeed let $(x_n)_n$, $(y_k)_k$ be two bounded sequences in l^M and $\mathscr{U}, \mathscr{V}$ two non-trivial ultrafilters on $\mathbb{N}$. Since c_0 does not embed in l^M, by passing to subsequences if necessary, we may prove that

$$\lim_{n\mathscr{U}}\lim_{k\mathscr{V}}\|x_n + y_k\| = \lim_{n\to\infty}\lim_{k\to\infty}\|x + y \oplus x'_n \oplus y'_k\|$$

and

$$\lim_{k\mathscr{V}}\lim_{n\mathscr{U}}\|x_n + y_k\| = \lim_{k\to\infty}\lim_{n\to\infty}\|x + y \oplus x'_n \oplus y'_k\|,$$

where $(x'_n)_n$, $(y'_k)_k$ are block basic sequences (see [**1**, **2**, **12**]). Then

$$\lim_{n\mathscr{U}}\lim_{k\mathscr{V}}\|x_n + y_k\| = \inf\left\{\rho > 0;\ \sum_{i=1}^{\infty} M\left(\frac{|x(i) + y(i)|}{\rho}\right) + \lim_{n\to\infty}\sum_{i=1}^{\infty} M\left(\frac{|x'_n(i)|}{\rho}\right) + \lim_{k\to\infty}\sum_{i=1}^{\infty} M\left(\frac{|y'_k(i)|}{\rho}\right) \leqslant 1\right\},$$

which is trivially equal to $\lim_{k\mathscr{V}}\lim_{n\mathscr{U}}\|x_n + y_k\|$. We have therefore obtained the stability of l^M.

Next we want to find a representation for the class of the weakly-null types in l^M. Let τ be an element of $\mathscr{T}_0(l^M)$. We know that τ can be represented in the following way: $\tau(x) = \lim_{n\to\infty}\|x \oplus x_n\|$ where $\|x_n\| = \tau(0)$ for all $n \in \mathbb{N}$. Then

$$\tau(x) = \inf\left\{\rho > 0; M_x\left(\frac{1}{\rho}\right) + \lim_{n\to\infty} M_{x_n}\left(\frac{1}{\rho}\right) \leqslant 1\right\}.$$

If $\tau(0) \neq 0$ (otherwise τ is the trivial type, i.e. $\tau(x) = \|x\|$ for all $x \in l^M$), we have $M_{x_n}(1/\tau(0)) = 1$ for all $n \in \mathbb{N}$, and thus the functions $N_n(t)$ defined by

$$N_n(t) = M_{x_n/\tau(0)}(t) \quad (n \in \mathbb{N}, 0 \leqslant t \leqslant 1)$$

belong to the class $C_{M,1}$ (see [**9**], p. 140). Since $C_{M,1}$ is a non-void norm-compact subset of $C[0,1]$, there exists a subsequence of $(N_n)_n$ which converges uniformly to an element N_τ in the class $C_{M,1}$. Thus

$$\tau(x) = \inf\left\{\rho > 0; M_x\left(\frac{1}{\rho}\right) + N_\tau\left(\frac{\tau(0)}{\rho}\right) \leqslant 1\right\}$$

for all x in l^M. Therefore $\tau(x)$ is the unique solution of the implicit equation

$$M_x\left(\frac{1}{\tau(x)}\right)+N_\tau\left(\frac{\tau(0)}{\tau(x)}\right)=1.$$

It is clear that any other function N of $C[0,1]$ satisfying this equation for all x in l^M coincides with N_τ. Indeed, since $\tau(\lambda x)$ goes to $\tau(0)$ when λ varies from 0 to ∞, the equality

$$N_\tau\left(\frac{\tau(0)}{\tau(x)}\right)=N\left(\frac{\tau(0)}{\tau(x)}\right)$$

for all x in l^M implies that $N=N_\tau$. Therefore we may define an injective map Φ from $\mathscr{T}_0(l^M)\backslash\{0\}$ into $C_{M,1}\times(0,\infty)$ such that $\tau\mapsto(N_\tau,\tau(0))$. By easy computations we can check that $N_{\lambda\tau}=N_\tau$,

$$N_{\tau*\sigma}(t)=N_\tau\left(\frac{t\tau(0)}{\tau*\sigma(0)}\right)+N_\sigma\left(\frac{t\tau(0)}{\tau*\sigma(0)}\right)$$

and
$$1=N_\tau\left(\frac{\tau(0)}{\tau*\sigma(0)}\right)+N_\sigma\left(\frac{\tau(0)}{\tau*\sigma(0)}\right)$$

for any $0\leqslant t\leqslant 1$, λ in $\mathbb{R}$ and τ,σ weakly-null types in $\mathscr{T}_0(l^M)$. In particular note that

$$(*_i\,\alpha_i\,\tau)\,0=\|\sum_i\alpha_i\,e_i\|_{l^{N_\tau}}$$

for all $(\alpha_i)\in\mathbb{R}^{(\mathbb{N})}$.

In $C_{M,1}$, as a subset of $C[0,1]$, there are two natural topologies, the topology of pointwise convergence and the topology of uniform convergence. Since $C_{M,1}$ is a compact subset of $C[0,1]$, it is well known that both topologies coincide on it. It is easily seen that if τ_n converges to τ in the P.C.T., then $N_{\tau_n}(t)$ converges to $N_\tau(t)$, for all $t\in[0,1]$. Hence $\|N_{\tau_n}-N_\tau\|_\infty$ converges to 0 and the map Φ is continuous. In [3] it is shown that the converse is also true: if $\|N_{\tau_n}-N_\tau\|_\infty$ converges to 0, then τ_n converges to τ uniformly on bounded subsets of l^M. We summarize the preceding results in the following proposition.

