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A NOTE ON PERTURBATION OF STABLE NORMS

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We prove that the stable norms form a nowhere dense subset of the class of all the equivalent norms in a Banach space. Moreover, we show that different stable norms in stable Banach spaces with some additional properties may be constructed.

The concept of stable Banach spaces was introduced by Krivine and Maurey [4] to extend a result of Aldous concerning L^1-subspaces [1] to a more general class of Banach spaces.

The definition of stability (see below) depends on the norm considered in the space. An equivalent non-stable norm on l^p, $1 \leqslant p < \infty$, was built in [3]. The same thing was done for Banach spaces with an unconditional basis in [6] and for L^1 in [2].

In this note we prove that the stable norms form a nowhere dense subset of the class of all the equivalent norms in a Banach space. Moreover, we show that different stable norms in stable Banach spaces with some additional properties may be constructed.

Let us recall some notations and definitions. All the vector spaces will be real. A Banach space X is stable if

$$\lim_{n,U}\lim_{m,V} \|x_n + y_m\| = \lim_{m,V}\lim_{n,U} \|x_n + y_m\|$$

whenever (x_n), (y_m) are bounded sequences in X and U, V are non-trivial ultrafilters on N.

The Banach-Mazur distance between isomorphic Banach spaces is defined by

$$\mathrm{d}(X, Y) = \inf\{\|T\| \, \|T^{-1}\| ; \ T : X \to Y \text{ is an onto isomorphism}\}.$$

It may be that $\mathrm{d}(X, Y) = 1$ but X and Y are not isometric.

We denote by M the associated quotient metric space consisting of equivalent classes, modulo distance equal to one, of all the norms on X which are equivalent to the original. It is clear that all the norms belonging to a class are either stable or non-stable.

1. PROPOSITION. *Let X be an infinite dimensional stable Banach space, the set $S = \{\|\cdot\| \in M ; \|\cdot\|$ is stable$\}$ is closed and nowhere dense in M.*

Proof. S is clearly closed. We are going to see that $\mathring{S} = \emptyset$. Indeed, let (e_n) be a basic sequence in X equivalent to the canonical l^p-basis for some p, $1 \leqslant p < \infty$ (see [4]). We define

$$N(y) = \|y\| + \sup\{|y_{2k}| + |y_{2k+1}| ; k < 1\}$$

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where $y = \Sigma y_n e_n \in Y = \overline{\text{span}}\ [e_n]$, and $||\cdot||$ denotes the original norm on $X \cdot N$ is a norm on Y equivalent to $||\cdot||$ and non-stable, since

$$\lim_{n,U} \lim_{m,V} N(e_{2n} + e_{2m+1}) \neq \lim_{m,V} \lim_{n,U} N(e_{2n} + e_{2m+1})$$

Multiplying by a constant, if necessary, we may suppose that, for some $C>0$ we have $N(y) \leqslant ||y|| \leqslant CN(y)$ when $y \in Y$. Now we define

$$|||x||| = \inf\{N(y) + ||x-y||;\ y \in Y\} \quad \text{for } x \in X.$$

The new norm $|||\cdot|||$ belongs to M and $|||y||| = N(y)$ if $y \in Y$, thus $|||\cdot|||$ is not stable. By considering the norms, $||\cdot||\cdot = ||\cdot||^{+} \in |||\cdot|||$, $\varepsilon > 0$, we arrive at the conclusion of the proposition.

In l^p, $1 \leqslant p < \infty$, there are stable norms as close to the classical l^p-norm as we want.

2. Proposition. *If* $\varepsilon > 0$, *the norm* $||\cdot||_* = ||\cdot||_p + \varepsilon||\cdot||_q$, $q>p$, *defined on* l^p *is stable and satisfies*

$$1 < d(||\cdot||_*,\ ||\cdot||_p) \leqslant 1 + \varepsilon.$$

Proof. We only have to prove $1 < d(||\cdot||_*, ||\cdot||_p)$. If $d(||\cdot||_*, ||\cdot||_p) = 1$, for every $\delta > 0$ we would find an onto isomorfism $T: l^p \to l^p$ such that

$$||x||_* \leqslant ||Tx||_p \leqslant (1+\delta)||x||_* \quad x \in l^p.$$

Let (e_n) be the canonical basis of l^p. The corresponding sequence (Te_n) would be bounded and then we could extract a subsequence (Tf_n) which converges coordinatewise. If n goes to infinity and $n < m$ are natural numbers which are quite far apart, we would have,

