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FUNCTIONAL ANALYSIS

Stability of Vector Valued Banach Sequence Spaces (*)

by

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Summary. In this paper we prove that the space of vector valued sequences $\Lambda(E_i)$, with Λ stable, symmetric, p-convex, sequence lattice and E_i stable p-Banach space is stable, in the sense of Krivine and Maurey.

The stable Banach spaces were introduced by Krivine and Maurey [4] in order to extend a theorem of Aldous concerning the subspaces of L^1 to a more general class of Banach spaces. The stable p-Banach spaces, $0 < p < 1$, were considered in [1] for the first time, and, the same result as Aldous–Krivine–Maurey was obtained there for this class of spaces.

In this paper we prove that the space of vector valued sequences $\Lambda(E_i)$, with Λ stable, symmetric, p-convex, sequence lattice and E_i, stable p-Banach spaces, is stable.

Our notation is standard and all vector spaces are real. A *p-convex norm*, $0 < p \leqslant 1$, on a vector space E is a map $x \mapsto \|x\| \in R_+$, so that,

$$\|ax\| = |a|\,\|x\|, \quad a \in R, \quad x \in E$$

$$\|x+y\|^p \leqslant \|x\|^p + \|y\|^p \quad x, y \in E$$

$$\|x\| > 0, \quad x \neq 0$$

A p-convex norm induces a locally bounded topology on E. We shall say E is a *p-Banach* if E is complete for this topology.

In the sequel $\mathscr{U}$ and $\mathscr{V}$ will denote non trivial ultrafilters on N. A separable p-Banach space E is *stable* iff

(*) To Professor L. Vigil on his 70th birthday.

$$\lim_{m\mathscr{V}} \lim_{n\mathscr{U}} \|a_n+b_m\| = \lim_{n\mathscr{U}} \lim_{m\mathscr{V}} \|a_n+b_m\|$$

whenever $(a_n)_n$, $(b_m)_m$ are bounded sequences in E. A general p-Banach space is stable if its separable subspaces are stable.

A type σ is a map $E \to R_+$ defined by

$$\sigma(x) = \lim_{n\mathscr{U}} \|x+a_n\|$$

where $(a_n)_n$ is a bounded sequence on X. Given $\sigma = \lim_{n\mathscr{U}} a_n$ and $\tau = \lim_{m\mathscr{V}} b_m$ two types on X, the type $\sigma * \tau$ is defined by

$$\sigma * \tau(x) = \lim_{n\mathscr{U}} \lim_{m\mathscr{V}} \| x+a_n+b_m \|$$

For stable E the "convolution" of the types is well defined and commutative.

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Let $(\Lambda, \|\cdot\|)$ be a complet p-convex lattice, $0 < p \leqslant 1$, with a symmetric and unconditional basis $(e_i)_1^\infty$, i.e. if $x = \sum_1^\infty x(i)\, e_i$, $y = \sum_1^\infty y(i)\, e_i$, $x, y \in \Lambda$, then,

i) $$\Big\|\sum_1^\infty x(i)\, e_i\Big\| \leqslant \Big\|\sum_1^\infty y(i)\, e_i\Big\|$$

whenever $|x(i)| \leqslant |y(i)|\ \forall i$, and $\|e_i\| = 1$, $\forall i$

ii) $$\Big\|\sum_1^\infty (|x(i)|^p + |y(i)|^p)^{1/p} e_i\Big\|^p \leqslant \Big\|\sum_1^\infty x(i)\, e_i\Big\|^p + \Big\|\sum_1^\infty y(i)\, e_i\Big\|^p$$

iii) $$\Big\|\sum_1^\infty x(i)\, e_i\Big\| = \Big\|\sum_1^\infty x(\pi(i))\, e_i\Big\|$$

for all permutation π on N.

Note that $(\Lambda, \|\cdot\|)$ is a p-Banach space and that if $(\Lambda, \|\cdot\|)$ is a Banach space with a 1-symmetric basis the above conditions are trivially verified.

Let $(E_i, \|\cdot\|)$ be a sequence of p-Banach spaces. $\Lambda(E_i)$ will denote the space of E_i-valued sequences, $f = (f(i))_i$, with $f(i) \in E_i$ and such that $\sum_1^\infty \|f(i)\|\, e_i \in \Lambda$ ($\|\cdot\|$ represents different p-norms in different spaces, but this does not origine any problem).