PROPOSITION 2·1. *The class $\mathscr{T}_0(l^M)\backslash\{0\}$ is homeomorphic to a subset of $C_{M,1}\times(0,\infty)$; the two natural topologies on $\mathscr{T}_0(l^M)$ coincide and so this space is locally compact for the T.U.C.B.*

Now we will characterize the image of the function Φ.

PROPOSITION 2·2. *Let N be an Orlicz function (we assume $N(1)=1$). The two following conditions are equivalent:*

(i) *there is a type τ such that $N=N_\tau$;*

(ii) *for each $\epsilon>0$, l^N is $(1+\epsilon)$-isomorphic to a closed subspace of l^M.*

Proof. (i) ⇒ (ii). This part could be also deduced from a result due to Krivine and Maurey appearing in [**6**, **13**]. Here we shall give a proof for the sake of completeness, using an argument of Woo [**14**].

Let τ be a type such that $N=N_\tau$ and $\tau(0)=1$. Given $\epsilon>0$, τ can be defined by a normalized block basic sequence $(x_n)_n$ such that the corresponding functions M_{x_n} satisfy

$$\sup_{0\leqslant t\leqslant 1}|M_{x_n}(t)-N(t)|\leqslant\frac{\epsilon}{2^n}\quad\text{for all } n\in\mathbb{N}.$$

Let us show that the closed span generated by the vectors $\{x_n\}$ is $(1+\epsilon)$-isomorphic to l^N. Indeed, let x be a norm one element $x = \sum_{n=1}^{\infty} \lambda_n x_n \in l^M$. As $1 = \sum_{i=1}^{\infty} M_{x_n}(|\lambda_n|)$, we have $\sum_{i=1}^{k} N(|\lambda_n|) < 1+\epsilon$ and, by convexity,

$$\sum_{i=1}^{k} N\left(\frac{|\lambda_n|}{1+\epsilon}\right) \leqslant \frac{1}{1+\epsilon} \sum_{i=1}^{k} N(|\lambda_n|) < 1.$$

If $(e_n)_n$ is the canonical basis of l^N, we obtain $\|\sum_{n=1}^{\infty} \lambda_n e_n\|_{l^N} < 1+\epsilon$. In a similar way, the statement $\|\sum_{n=1}^{\infty} \lambda_n e_n\|_{l^N} = 1$ implies that $\|\sum_{n=1}^{\infty} \lambda_n x_n\|_{l^M} < 1+\epsilon$.

(ii) $\Rightarrow$ (i). For this implication we use the following fact.

LEMMA 2·3. *Assume that l^N $(1+\epsilon)$-embeds in l^M for any positive ϵ. Given $\epsilon > 0$ there is a normalized block basic sequence $(x_n)_n$ in l^M such that*

$$\left\|\sum_{n=1}^{\infty} \lambda_n x_n\right\|_{l^M} \leqslant \left\|\sum_{n=1}^{\infty} \lambda_n e_n\right\|_{l^N} \leqslant (1+\epsilon)\left\|\sum_{n=1}^{\infty} \lambda_n x_n\right\|_{l^M}.$$

(We will sketch the proof of the lemma later.)

We come back to the proof of the proposition. Given $\epsilon > 0$, consider the block basic sequence $(x_n)_n$ and let τ_ϵ be the type defined by $(x_n)_n$, i.e. $\tau_\epsilon(x) = \lim_{n\to\infty} \|x \oplus x_n\|$. If $(\alpha_i) \in \mathbb{R}^{\mathbb{N}}$ we have

$$\|\sum_i \alpha_i e_i\|_{l^N} \leqslant (*_i\, \alpha_i \tau_\epsilon)(0) \leqslant (1+\epsilon)\|\sum_i \alpha_i e_i\|_{l^N}.$$

As $\tau_\epsilon \to \tau$ for T.U.C.B. when ϵ goes to 0, $(*_i\, \alpha_i \tau_\epsilon)(0)$ converges to $(*_i\, \alpha_i \tau)(0) = \|\sum_i \alpha_i e_i\|_{l^N}$. Hence $l^N = l^{N_\tau}$ with the same norm. If $1 \leqslant m \leqslant k$ (and $m, k \in \mathbb{N}$), let $t_{m,k}$ be the real number such that $N(t_{m,k}) = m/k$. Then

$$N(t_{m,k}) + (k-m)\,N(t_{1,k}) = N_\tau(t_{m,k}) + (k-m)\,N_\tau(t_{1,k})$$

for all $m, k \in \mathbb{N}$ with $m \leqslant k$, which implies $N = N_\tau$.

Sketch of the proof of Lemma 2·3. If the functions $N(t)$ and t are not equivalent at zero as Orlicz functions, the result follows from the Bessaga-Pelczynski theorem, because the canonical basis of l^N is weakly null (see [**9**], propositions I·1, 12).

Suppose now that they are equivalent at zero. Let ϵ be a positive number, which will be fixed later. Let $(x_n)_n$ be a sequence in l^M $(1+\epsilon)$-equivalent to the canonical basis of l^N. We may suppose $x_n = x + x'_n$ for each $n \in \mathbb{N}$, where $(x'_n)_n$ is a block basic sequence. We have therefore

$$\|2x + x'_1 + x'_2\| \leqslant \|e_1 + e_2\|_{l^N} = \|e_1 - e_2\| \leqslant (1+\epsilon)\,\|x'_1 - x'_2\| = (1+\epsilon)\,\|x'_1 + x'_2\|.$$

By [**4**], proposition 3·1 we get that $\|2x\| < \eta$ whenever $\epsilon < \epsilon(\eta)$. Hence simple computations show that $(x'_n)_n$ is $(1+\eta')$-equivalent to the canonical basis of l^N ($\eta' \approx \epsilon + K\eta$ where K is the equivalence constant between l^N and l^1).

