PUBLICACIONES DEL DEPARTAMENTO DE ANALISIS MATEMATICO

COLLOQUIUM DEL DEPARTAMENTO DE
ANALISIS MATEMATICO.
CURSO 1993-94

SECCION 1 Número 32

FACULTAD DE MATEMATICAS
UNIVERSIDAD COMPLUTENSE

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DEPARTAMENTO DE ANALISIS MATEMATICO
U.C.M.
SEMINARIOS

EMBEDDING ℓ_∞-CUBES IN THE ORBIT OF AN ELEMENT IN COMMUTATIVE AND NON COMMUTATIVE ℓ_p^n-SPACES

by

Jesús Bastero, Ana Peña and Gideon Schechtman

Embedding ℓ_∞-cubes in the orbit of an element in commutative and non conmutative ℓ_p^n-spaces

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Jesús Bastero*, Ana Peña** and Gideon Schechtman

A ℓ_∞-cube is a finite discrete metric space, i.e., a metric space (M, d) with cardinality N and such that $d(x, y) = 1$ if $x \neq y$, and $= 0$, otherwise. We say that the ℓ_∞-cube (M, d) $(1+\varepsilon)$-embeds into the normed space $(E, \|.\|)$ if there are N points $\{x_1, \ldots, x_N\}$ in E such that

$$1 - \varepsilon \leq \|x_i - x_j\| \leq 1 + \varepsilon$$

for all $i \neq j$.

There are several well known results about the $(1 + \varepsilon)$-embedings of ℓ_∞-cubes in finite dimensional normed spaces. We report here some of them.

In [B-B-K] the following result is proven:

"There exists a numerical constant $C > 0$ such that the ℓ_∞-cube of cardinality N is $(1 + \varepsilon)$ embedded in any finite dimensional 1-subsymetric space E, provided that $\dim E > \frac{C}{\varepsilon^2} \log N$ " (the result is the best possible, asymptotically in $\dim E$).

Some extensions of this result appear in [B-B], where sharp estimates are given for the case of the 1-unconditional space $\ell_p^n(\ell_q^m)$ $1 \leq p, q < \infty$.

In [B-P-S] the authors extended this result for the operator ideals C_E^n, where E is a 1-symmetric n-dimensional space. Actually they proved that

"Given $0 < \varepsilon < 1$, there exits a constant $C(\varepsilon) > 0$ such that for all N satisfying $\log N \leq C(\varepsilon) n^2$ we can find N points $T_1, ..., T_N$ in C_E^n, satisfying $1 - \varepsilon \leq \|T_i - T_j\|_{C_E^n} \leq 1$, for all $i \neq j$".

In the papers concerning the commutative case ([B-B] and [B-B-K]) the authors find the points by using random embeddings defined by mean of vector valued Rademacher functions. In the non conmutative case ([B-P-S]) the points are obtained among the orthogonal projections associted to subspaces randomly choosen in the corresponding Grassmanian manifold.

Our goal is this paper is to obtain the results coresponding to the cases ℓ_p^n and C_p^n by considering a group of isometries acting on ℓ_p^n, respectively C_p^n. We achieve the

* Partially supported by CICYT PS90-0120

** Partially supported by a DGA grant and by CICYT PS90-0120

known results included in [B-B-K] and [B-P-S] with some additional information. In fact, with the use of isometries we can $(1+\varepsilon)$-embed a ℓ_∞-cube with cardinality N into the orbit of any element x (N depending on x), i.e. into the set of Tx's, when T runs over all the isometries considered.

As usual ℓ_p^n will denote the space $\mathbb{R}^n$ with the norm $\|x\|_p = \left(\sum_1^n |x_i|^p\right)^{1/p}$, $x \in \mathbb{R}^n$; e_i, $1 \le i \le n$, will be the canonical basis in $\mathbb{R}^n$.

The class C_p^n $(1 \le p \le \infty)$ is the space of all linear operators from $\mathbb{R}^n$ into $\mathbb{R}^n$, endowed with the norm $\|A\|_{C_p^n} = \left(\sum_{i=1}^n |s_i(A)|^p\right)^{1/p}$, where $\{s_i(A)\}_{i=1}^n$ are defined as the singular values of A, that is, the eigenvalues of $\sqrt{A^*A}$. It's well known that $\|.\|_{C_\infty^n}$ coincides with the operator norm, denoted by $\|.\|_\infty$, $\|.\|_{C_2^n}$ with the Hilbert-Schmizt norm and $\|.\|_{C_1^n}$ with the trace class norm. (see [G-K] for further information about Schatten classes).