For $f \in \Lambda(e_i)$ we define

$$\|f\|_{\Lambda(E_i)} = \Big\|\sum_1^\infty \|f(i)\|\, e_i\Big\|_\Lambda.$$

It is easy to prove that $(\Lambda(E_i), \|\cdot\|)$ is a p-Banach space. We denote by $\bar{f} = \sum_1^\infty \|f(i)\|\, e_i$ whenever $f = (f(i))_i$, and so, $\|f\|_{\Lambda(E_i)} = \|\bar{f}\|_\Lambda$. The main result is the following

1. THEOREM. *If Λ and E_i are stable $i \in N$, then $\Lambda(E_i)$ is also stable.*

To prove the theorem, we need the notion of equisummable family and some other auxiliary results as it is done in [6] dealing with another problem.

A family $\mathscr{H} \subseteq \Lambda(E_i)$ is *equisummable* if, given any $\varepsilon > 0$, there exists a $v \in N$ so that, for each $h \in \mathscr{H}$ we have a subset $I \subset N$ with $|I| \leqslant v$ verifying $\|h - h\chi_I\| < \varepsilon$, where $h\chi_I(i) = h(i)$, if $i \in I$ and $= 0$, otherwise. Because of the symmetry and the uncondicionality of the basis (e_i) it is easy to see that also $\|h - h\chi_J\| < \varepsilon$, where $J = \{i_1, \ldots, i_v\}$ is the set formed by the indexes such that $\|h(i_1)\| \geqslant \ldots \geqslant \|h(i_v)\| \geqslant \|h(i)\|$ if $i \neq i_1, \ldots, i_v$.

In stable p-Banach spaces it is known that the types are realised uniformly on compact subsets (see proposition III.1 of [4]). In the following lemma we stablish an uniform realization on other subsets, for suitable types on Λ.

2. LEMMA. *Let $\mathscr{A}$ be a bounded family of Λ such that their elements have no more than v coordinantes different from zero. Let $(x_n)_n$ a bounded sequence of Λ and $\sigma = \lim_{n\mathscr{U}} x_n$. If $\|x_n\|_\infty = \sup_i \|x_n(i)\| \to_n 0$, when $n \to \infty$, $\sigma(y) = \lim_{n\mathscr{U}} \|y + x_n\|$ uniformly on $\mathscr{A}$.*

Proof. We prove the lemma when Λ is a Banach space, and it may obtain the p-Banach case with similar arguments to those which appear in [1], lemma 6. For each element $y \in \mathscr{A}$, let $\tilde{y}$ be any fixed vector in Λ obtained from y by a permutation of its coordinates, in such way that, $\tilde{y}(i) = 0$ if $i > v$. Because of symmetry of Λ, for each x_n, we can find suitable permations of N, $\{i_m\}_1^\infty$ and $\{k_m\}_1^\infty$, such that

$$\big|\|y+x_n\| - \|\tilde{y}+x_n\|\big| \leqslant \Big\|\sum_1^v \big(|y(i_j)+x_n(i_j)| - |y(i_j)+x_n(k_j)|\big)\, e_j + \sum_{v+1}^{2v} \big(|x_n(i_j)| - |x_n(k_j)|\big)\, e_j\Big\| \leqslant 4v\, \|x_n\|_\infty$$

Then $\sigma(y) = \lim_{n\mathscr{U}} \|y + x_n\| = \lim_{n\mathscr{U}} \|\tilde{y} + x_n\|$ uniformly on $\mathscr{A}$ by applying Proposition III.1 of [4].

For every bounded sequence $(f_n)_n$ in $\Lambda(E_i)$ we may define a type σ on Λ as $\sigma = \lim_{n\mathscr{U}} \bar{f}_n$, i.e.

$\sigma(x) = \lim_{n\mathscr{U}} \|x + \bar{f}_n\|_\Lambda$, $x \in \Lambda$, being $\mathscr{U}$ a non trivial ultrafilter on N.

3. LEMMA. *Let $\mathscr{H}$ be a bounded family of $\Lambda(E_i)$ such that their elements have no more tha v coordinates different from zero. Let $(f_n)_n$ and $(g_m)_m$ be*

two bounded sequences of $\Lambda(E_i)$, so that f'_n s and g'_m s are finitely non zero and $\lim_{n\to\infty}\|f_n\|_\infty = \lim_{m\to\infty}\|g_m\|_\infty = 0$ $(\|f\|_\infty = \sup_i \|f(i)\|)$. *If* $\sigma = \lim_{n\mathscr{U}} \bar{f}_n$, $\tau = \lim_{m\mathscr{V}} \bar{g}_m$, *then*

$$\sigma * \tau(\bar{h}) = \lim_{n\mathscr{U}} \lim_{m\mathscr{V}} \|h + f_n + g_m\|_{\Lambda(E_i)}$$

uniformly on $h \in \mathscr{H}$.