Remarks 2·4. (i) If $N \in C_M = \bigcap_{\lambda>0} C_{M,\lambda}$ then there exist $\tau \in \mathscr{T}_0(l^M)$ such that $N = N_\tau$. This follows from the proof of the theorems I·4, I·8 of [**9**].

(ii) If l^N is isomorphic to a subspace of l^M then there exists a type $\tau \in \mathscr{T}_0(l^M)$ such that N and N_τ are equivalent at zero Orlicz functions. (The same idea appearing in the proposition proves this assertion.)

(iii) If an Orlicz function $N(t)$ is equivalent to but different from the function t^p, we know that $N \notin C_{M,1}$ for $M(t) = t^p$. Then the norm induced by l^N is distorted in the space (see [**8**] for the distortion problem).

(iv) H. Hudzyck pointed to us that we may estimate from above and below the types, by using the Simonenko indices in Orlicz spaces. Indeed, let

$$p_M = \inf_{0<t} \frac{tM'(t)}{M(t)} \quad \text{and} \quad q_M = \sup_{0<t} \frac{tM'(t)}{M(t)}$$

the corresponding Simonenko indices (see [**10**]), where $M'(t)$ denotes the right-hand side derivative of the function M. Then, by integrating, we get that

$$\alpha^q M(t) \leqslant M(\alpha t) \leqslant \alpha^p M(t)$$

for all $t > 0$ and $0 \leqslant \alpha \leqslant 1$. If $\tau \in \mathscr{T}_0(l^M)$, $\tau(0) = 1$ and $N_\tau = N$, we can deduce the same inequalities for the function N. Since $1 \leqslant \tau(x)$, we eventually arrive at

$$(1+M_x(1))^{1/q} \leqslant \tau(x) \leqslant (1+M_x(1))^{1/p}.$$

(v) In [**13**] a representation for some special kinds of types in non-atomic rearrangement invariant function spaces is obtained by using variables in extended measure spaces (see also [**5**]).

(vi) The results stated earlier for Orlicz sequence spaces may be easily extended to the more general case of modular sequence spaces.

3. *Lorentz sequence spaces*

We can refine the general representation of the types given earlier for symmetric spaces, by using the results in [**2**] and [**12**]. In fact, if E is a Banach space with a 1-symmetric basis and not containing c_0, we may split the block basic sequence into two parts, one asymptotically converging to a vector $b \in E$ and the other one converging to 0 in the l^∞-norm (see lemma 4 in [**2**] or lemma 5 in [**12**]). Then, for $\tau \in \mathscr{T}(E)$, we have

$$\tau(x) = \lim_{n\to\infty} \|(x+a) \oplus b \oplus x_n\|$$

where 'a', 'b' are fixed vectors in E, $(x_n)_n$ is a block basic sequence with $\|x_n\| = C$ for all n, and $\lim_{n\to\infty} \|x_n\|_\infty = 0$. (As we said before, the vector a is unique, namely a is the coordinate-wise limit of the sequence defining the type.) We apply this representation for Lorentz sequence spaces and we also may obtain a more specific description of the types.

PROPOSITION 3·1. *If τ is a type on $d(\omega, p)$ then there exist two vectors a, b in $d(\omega, p)$ and a real non-negative number μ such that $\tau(x) = (\|(x+a) \oplus b\|^p + \mu^p)^{1/p}$. Moreover a and μ are unique, b is unique up to rearrangement and $a = 0$ if and only if τ is symmetric.*

Proof. By the preceding observations we may assume that

$$\tau(x) = \lim_{n\to\infty} \|(x+a) \oplus b \oplus x_n\|.$$

Suppose that x, a and b have finite support. When n is large enough we can pick a natural number k satisfying

$$\|(x+a) \oplus b \oplus x_n\|^p = \|(x+a) \oplus b\|^p + \sum_{i=1}^{\infty} x_n(i)^{*p}\, \omega_{i+k}.$$

But

$$\left|\sum_{i=1}^{\infty} x_n(i)^{*p}\,\omega_i - \sum_{i=1}^{\infty} x_n(i)^{*p}\,\omega_{i+k}\right| \leqslant \|x_n\|_\infty^p \sum_{i=1}^{k} \omega_i \to 0$$

when n goes to infinity. Hence

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_n(i)^{*p}\,\omega_i = \lim_{n\to\infty} \|x_n\|^p = \mu^p$$

and so

$$\tau(x) = (\|(x+a)\oplus b\|^p+\mu^p)^{1/p}.$$

In the general case we get the same expression by approximation. Suppose now that we have another representation, say

$$\tau(x) = (\|(x+a)\oplus b'\|^p+\mu'^p)^{1/p}.$$

If $b^*(1) \geqslant \ldots \geqslant b^*(k) > b'^*(1) \geqslant b^*(k+1)$ for some $k\in\mathbb{N}$ then taking $x+a = \sum_{i=1}^r t_i e_i$, with the t_i belonging to the open interval $(b'^*(1), b^*(k))$ and $r\in\mathbb{N}$ we would have

$$\sum_{i=1}^{k} b^*(i)^p\,\omega_i + \sum_{i=1}^{r} t_i^p\,\omega_{k+i} + \sum_{k+1}^{\infty} b^*(i)^p\,\omega_{r+i} + \mu^p = \sum_{i=1}^{r} t_i^p\,\omega_i + \sum_{k+1}^{\infty} b'^*(i)^p\,\omega_{r+i} + \mu'^p$$

which would imply $\omega_i = \omega_{k+i}$, for $1 \leqslant i \leqslant r$. Therefore we would have $\omega_1 = \omega_i$ for all $i\in\mathbb{N}$ which contradicts the assumption on the sequence $(\omega_i)_i$. Reiterating this argument we obtain $b^* = b'^*$ and so $\mu = \mu'$.