The main ingredient in our proof is deviation inequalities associated with normal Levy families.

A family $(X_n, d_n, \mathbb{P}_n)_{n\in\mathbb{N}}$ of probability metric spaces is called a normal Levy family with constants c_1, c_2, if given any continuous function $f : X_n \to \mathbb{R}$ with modulus of continuity $w_f(.)$ and median M_f we have

$$\mathbb{P}\left(|f - M_f| > w_f(\varepsilon)\right) < c_1 \exp(-c_2\varepsilon^2 n)$$

for all $\varepsilon > 0$.

The normal Levy families we use in this paper are:

a) $(\prod_n \times \{-1,1\}^n, d_n, \mathbb{P}_n)_n$, where $\prod_n$ is the group of all permutations of the set $\{1, 2...n\}$, $\mathbb{P}_n$ is the uniform probability on the product space and

$$d_n((\pi,\varepsilon),(\pi',\varepsilon')) = \frac{1}{2n}\|\varepsilon - \varepsilon'\|_1 + \frac{1}{n}|\{i; \pi(i) \ne \pi'(i)\}|$$

$\pi, \pi' \in \prod_n$ and $\varepsilon, \varepsilon' \in \{-1,1\}^n$, ($|.|$ denotes the cardinality of the corresponding set).

b) Let O_n be the ortogonal group. Consider the normal subgroup $SO_n = \{T \in O_n; \det(T) = 1\}$ provided with its normalized Haar measure $\mathbb{P}_n$ and with the natural metric $d_n(T,S) = \left(\sum_{i=1}^{n-1} \|a^i - b^i\|_2^2\right)^{1/2}$, if $T = [a^1, ..., a^n]$ and $S = [b^1, ..., b^n]$ are two elements in SO_n. The distance d_n is left-invariant but not right-invariant under the action of SO_n on itself. The family $(SO_n, d_n, \mathbb{P}_n)_n$ is a normal Levy family (see [G-M], [M-S]). It is perhaps worth pointing out at this point that the corresponding family $(O_n, \|\cdot\|_{C_2^n}, \mathbb{P}_n)_n$ is not a normal Levy family.

Each element $(\pi,\varepsilon) \in \prod_n \times \{-1,1\}^n$ defines an isometry $T = T_{\pi,\varepsilon}$ on ℓ_p^n by $Tx = \sum_{i=1}^n x_{\pi(k)}\varepsilon_k e_k$, $x \in \ell_p^n$.

In the same manner we will consider SO_n as a group of isometries which acts on C_p^n by $U(A) = UA$, if $A \in C_p^n$ and $U \in SO_n$.

We begin with some preparatory lemmas ($I\!E$ will denote the expectation in a probability space).

Lemma 1. *Let A be a fixed element in C_p^n, $n \geq 1$, $1 \leq p < \infty$. For this element let ψ_A be the function on SO_n defined by $\psi_A(U) = \|UA - A\|_{C_p^n}$ then*

i) $I\!E\psi_A \geq \|A\|_{C_p^n}$.

ii) The modulus of continuity of ψ_A verifies:

$$w_{\psi_A}(\delta) \leq \begin{cases} 2^{1/2} n^{1/p-1/2} \|A\|_\infty \delta, & \text{if } 1 \leq p \leq 2\,; \\ 2^{1-1/p} \|A\|_\infty \delta^{2/p}, & \text{if } 2 < p < \infty, \end{cases}$$

Proof: i) If n is an even natural number $-U \in SO_n$ for any $U \in SO_n$. Hence

$$\begin{aligned} I\!E\psi_A &= \frac{1}{2}\left[\int_{SO_n} \|UA - A\|_{C_p^n} dI\!P_n + \int_{SO_n} \|UA + A\|_{C_p^n} dI\!P_n\right] \\ &\geq \frac{1}{2}\int_{SO_n} 2\|A\|_{C_p^n} dI\!P_n = \|A\|_{C_p^n}. \end{aligned}$$