Proof. Suppose that Λ and E_i are Banach spaces. If $f_n(i) = 0$ for all $i > N_n$ and $h(i) = 0$ for $i \notin I$,

$$\left| \|h+f_n+g_m\|_{\Lambda(E_i)} - \|\bar{h}+\bar{f}_n+\bar{g}_m\|_\Lambda \right| < \left\| \sum_1^\infty (\|h(i)+f_n(i)+g_m(i)\| -$$

$$- \|h(i)\| - \|f_n(i)\| - \|g_m(i)\|)\, e_i \right\| \leqslant \left\| \sum_I (\|f_n(i)+g_m(i)\| +$$

$$+ \|f_n(i)\| + \|g_m(i)\|)\, e_i \right\| + \left\| \sum_1^{N_n} 2\, \|g_m(i)\|\, e_i \right\| \leqslant$$

$$\leqslant 2\nu\,(\|f_n\|_\infty + \|g_m\|_\infty) + 2N_n\, \|g_m\|_\infty .$$

Then, $\lim_{n\to\infty}\lim_{m\to\infty} \left| \|h+f_n+g_m\|_{\Lambda(E_i)} - \|\bar{h}+\bar{f}_n+\bar{g}_m\|_\Lambda \right| = 0$ uniformly on $\mathscr{H}$. Hence, the desired result follows from Lemma 2.

Next we must use a sort of "splitting lemma" in p-Banach spaces. We need to break each element of a bounded sequence in two disjoint parts forming two new sequences; the first one will be an equisummable sequence and the other one will have its coordinates converging to zero uniformly.

4 Lemma. *Let $(x_n)_n$ be a bounded sequence in Λ, then, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ so that $x_{n_k} = x'_{n_k} + x''_{n_k}$, $k \in N$ and*

i) *the supports of x'_{n_k} and x''_{n_k} are disjoint,*

ii) *$(x'_{n_k})_k$ is an equisummable family,*

iii) $\|x''_{n_k}\|_\infty \xrightarrow[k\to\infty]{} 0$.

Proof. In order to prove the lemma we may assume $\|x_n\| \leqslant 1$, $\forall n \in N$. For each $\varepsilon > 0$ and $j \in N$ there exists an integer M, depending upon j and ε, such that if $\|x\| \leqslant 1$ and the module of their coordinates is non increasing (i.e. $|x(1)| \geqslant |x(2)| \geqslant \ldots$), $\left|\{i;\, \varepsilon^{j+1} \leqslant |x(i)| < \varepsilon^j\}\right| \leqslant M$. Indeed, this statement holds because of c_0 is not contained in Λ and Λ is a sequence lattice. We apply this assert to the sequence $(x_n^*)_n$, where x_n^* is the non increasing rearrengement in module of the coordinates of x_n. It follows that we can choose a subsequence $(x_{n_k}^*)$ of $(x_n^*)_n$ satisfying that, for each integer $j > 0$,

$$\left| \{i;\, 2^{-j-1} \leqslant |x_{n_k}^*(i)| < 2^{-j}\} \right| = c_j$$

for all $k \in N$. Now we split $x_{n_k}^*$ in two disjoint parts $x_{n_k}^{*\prime}$ composed by

the coordinates bigger than 2^{-k} and $x_{n_k}^{*\prime\prime}$ by the rest. Obviously, $\|x_{n_k}^{*\prime\prime}\|_\infty \xrightarrow[k\to\infty]{} 0$ and, since the basis $(e_i)_i$ is boundedly complete (see [5] Theorem 1.c.10), for each $\varepsilon > 0$, there exists a $\nu \in N$ such that

$$\left\| x_{n_k}^{*\prime} - \sum_1^\nu x_{n_k}^{*\prime}(i)\, e_i \right\| < \varepsilon$$

for all $k \in N$. Hence, it follows immediately the conclusions of the lemma. #

Remarks.

1. — If the sequence $(x_n)_n$ is composed by non increasing positive vectors, i.e., $x_n(i) \geqslant x_n(i+1) \geqslant 0$ for all $n, i \in N$, it is possible to improve the conclusions of the lemma, so that, the corresponding subsequence $(x'_{n_k})_k$ is norm convergent. The arguments are essentially the same, but we must force the convergence by coordinates with a convenient speediness. (see [6]).

2. — Yves Raynaud pointed to us that by substituting the symmetry of Λ for the condition "Λ does not contain 1_n^∞ — uniformly", the conclusion of Lemma 4 holds. Since there exist Banach spaces Λ with 1-symmetric basis such that Λ does not contains c_0, but Λ contains 1_n^∞ — uniformly, the conclusion of the Lemma is not equivalent to "Λ does not contain 1_n^∞ — uniformly", even Λ is symmetric.