COROLLARY 3·2. (i) *For* $1 \leqslant p \leqslant \infty$ *the space* $d(\omega, p)$ *is stable.*
(ii) *A type* τ *on* $d(\omega, p)$ *is a* l^q*-type if and only if* $a = b = 0$ *and* $q = p$.

Proof. (i) Let $(x_n)_n$ and $(y_k)_k$ be two bounded sequences in $d(\omega, p)$ and let τ and σ be the types defined by $\tau(x) = \lim_{n\to\infty} \|x+x_n\|$ and $\sigma(x) = \lim_{k\to\infty} \|x+y_k\|$. These types also have the following representation

$$\tau(x) = (\|(x+a)\oplus b\|^p+\mu^p)^{1/p}$$

and

$$\sigma(x) = (\|(x+a')\oplus b'\|^p+\mu'^p)^{1/p}.$$

Then

$$\lim_{n\to\infty}\lim_{k\to\infty} \|x_n+y_k\| = \lim_{n\to\infty} \sigma(x_n) = \lim_{n\to\infty} (\|(x_n+a')\oplus b'\|^p+\mu'^p)^{1/p}$$

$$= (\|(x+a+a')\oplus b\oplus b'\|^p+\mu^p+\mu'^p)^{1/p} = \lim_{k\to\infty}\lim_{n\to\infty} \|x_n+y_k\|.$$

(ii) Recall that a type τ is a l^q-type if it is symmetric and $\lambda\tau * \beta\tau = (\lambda^q+\beta^q)^{1/q}\tau$ for all $\lambda, \mu \geqslant 0$. If $\tau(x) = (\|x\oplus b\|^p+\mu^p)^{1/p}$ we have

$$\lambda\tau * \beta\tau(x) = (\|x\oplus\lambda b\oplus\beta b\|^p+(\lambda^p+\beta^p)\,\mu^p)^{1/p}$$

and

$$(\lambda^q+\beta^q)^{1/q}\tau(x) = (\|x\oplus(\lambda^q+\beta^q)^{1/q} b\|^p+(\lambda^q+\beta^q)^{p/q}\mu^p)^{1/p}.$$

From the preceding proposition we have $\lambda b\oplus\beta b = (\lambda^q+\beta^q)^{1/q} b$ and $(\lambda^q+\beta^q)^{p/q}\mu^p = (\lambda^p+\beta^p)\,\mu^p$, so that $b = 0$ and either $p = q$ or $\mu = 0$.

In general, the class of weakly-null types in $d(\omega, p)$ is not locally compact for the T.U.C.B. In the next proposition we obtain a characterization of the Lorentz sequence spaces for which $\mathscr{T}_0(d(\omega, p))$ is locally compact for the T.U.C.B. Let us use the notation $W(n) = \sum_{i=1}^n \omega_i$.

PROPOSITION 3·3. *The following assertions are equivalent:*

(i) $\mathcal{T}_0(d(\omega,p))$ *is locally compact for the T.U.C.B.;*

(ii) $\lim_{n\to\infty} \frac{W(2n)}{W(n)} = 2$;

(iii) *non-trivial ultrapowers of the space* $d(\omega,p)$ *can be written as*

$$d(\omega,p)^{\mathbb{N}}/\mathcal{U} = d(\omega,p:I) \oplus_p L^p,$$

where I is an (uncountable) set of indices, L^p *is a (non-separable) abstract* L^p*-space and* $\oplus_p$ *means the direct sum in the* l^p*-sense* ($\|x \oplus y\|^p = \|x\|^p + \|y\|^p$).

Proof. (i) ⇒ (ii). Let $(\tau_n)_n$ be the sequence of types defined by $\tau_n(x) = \|x \oplus b_n\|$ where

$$b_n = \left(\sum_{i=1}^n \omega_i\right)^{-1/p} \sum_{i=1}^n e_i$$

(and where $(e_i)_i$ is the canonical basis). It is clear that $\|b_n\| = 1$, $\lim_{n\to\infty} \|b_n\|_\infty = 0$ and that, for all $x \in d(\omega,p)$, $\lim_{n\to\infty} \tau_n(x) = (\|x\|^p + 1)^{1/p}$. If we consider the type $\tau(x) = (\|x\|^p + 1)^{1/p}$ we have that τ_n converges to τ in P.C.T. and so it also converges in the T.U.C.B. Then $\lim_{n\to\infty} \|b_n \oplus b_n\| = 2^{1/p}$. On the other hand we have

$$\|b_n \oplus b_n\| = \left(\sum_{i=1}^n \omega_i\right)^{-1/p} \left\|\sum_{i=1}^{2n} e_i\right\|$$

and eventually $\lim_{n\to\infty} W(2n)/W(n) = 2$.

(ii) ⇒ (iii). We use the following claim which will be proved later.

Claim. Condition (ii) is also equivalent to (ii) bis: $\lim_{n\to\infty} \omega_{kn}/\omega_n = 1$ for all $k \in \mathbb{N}$.