If n is odd, for every $i = 1, \dots, n$ we consider the following $n \times n$ matrix

$$P_i = [-e_1, -e_2, \dots, e_i, \dots, -e_n] \in SO_n,$$

then

$$\begin{aligned} I\!E\psi_A &= \frac{1}{2n-2}\left[\sum_{i=1}^{n} \int_{SO_n} \|P_i UA - A\|_{C_p^n} dI\!P_n + (n-2)\int_{SO_n} \|UA - A\|_{C_p^n} dI\!P_n\right] \\ &\geq \frac{1}{2n-2}\int_{SO_n} \left\|\sum_{i=1}^{n} P_i UA + (n-2)UA - (2n-2)A\right\|_{C_p^n} dI\!P_n \\ &= \|A\|_{C_p^n} \end{aligned}$$

ii) Let U, V any two elements in SO_n.

If $1 \leq p \leq 2$ then

$$\begin{aligned} |\psi_A(U) - \psi_A(V)| &\leq \|UA - VA\|_{C_p^n} \leq n^{1/p-1/2} \|(U - V)A\|_{C_2^n} \\ &\leq n^{1/p-1/2} \|A\|_\infty \|U - V\|_{C_2^n}. \end{aligned}$$

If $2 < p < \infty$, then,

$$\begin{aligned} |\psi_A(U) - \psi_A(V)| &\leq \|UA - VA\|_{C_p^n} \leq \|A\|_\infty \|U - V\|_{C_p^n} \\ &\leq 2^{1-2/p} \|A\|_\infty \|U - V\|_{C_2^n}^{2/p} \end{aligned}$$

since $s_i(U - V) \leq 2$, $i = 1, 2, \dots, n$. The rest of the proof is an easy consequence of the following:

Lemma 2. *If I is de identity $(n \times n)$-matrix then $\|I-P\|_{C_2^n} \leq \sqrt{2} d_n(I,P)$ for any P in SO_n.*

Proof: Let $P \in SO_n$ with entries (P_{ij}) $1 \leq i,j \leq n$. It is quite clear that

$$\|I-P\|^2_{C_2^n} = 2\sum_{i=1}^{n}(1-P_{ii})$$

and

$$d_n(I,P)^2 = 2\sum_{i=1}^{n-1}(1-P_{ii}).$$

Let $Q = [e_1, ..., e_{n-1}, -e_n]$. Since PQ is an ortogonal matrix and $\det(PQ) = -1$, then $\mathrm{tr}(PQ) \leq n-2$ and so $\mathrm{tr}(P) - 2P_{nn} \leq n-2$. Hence the result holds.

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Note 1. If we applied the preceding lemma for ψ_A, then, for any $t > 0$:

$$\mathbb{P}_n\{U \in SO_n; |\psi_A(U) - M_{\psi_A}| > t\} \leq c_1 \exp\{-c_2 \frac{t^2}{\|A\|^2_\infty} n^{2-2/p}\}$$

if $1 \leq p \leq 2$, and

$$\mathbb{P}_n\{U \in SO_n; |\psi_A(U) - M_{\psi_A}| > t\} \leq c_1 \exp\{-c_2 \frac{t^p}{\|A\|^p_\infty} n\}$$

for $2 < p < \infty$ (the constants c_1, c_2 may represent different values in the different occurences).

Next, we are going to study a similar lemma for the case of ℓ_p^n, $1 \leq p < \infty$. Let $G = \prod_n \times \{-1,1\}^n$, with the probability $\mathbb{P}_n$ and the distance d_n given before.

Lemma 3. *Let $x = (x_1, ..., x_n)$ be a fixed element in ℓ_p^n. Let ψ_x a function defined on G by $\psi_x(T) = \|Tx - x\|_p$ for any $T \in G$. Then*

i) $\mathbb{E}\psi_x \geq \|x\|_p$.

ii) The modulus of continuity verifies:

$$w_{\psi_x}(\delta) \leq 2n^{1/p}\|x\|_\infty \delta^{1/p}$$

Proof: i) By definition

$$\mathbb{E}\psi_x = \frac{1}{2^n}\frac{1}{n!}\sum_{\varepsilon}\sum_{\pi}\|\sum_{i=1}^{n}(x_{\pi(i)}\varepsilon_i - x_i)e_i\|_p.$$