$$||T(f_n - f_{n+1}) + T(f_m - f_{m+1})||_p^p \approx ||T(f_n - f_{n+1})||_p^p + ||T(f_m - f_{m+1})||_p^p$$

asymptotically. Hence,

$$2^{1/p}(2^{1/p} + \varepsilon 2^{1/q}) \leqslant (1+\delta)\ (4^{1/p} + \varepsilon 4^{1/q}) \text{ for every } \delta > 0,$$

which is not possible. ∎

We say a Banach space X is of cotype q, $0<q\leqslant\infty$, if there is a constant $M>0$ so that for every finite set of vectors $\{x_n\}$ in X we have

$$M\left(\sum_1^n ||x_j||^q\right)^{1/q} \leqslant 2^{-n} \sum_{\varepsilon_j=\pm 1} \left\|\sum_1^n \varepsilon_j x_j\right\|$$

(see [5] for all the notions related to this concept).

In the next proposition we prove that in a stable Banach space having a finite cotype there are stable norms as far away from the original norm as we want.

Let $(X, \|\cdot\|)$ be a stable Banach space and (e_n) a basic sequence in X. For each $k \in N$ we may consider

$$X = \text{span } [e_1, e_2, \ldots e_n] \oplus X_k$$

Define

$$\|x\|_k = \max \{|a_i|; \quad 1 \leqslant i \leqslant k\} + \|x_k\|$$

for $x = \Sigma a_i e_i + x_k \in X$. $\|\cdot\|$ is a stable norm on X (see [4]) equivalent to $\|\cdot\|$. Then we have the following.

3. Proposition. *If X has a finite cotype then*

$$\sup \{d(\|\cdot\|_k, \|\cdot\|); \quad k \in N\} = \infty.$$

Proof. Suppose $d(\|\cdot\|_k, \|\cdot\|) < C$ for all $k \in N$. Then, there are onto isomorphisms $T_k : X \to X$ verifying

$$\|x\|_k \leqslant \|T_k x\| \leqslant C\|x\|_k \quad \text{for } x \in X.$$

Since X is of cotype q, for some $q < \infty$, there exists a constant $M > 0$ such that

$$M\left(\sum_1^k \|T_k e_j\|^q\right)^{1/q} \leqslant 2^{-k} \sum_{\varepsilon_j = \pm 1} \left\|\sum_1^k \varepsilon_j T_k(e_j)\right\| \leqslant C$$

and this contradicts that $\|T_k(e_j)\| \geqslant 1$, $1 \leqslant j \leqslant k$, for all $k \in N$. ∎

When X has no cotype, or equivalently X contains l_n^∞'s uniformly, we do not know if the above result is true. In some special cases, for instance in $l^p(l_n^\infty)$, $1 \leqslant p < \infty$, we may also construct stable norms equivalent to the original and having distance greater than 1. Indeed, if we denote by (e_n) the canonical basis of $l^p(l_n^\infty)$,

$$\|x\|' = \max \left\{|x_1|, \left\|\sum_2^\infty x_j e_j\right\|\right\}; \quad x = \Sigma x_j e_j \in l^p(l_n^\infty)$$

defines a new norm, equivalent to the canonical norm $\|\cdot\|$ on $l^p(l_n^\infty)$ and stable. We are going to show that $d(\|\cdot\|, \|\cdot\|') > 1$. Let $T : X \to X$ be an onto isomorphism such that

$$(*) \quad \|x\|' \leqslant \|Tx\| \leqslant C\|x\|'; \quad x \in l^p(l_n^\infty).$$

The sequence $(Te_{k(n)})$ where $k(n) = 1/2n(n-1) + 1$ is bounded. By passing to a suitable subsequence we may suppose it converges coordinatewise. Let (Tf_n) be this subsequence. If we apply $(*)$ to the vectors e_1, $f_n - f_{n+1}$, $e_1 + f_n - f_{n+1}$ we obtain

$$1 \leqslant \|Te_1\|, \; 2^{1/p} \leqslant \|T(f_n - f_{n+1})\| \text{ and } \|Te_1 + T(f_n - f_{n+1})\| \leqslant 2^{1/p} C$$

When n goes to infinity the vectors Te_1 and $T(f_n - f_{n+1})$ are asymptotically disjoint and then we have

$$\|Te_1 + T(f_n - f_{n+1})\|_p^p \approx \|Te_1\|_p^p + \|T(f_n - f_{n+1})\|_p^p$$

Hence, we get $C \geq (3/2)^{1/p}$ and thus $d(\|\cdot\|, \|\cdot\|') > 1$. ∎

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