Let $(x_n)_n$, $(y_n)_n$ be two bounded sequences in $d(\omega,p)$ such that

$$\lim_{n\to\infty} \|x_n\|_\infty = \lim_{n\to\infty} \|y_n\|_\infty = \lim_{n\to\infty} \||x_n| \wedge |y_n|\| = 0.$$

By passing to subsequences if necessary, we shall see that

$$\lim_{n\to\infty} \|x_n + y_n\|^p = \lim_{n\to\infty} \|x_n\|^p + \lim_{n\to\infty} \|y_n\|^p.$$

We may assume that x_n and y_n are disjoint vectors with bounded supports. It is well known that $d(\omega,p)$ is p-convex, i.e.

$$\|x_n + y_n\|^p \leqslant \|x_n\|^p + \|y_n\|^p.$$

On the other hand

$$\|x_n + y_n\|^p = \sum_{i=1}^\infty (x_n + y_n)^*(i)^p \omega_i = \sum_{i=1}^\infty x_n^*(i)^p \omega_{\sigma_n(i)} + \sum_{i=1}^\infty y_n^*(i)^p \omega_{\pi_n(i)},$$

where σ_n and π_n are two increasing maps from $\mathbb{N}$ into $\mathbb{N}$ with disjoint range. Let N be a fixed natural number. For each $n \in \mathbb{N}$ we define $I_N^n = \{i \geqslant 1; Ni < \sigma_n(i) - i\}$. It is clear that $y_n^*(\sigma_n(i) - i) \geqslant x_n^*(i)$, and so $x_n^*(i) \leqslant y_n^*(Ni)$ for all $i \in I_N^n$. Then

$$\sum_{i\in I_N^n} x_n^*(i)^p \omega_i \leqslant \sum_{i\in I_N^n} y_n^*(Ni)^p \omega_i \leqslant \sum_{i=1}^\infty y_n^*(Ni)^p \omega_i = \|D_N y_n\|^p,$$

where the operator D_N is defined by $D_N(y) = (y_{Ni})_i$ (see in [**9**], II·2·b·1 the definition of Boyd indices). We shall establish that

$$\lim_{n\to\infty} \|D_N y_n\| \leqslant \frac{1}{N} \lim_{n\to\infty} \|y_n\|. \qquad (*)$$

Since $\lim_{n\to\infty} \omega_{Nn}/\omega_n = 1$ and $\lim_{n\to\infty} \|y_n\|_\infty = 0$, it is easy to see that

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} y_n^*(Ni)^p \omega_i = \lim_{n\to\infty} \sum_{i=1}^{\infty} y_n^*(Ni)^p \omega_{Ni}.$$

It is also true that

$$\sum_{i=1}^{\infty} y_n^*(Ni)^p \omega_{Ni} \leqslant \sum_{i=1}^{\infty} y_n^*(Ni-1)^p \omega_{Ni-1} \leqslant \dots \leqslant \sum_{i=1}^{\infty} y_n^*(Ni-N+1)^p \omega_{Ni-N+1}.$$

Then

$$\sum_{i=1}^{\infty} y_n^*(Ni)^p \omega_{Ni} \leqslant \frac{1}{N} \sum_{j=0}^{N-1} \sum_{i=1}^{\infty} y_n^*(Ni-j)^p \omega_{Ni-j} = \frac{1}{N} \sum_{i=1}^{\infty} y_n^*(i)^p \omega_i = \frac{1}{N} \|y_n\|,$$

which proves $(*)$; here

$$\lim_{n\to\infty} \sum_{i\in I_N^n} x_n^*(i)^p \omega_i \leqslant \frac{1}{N} \lim_{n\to\infty} \|y_n\|.$$

As

$$\sum_{i\notin I_N^n} x_n^*(i)^p \omega_{\sigma_n(i)} \geqslant \sum_{i\notin I_N^n} x_n^*(i)^p \omega_{(N+1)i} = \sum_{i=1}^{\infty} x_n^*(i)^p \omega_{(N+1)i} - \sum_{i\in I_N^n} x_n^*(i)^p \omega_{(N+1)i}$$

$$\geqslant \sum_{i=1}^{\infty} x_n^*(i)^p \omega_{(N+1)i} - \sum_{i\in I_N^n} x_n^*(i)^p \omega_i$$

we have

$$\lim_{n\to\infty} \sum_{i\notin I_N^n} x_n^*(i)^p \omega_{\sigma_n(i)} \geqslant \lim_{n\to\infty} \sum_{i=1}^{\infty} x_n^*(i)^p \omega_{(N+1)i} - \frac{1}{N} \lim_{n\to\infty} \|y_n\|$$

and therefore we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_n^*(i)^p \omega_{\sigma_n(i)} \geqslant \lim_{n\to\infty} \sum_{i=1}^{\infty} x_n^*(i)^p \omega_i - \frac{1}{N} \lim_{n\to\infty} \|y_n\|.$$

An analogous inequality is true for $\lim_{n\to\infty} \sum_{i=1}^{\infty} y_n^*(i)^{p'} \omega_{\pi_n(i)}$, so we eventually have

$$\lim_{n\to\infty} \|x_n + y_n\|^p \geqslant \lim_{n\to\infty} \|x_n\|^p + \lim_{n\to\infty} \|y_n\|^p - \frac{1}{N}(\lim_{n\to\infty} \|x_n\|^p + \lim_{n\to\infty} \|y_n\|^p)$$

for each natural number N, which implies (4·2).

Now as $d(\omega, p)$ does not contain c_0 we may write (see [**3**])

$$d(\omega, p)^{\mathbb{N}}/\mathscr{U} = \tilde{E}_0 \oplus_p \tilde{E}_1,$$

where $\tilde{E}_0$, the space of elements of the ultrapower represented by a sequence of vectors with uniformly bounded length, is order isomorphic to a space $d(\omega, p; I)$ and $\tilde{E}_1$ is the space of elements represented by a sequence $(x_n)_n$ with $\lim_{n\to\infty} \|x_n\|_\infty = 0$. By the preceding, under assertion (ii) $\tilde{E}_1$ is an abstract L^p space (see [**9**], II·1·b·1).