We return to the proof of Theorem 1. Let $(f_n)_n$ and $(g_m)_m$ be two bounded sequences of $\Lambda(E_i)$ and let $\mathscr{U}$, $\mathscr{V}$ be non trivial ultrafilters on N. We must prove that

$$\lim_{n\mathscr{U}} \lim_{m\mathscr{V}} \|f_n + g_m\|_{\Lambda(E_i)} = \lim_{m\mathscr{V}} \lim_{n\mathscr{U}} \|f_n + g_m\|_{\Lambda(E_i)}$$

We recall that it suffices to prove the above equality when f_n and g_m have finite support. In view of Lemma 4 we can get corresponding subsequences $(f_{n_k})_k$ and $(g_{m_j})_j$ of $(f_n)_n$ and $(g_m)_m$, respectively, such that

i) $f_{n_k} = f'_{n_k} + f''_{n_k}$ with $\operatorname{supp} f'_{n_k} \cap \operatorname{supp} f''_{n_k} = \emptyset$, $k \in N$.

ii) $\|f''_{n_k}\|_\infty \to 0$, if $k \to \infty$,

iii) $\{f'_{n_k};\, k \in N\}$ is a bounded equisummable family and analogous conditions for $(g_{m_j})_j$.

Hence, by passing to subsequences, we have

$$\lim_{n\mathscr{U}} \lim_{m\mathscr{V}} \|f_n + g_m\|_{\Lambda(E_i)} = \lim_{k\mathscr{U}} \lim_{j\mathscr{V}} \|(f'_{n_k} + g'_{m_j}) + (f''_{n_k} + g''_{m_j})\|_{\Lambda(E_i)}$$

and

$$\lim_{m\mathscr{V}} \lim_{n\mathscr{U}} \|f_n + g_m\|_{\Lambda(E_i)} = \lim_{j\mathscr{V}} \lim_{k\mathscr{U}} \|(f'_{n_k} + g'_{m_j}) + (f''_{n_k} + g''_{m_j})\|_{\Lambda(E_i)}.$$

Since $\mathscr{H}_1 = \{f'_{n_k};\, k \in N\}$ and $\mathscr{H}_2 = \{g'_{m_j};\, j \in N\}$ are equisummable families, we can suppose that there is a $\nu \in N$ such that

$$\left|\{i; f'_{n_k}(i) \neq 0\}\right| \leqslant \nu \quad \text{and} \quad \left|\{i; g'_{m_j}(i) \neq 0\}\right| \leqslant \nu$$

for all $k, j \in N$.

Let $\sigma = \lim_{k\mathcal{U}} \overline{f''_{n_k}}$ and $\tau = \lim_{j\mathcal{V}} \overline{g''_{m_j}}$ be the corresponding types on Λ, since $\mathcal{H} = \{f'_{n_k}+g'_{m_j}; k, j \in N\}$ is a bounded equisummable family in $\Lambda(E_i)$, we get

$$\lim_{n\mathcal{U}} \lim_{m\mathcal{V}} \|f_n+g_m\|_{\Lambda(E_i)} = \lim_{k\mathcal{U}} \lim_{j\mathcal{V}} \sigma * \tau(\overline{f'_{n_k}+g'_{m_j}})$$

and

$$\lim_{m\mathcal{V}} \lim_{n\mathcal{U}} \|f_n+g_m\|_{\Lambda(E_i)} = \lim_{j\mathcal{V}} \lim_{k\mathcal{U}} \tau * \sigma\,(\overline{f'_{n_k}+g'_{n_k}}).$$

In order to continue the proof we need the following lemma

5. Lemma. *Let $(f_n)_n$ be a sequence in $\Lambda(E_i)$ such that $\left|\{i; f_n(i) \neq 0\}\right| \leqslant \nu$ for all $n \in N$, then, there exist a $\lambda \in N$ and a subsequence $(f'_n)_n$ of $(f_n)_n$ such that*

i) $f'_n = f'^{(1)}_n + f'^{(2)}_n \qquad n \in N.$

ii) *$f'^{(1)}_n$ belongs to span $[e_1, \ldots, e_\lambda]$, for all $n \in N$.*

iii) *The sequence $(f'^{(2)}_n)_n$ is a block sequence of $(e_n)^{\infty}_{\lambda+1}$.*

Proof. Let us first considerer that $\{n; f_n(i) \neq 0\}$ is finite for all $i \in N$. Choose $\lambda = 0$, $f'_1 = f_1$ and put $f'^{(2)}_1 = f'_1$. Pick next f'_2, so that, $\max\{i; f'_1(i) \neq 0\} < \min\{i; f'_2(i) \neq 0\}$ and put $f'^{(2)}_2 = f_2$. We continue the inductive construction of (f'_n) in an obvious way.