(iii) ⇒ (ii). Assertion (iii) implies that if the sequence $(x_n)_n$ defines the type τ and $(y_n)_n$ defines the type σ and if these two sequences are disjoint then $(x_n+y_n)_n$ defines the type $\tau * \sigma$. Suppose that $(\sigma_n)_n$ is a sequence in $\mathscr{T}_0(d(\omega,p))$ which converges to σ for P.C.T. but not for T.U.C.B. Then, by passing to subsequences if necessary, there exists a bounded sequence $(x_n)_n$ in $d(\omega,p)$ so that $|\sigma_n(x_n)-\sigma(x_n)| \geqslant \epsilon > 0$ for all $n \in \mathbb{N}$. We may suppose that the sequence $(x_n)_n$ defines a type τ. By a diagonal argument, we can find a bounded sequence $(y_n)_n$ in $d(\omega,p)$ having the following properties: (*a*) $(y_n)_n$ defines the type σ, (*b*) $\lim_{n\to\infty} |\sigma_n(x_n) - \|x_n+y_n\|| = 0$, (*c*) the sequences $(x_n)_n, (y_n)_n$ are almost disjoint. Hence we have $\lim_{n\to\infty} \|x_n+y_n\| = \tau*\sigma(0)$ and then

$$\lim_{n\to\infty} \sigma_n(x_n) = \tau*\sigma(0) = \lim_{n\to\infty} \sigma(x_n)$$

which is a contradiction.

Proof of the claim. First we remark that condition (ii) on W implies in fact that

$$\lim_{n\to\infty} \frac{W(\alpha n)}{W(n)} = \alpha$$

for all $\alpha > 0$. Indeed, for every $k \in \mathbb{N}$,

$$\frac{W(2kn)-W(kn)}{W(kn)} \leqslant \frac{W((k+1)\,n)-W(n)}{W(kn)} \leqslant \frac{W(kn)}{W(kn)} = 1.$$

Then
$$\lim_{n\to\infty} \frac{W((k+1)\,n)-W(n)}{W(kn)} = 1,$$

and by induction
$$\lim_{n\to\infty} \frac{W(kn)}{W(n)} = k \quad \text{for } k\in\mathbb{N}.$$

Thus
$$\lim_{n\to\infty} \frac{W(kn)}{W(ln)} = \frac{k}{l} \quad \text{for } k,l\in\mathbb{N},$$

and so
$$\lim_{n\to\infty} \frac{W(\alpha n)}{W(n)} = \alpha \quad \text{for } \alpha\in\mathbb{Q}^+.$$

Now an approximation argument gives the result for $\alpha \in \mathbb{R}^+$.

(ii) ⇒ (ii) bis. We have $\omega_{kn}/\omega_n \leqslant 1$ and then

$$\frac{\omega_{kn}}{\omega_n} \geqslant \frac{(W((k+1)\,n)-W(kn))/n}{W(n)/n} = \frac{W((k+1)\,n)-W(kn)}{W(n)}.$$

(ii) bis ⇒ (ii). Since the sequence $(W(n))_n$ increases to ∞ we obviously have

$$\lim_{n\to\infty} \frac{W(2n)}{W(n)} = \lim_{n\to\infty} \frac{W(2n)-W(2n-2)}{W(n)-W(n-1)} = \lim_{n\to\infty} \frac{\omega_{2n}-\omega_{2n-1}}{\omega_n} = 2.$$

Remarks. (1) Note that condition (ii) is also equivalent to the following statement: there exists $k\in\mathbb{N}$ with $k \geqslant 2$ such that $\lim_{n\to\infty} \omega_{kn}/\omega_n = 1$.

(2) For each $x > 0$, set $\omega^{-1}(x) = \inf\{n ; \omega_n \leqslant x\}$. Then (ii) is also equivalent to the following assertion:

$$\text{for all } \gamma \in (0,1),\ \lim_{n\to\infty} n^{-1}\omega^{-1}(\gamma\omega_n) = \infty.$$

(This is merely a transcription of (ii) bis.)

This has to be compared to the condition for $d(\omega, p)$ to be isomorphic to an Orlicz sequence space, namely there exists $\gamma > 0$ such that

$$\sum_n \frac{1}{\omega^{-1}(\gamma\omega_n)} < \infty$$

(see [**9**], I·4·c·2).

Proposition 3·4. *There exists a Lorentz sequence space $d(\omega, p)$ for which $\mathcal{T}_0(d(\omega, p))$ is locally compact for the T.U.C.B. but which is not isomorphic to any Orlicz sequence space.*

Proof. We consider a decreasing function ω defined on (e^e, ∞) such that (a) $\omega(x \log\log x) = \gamma_0\,\omega(x)$ for some fixed $\gamma_0 \in (0,1)$ and (b) the function $h(x) = \log\omega(e^x)$ is convex.

In order to define the function h, we set $\phi(t) = t + \log\log t$ (for $t \geqslant e$), which is a continuous bijection of $[e, \infty)$ and choosing $t_0 > e$ we set $t_1 = \phi(t_0), \ldots, t_{k+1} = \phi(t_k)$ for $k \geqslant 0$ and $t_{-1} = \phi^{-1}(t_0), \ldots, t_k = \phi^{-1}(t_{k+1})$ for $k < 0$; we also set $\tau_k = k\log\gamma_0$ for $k \in \mathbb{Z}$.

Let h_N be affine on the interval $[t_{N-1}, t_N]$, with $h_N(t_N) = \tau_N, h_N(t_{N-1}) = \tau_{N-1}$. We then define h_N on $[t_{N-2}, t_{N-1}]$ by $h_N(t) = h_N(\phi(t)) - \log\gamma_0$ and recursively on the interval $(e, t_N]$. As ϕ is concave, increasing and $\phi'(t) \geqslant 1$, it is easy to see that h_N is a decreasing convex function. By using an Ascoli argument we may define

$$h(t) = \lim_{k\to\infty} h_{N_k}(t)$$

(for some subsequence of h_{N_k}) which is clearly decreasing convex and satisfies

$$h(\phi(t)) = h(t) + \log\gamma_0 \quad (t \in (e, \infty)).$$

Consider $\omega(x) = \exp h(\log x)$ for $x \in (e^e, \infty)$. We clearly have

$$\omega^{-1}(\gamma_0\,\omega(x)) = x\log\log x.$$

Set $\psi(x) = x\log\log x$. We obtain $\omega^{-1}(\gamma_0^k\,\omega(x)) = \psi^{(k)}(x)$, where $\psi^{(k)}$ is the kth iterate of ψ. By induction it is easy to see that

$$\psi^{(k)}(x) \leqslant C_k x(\log\log x)^k \leqslant D_k x\log x.$$

Thus

$$\sum_n \frac{1}{\omega^{-1}(\gamma_0^k\,\omega_n)} = \sum_n \frac{1}{\psi^{(k)}(n)} = \infty,$$

and hence $d(\omega, p)$ is not isomorphic to any Orlicz space.

On the other hand we have

$$\liminf_{n\to\infty}\left(\frac{\omega_{2^k n}}{\omega_n}\right) \geqslant \gamma_0 \quad \text{for each } k \in \mathbb{N} \text{ with } k \geqslant 1.$$

Moreover, because of the convexity of the function h,

$$\frac{\omega_{2^k n}}{\omega_n} \leqslant \left(\frac{\omega_{2^k n}}{\omega_{2^{k-1} n}}\right)^k$$

for every $k \in \mathbb{N}$, and the sequence $(\omega_{2n}/\omega_n)_n$ is non-decreasing. Hence we ha
$\lim_{n\to\infty} \omega_{2n}/\omega_n \geqslant \gamma_0^{1/k}$ for each $k \geqslant 1$, and so $\lim_{n\to\infty} \omega_{2n}/\omega_n = 1$.

4. *Dual of Lorentz sequence spaces*

Let $p > 1$ and let $d(\omega, p)^*$ be the dual space of $d(\omega, p)$. We shall show that $d(\omega, p$
satisfies the property in Proposition 3·1 for $d(\omega, p)$. If X is a sequence space,
us denote by $\mathscr{T}_{00}(X)$ the class of the types defined by a sequence $(x_n)_n$ such that $\|x_n$
tends to 0 when n increases to ∞.

PROPOSITION 4·1. *Let X be a Banach space with a 1-symmetric basis and
containing c_0. Let X_0^* be the closure in the dual X^* of the space of functionals with fir
support. If every type $\tau \in \mathscr{T}_{00}(X)$ is of the form $\tau(x) = (\|x\|^p + a^p)^{1/p}$ for some $a > 0$, th
every type $\tau \in \mathscr{T}_{00}(X_0^*)$ is of the form $\tau(x^*) = (\|x^*\|^q + a^q)^{1/q}$, where q is the conjug
exponent to p (i.e. $p^{-1} + q^{-1} = 1$).*

Proof. Let x^* be a fixed element in X^* with bounded support and let $(x_n^*)_n$ b
bounded sequence in X^* defining a type $\tau \in \mathscr{T}_{00}(X_0^*)$, with $\lim_{n\to\infty} \|x_n^*\|_\infty = 0$ a
$\lim_{n\to\infty} \|x_n^*\| = a$. We can choose $x \in X$ with $\|x\| = 1$ and $\langle x, x^*\rangle = \|x^*\|$, and, for ea
$n \geqslant 1$, $x_n \in X$ with $\|x_n\| = 1$ and $\langle x_n, x_n^*\rangle \geqslant (1-\epsilon)\|x_n^*\|$. As X does not contain
we may suppose $x_n = x_\infty + x_n' + x_n''$, where x_∞ is a fixed vector, $(x_n')_n$ a sequence
vectors with uniformly bounded supports converging coordinate-wise to 0 a
$\lim_{n\to\infty} \|x_n''\|_\infty = 0$. Then $\lim_{n\to\infty} \langle x_\infty + x_n', x^*\rangle = 0$. Thus we may suppose th
$x_n = x_n''$. Now

$$\lim_{n\to\infty} [\langle \alpha x + \beta x_n'', x^* + x_n^*\rangle - \alpha\langle x, x^*\rangle - \beta\langle x_n'', x_n^*\rangle] = 0$$

and
$$\alpha\langle x, x^*\rangle + \beta\langle x_n'', x_n^*\rangle \geqslant \alpha\|x^*\| + (1-\epsilon)\beta\|x_n^*\|.$$

Therefore

$$\liminf \langle \alpha x + \beta x_n'', x^* + x_n^*\rangle \geqslant \alpha\|x^*\| + (1-\epsilon)\beta a = [\|x^*\|^q + (1-\epsilon)^q a^q]^{1/q}$$

for an appropriate choice of $\alpha, \beta \geqslant 0$ with $\alpha^p + \beta^p \leqslant 1$. But

$$\lim_{n\to\infty} \|\alpha x^* + \beta x_n^*\| = (\alpha^p + \beta^p)^{1/p} \leqslant 1$$

by hypothesis on $\mathscr{T}_{00}(X)$, and thus we obtain

$$\liminf \|x^* + x_n^*\| \geqslant [\|x^*\|^q + (1-\epsilon)^q a^q]^{1/q}$$

for every $\epsilon > 0$, and hence $\liminf \|x^* + x_n^*\| \geqslant [\|x^*\|^q + a^q]^{1/q}$.