Now asumme that there is a first index i such that $f_n(i) \neq 0$ for an infinity of n's. Pick the corresponding subsequence, namely again $(f_n)_n$, such that $f_n(i) \neq 0$. $\forall n$. We have two possibilities for this subsequence: either $\{n; f_n(j) \neq 0\}$ is finite for all $j > i$ or not. In the first case, we may return to the preceeding solved situation, by putting $\lambda = i$, $f^{(1)}_n = f_n \cdot \chi_{\{i\}}$ and $f^{(2)}_n = f_n - f^{(1)}_n$. Otherwise, we select the corresponding subsequence and so on, (the existence of λ is insured because $\left|\{i; f_n(i) \neq 0\}\right| < \nu$, $\forall n \in N$). #

We turn again to the proof of the theorem. By applying the preceding lemma to the sequences (f'_{n_k}) and (g'_{m_j}) we get further subsequences, denoted by $(f'_{n_k})_k$ and $(g'_{n_j})_j$, and $\lambda \in N$ such that

$$\overline{f'_{n_k}+g'_{m_j}} = \overline{f'^{(1)}_{n_k}+g'^{(1)}_{m_j}} + \overline{f'^{(2)}_{n_k}+g'^{(2)}_{m_j}}$$

$$\overline{f'^{(1)}_{n_k}+g'^{(1)}_{m_j}} \in \text{span } [e_1, \ldots, e_\lambda] \text{ for all } k, j \in N$$

$(f'^{(2)}_{n_k})$ $(f'^{(2)}_{n_k})_k$ and $(g'^{(2)}_{m_j})_j$ are two block sequences of $(e_n)^{\infty}_{\lambda+1}$.

Then, there exist $\lim_{k\mathcal{U}} \lim_{j\mathcal{V}} \overline{f'^{(1)}_{n_k}+g'^{(1)}_{m_j}}$ and $\lim_{j\mathcal{V}} \lim_{k\mathcal{U}} \overline{f'^{(1)}_{n_k}+g'^{(1)}_{m_j}}$. Since the E_i

$$\lim_{k\mathcal{U}} \lim_{j\mathcal{V}} \overline{f'^{(1)}_{m_k}+g'^{(1)}_{m_j}} = \lim_{j\mathcal{V}} \lim_{k\mathcal{U}} \overline{f'^{(1)}_{n_k}+g'^{(1)}_{m_j}}$$

Moreover, if k and j are quite far apart

$$\overline{f'^{(2)}_{n_k}+g'^{(2)}_{m_j}} = \overline{f'^{(2)}_{n_j}} + \overline{g'^{(2)}_{m_j}}$$

and then, by applying the stability of Λ, the type

$$\lim_{k\mathcal{U}} \lim_{j\mathcal{V}} \overline{f'^{(2)}_{n_k}+f'^{(2)}_{m_j}} = (\lim_{k\mathcal{U}} \overline{f'^{(2)}_{n_k}}) * (\lim_{j\mathcal{V}} \overline{g'^{(2)}_{m_j}}) = \lim_{j\mathcal{V}} \lim_{k\mathcal{U}} \overline{f'^{(2)}_{n_k}+g'^{(2)}_{m_j}}$$

Hence

$$\lim_{n\mathcal{U}} \lim_{m} \|f_n+g_m\|_{\Lambda(E_i)} = \sigma * \tau * (\lim_{k\mathcal{U}} \lim_{j\mathcal{V}} \overline{f'^{(1)}_{n_k}+g'^{(1)}_{m_j}}) *$$

$$* (\lim_{k\mathcal{U}} f'^{(2)}_{n_k}) * (\lim_{j\mathcal{V}} g'^{(2)}_{m_j}) = \lim_{m\mathcal{V}} \lim_{n\mathcal{U}} \|f_n+g_n\|_{\Lambda(E_i)} \qquad \#$$

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Х. Бастеро, Х. М. Мира, **Устойчивость пространств векторных значимых последовательностей**

В настоящей статье доказывается, что пространство векторных значимых последовательностей $\Lambda(E_i)$, с Λ устойчивыми симметричными p-выпуклыми решетками последовательности и E_i устойчивыми p-банаховыми пространствами, устойчиво в смысле Кривина и Морея.