Conversely choose y_n in X almost norming $x^* + x_n^*$, i.e. $\|y_n\| = 1$ and

$$\langle x^* + x_n^*, y_n\rangle \geqslant (1-\epsilon)\|x + x_n^*\|.$$

We decompose $y_n = x_n + z_n$ where x_n has the same support as x^* and z_n is disj
from x^*. We have $\lim_{n\to\infty} \langle x^*, x_n\rangle = \langle x^*, x\rangle$, where x is the coordinate-wise limi
the sequence $(x_n)_n$, so we may suppose $x_n = x$. As before we may also assume t
$\lim_{n\to\infty} \|z_n\|_\infty = 0$. Then

$$\langle x^* + x_n^*, y_n\rangle = \langle x^*, x\rangle + \langle x_n^*, z_n\rangle \leqslant \|x^*\|\,\|x\| + \|x_n^*\|\,\|z_n\|$$
$$\leqslant (\|x^*\|^q + \|x_n^*\|^q)^{1/q}(\|x\|^p + \|z_n\|^p)^{1/p}.$$

As $$\lim_{n\to\infty} [\|y_n\| - (\|x\|^p + \|z_n\|^p)^{1/p}] = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n^*\| = a,$$

we obtain $$(1-\epsilon) \limsup \|x^* + x_n^*\| \leqslant (\|x^*\|^q + a^q)^{1/q}$$

for every $\epsilon > 0$ and eventually

$$\limsup \|x^* + x_n^*\| \leqslant (\|x^*\|^q + a^q)^{1/q}.$$

Corollary 4·2. *If $p > 1$, then types τ on $d(\omega, p)^*$ are of the form*

$$\tau(x^*) = (\|(x^* + a^*) \oplus b^*\|^q + \mu^q)^{1/q}$$

for $a^*, b^* \in d(\omega, p)^*$ and $\mu \in \mathbb{R}^+$.

Proof. The existence of a^*, b^*, μ follows from Proposition 3·1, the remarks preceding it and Proposition 4·1.

Remark. We do not know if b^* and μ are uniquely determined.

Corollary 4·3. (i) *For $1 < p < \infty$ the space $d(\omega, p)^*$ is stable.*
(ii) *A type on $d(\omega, p)^*$ is a l^r-type if and only if $b^* = 0$ and $r = q$.*

Proof. Assertion (i) is shown as assertion (i) of Corollary 3·2.
To prove (ii), set $b_k^* = b^* \oplus b^* \oplus \dots \oplus b^*$ (sum of k terms). We have

$$\|x_k + b_k^*\|^q + k\mu^q = \|x \oplus k^{1/r} b^*\|^q + k^{q/r} \mu^q.$$

As the left-hand side is larger than $k(\|b^*\|^q + \mu^q)$ and the right-hand one is at most of order $k^{q/r}$ we see that $q \geqslant r$. If $q > r$, letting $x = 0$ we obtain

$$\|b_k^*\| \underset{k\to\infty}{\sim} k^{1/r}(\|b^*\|^q + \mu^q)^{1/q}.$$

But we have

$$\|k^{-1/r} b_k^* \oplus j^{-1/r} b_j^*\|^q + k(k^{-1/r}\mu)^q + j(j^{-1/r}\mu)^q = \|2^{1/r} b^*\|^q + (2^{1/r}\mu)^q.$$

Now when $j \to \infty$ and then $k \to \infty$ the first member tends to $2(\|b^*\|^q + \mu^q)$ and the second equals $2^{q/r}(\|b^*\|^q + \mu^q)$ which is a contradiction. So $q = r$.

We obtain $\|b^* \oplus b^*\|^q = 2\|b^*\|^q$, which forces the sequence $\omega = (\omega_i)_i$ to be constant on the support of the non-increasing rearrangement of the vector $b^* \oplus b^*$. (Note that $b^* \oplus b^*$ is normed by a vector $x \oplus x$ of $d(\omega, p)$ satisfying $\|x \oplus x\|^p = 2\|x\|^p$.) Now the same is true for the sequence $\omega^{(N)} = (\omega_{N+i})_i$, for which we can choose a vector z^* in $d(\omega, p)^*$ so that

$$\|z^* \oplus b^*\|^q = \|z^*\|^q + \|b^*\|^q_{d(\omega^{(N)}, p)^*}$$

and the same for $\|z^* \oplus b^* \oplus b^*\|$ (choose $z = \lambda(e_1 + \dots e_N)$ with large λ). Thus if $b^* \neq 0$ then ω_n is constant, a contradiction.

Corollary 4·4. *The following assertions are equivalent:*
(i) *$\mathcal{T}_0(d(\omega, p)^*)$ is locally compact for the T.U.C.B.;*
(ii) *$d(\omega, p)^{*\mathbb{N}}/\mathcal{U} = d(\omega, p; I)^* \oplus_q L^q$;*
(iii) *$\mathcal{T}_0(d(\omega, p))$ is locally compact for the T.U.C.B.*

Proof. The implication (ii) $\Rightarrow$ (i) is analogous to the case of $d(\omega,p)$, and (i) $\Rightarrow$ (ii) is easy. (ii) implies that $d(\omega,p)^*$ is super-reflexive, so that $d(\omega,p)^{*\mathbb{N}}/\mathcal{U} = (d(\omega,p)^{\mathbb{N}}/\mathcal{U})^*$. Hence by duality

$$d(\omega,p)^{\mathbb{N}}/\mathcal{U} = d(\omega,p;I) \oplus pL^p$$

which is condition (iii) of Proposition 3·3. Thus (ii) $\Rightarrow$ (iii) and the converse is proved similarly.

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