For every $\varepsilon = (\varepsilon_1, ..., \varepsilon_n) \in \{-1,1\}^n$, we consider $\varepsilon' = -\varepsilon$ and by associating these two elements we obtain that

$$\begin{aligned} \mathbb{E}\psi_x &= \frac{1}{2^{n+1}}\frac{1}{n!}\sum_{\pi}\sum_{\varepsilon} \|T_{\pi\varepsilon}x - x\|_p + \|T_{\pi\varepsilon}x + x\|_p \\ &\geq \frac{1}{2^{n+1}}\frac{1}{n!}\sum_{\pi}\sum_{\varepsilon} 2\|x\|_p = \|x\|_p. \end{aligned}$$

ii) Let $S = S_{\pi,\varepsilon}$ and $T = T_{\sigma,\mu} \in G$ such that $d_n(S,T) \leq \delta$. Denote by I, J the following sets: $I = \{k; \pi(k) \neq \sigma(k)\}$ and $J = \{k; \varepsilon_k \neq \mu_k\}$. Hence,

$$\begin{aligned} |\psi_x(T) - \psi_x(S)| \leq \|Tx - Sx\|_p &= \left\| \sum_{I\cup J} (x_{\pi(i)}\varepsilon_i - x_{\sigma(i)}\mu_i)e_i \right\|_p \\ &\leq 2\|x\|_\infty |I \cup J|^{1/p} \leq 2n^{1/p}\|x\|_\infty d_n(T,S)^{1/p} \end{aligned}$$

where we denote by $|I \cup J|$ the cardinal of the set $I \cup J$, and the result follows inmediately.

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Note 2. With the same notation as before

$$\mathbb{P}_n\{T \in G; |\psi_x(T) - M_{\psi_x}\| > t\} \leq c_1 \exp\{-c_2 \frac{t^{2p}}{2^{2p}n\|x\|_\infty^{2p}}\}.$$

It is well known that similar expressions as those appearing in the notes 1 and 2 with the expectation instead of the median of corresponding functions can be obtained. We are now ready to state and prove the theorem.

Theorem 4. *There exits a constant $C > 0$ such that, for any $\varepsilon > 0$, $A \in C_p^n$ and $x \in \ell_p^n$, we can find a subset of N points $\{U_1, ..., U_N\}$ in SO_n and a subset of M points $\{T_1, ..., T_M\}$ in G verifying*

i) $1 - \varepsilon \leq \|U_iA - U_jA\|_{C_p^n} \leq 1 + \varepsilon$, for all $i \neq j$, provided that

$$n^{2-2/p}\|A\|_{C_p^n}^2 \|A\|_\infty^{-2} > \frac{C}{\varepsilon^2}\log N,$$

if $1 \leq p \leq 2$ and

$$n\|A\|_{C_p^n}^p \|A\|_\infty^{-p} > C\frac{2^p}{\varepsilon^p}\log N,$$

if $2 \leq p \leq \infty$

ii) $1-\varepsilon \le \|T_i x - T_j x\|_{l_p^n} \le 1+\varepsilon$ *for all* $i \ne j$, *provided that*

$$\|x\|_p^{-2p}\|x\|_\infty^{2p} n^{-1} > C\frac{2^{2p}}{\varepsilon^{2p}}\log M.$$

Proof: Let's give the proof only for the C_p^n-case (the other is similar). We define the function $F_A : SO_n \times SO_n \to I\!R$ by $F_A(U,V) = \|UA - VA\|_{C_p^n}$ Let N be a natural number and consider now the set

$$\Delta = \{(U_1,\ldots,U_N) \in SO_n \times \ldots \times SO_n; \\ |F_A(U_i,U_j) - I\!E F_A| \le \varepsilon I\!E F_A,\ \ 1 \le i,j \le N, i \ne j\}.$$

The problem is to find out the biggest number N such that $\Delta \ne \emptyset$. For that, we compute the probability of Δ^c (the complementary set). Since

$$I\!P_n \times \ldots \times I\!P_n(\Delta^c) \le I\!P_n \times I\!P_n \left\{ \bigcup_{i,j\in\{1,\ldots,N\}, i\ne j} \Delta_{i,j}^c \right\}$$

where $\Delta_{i,j} = \{(U_i,U_j) \in SO_n \times SO_n; |F_A(U_i,U_j) - I\!E F_A| \le \varepsilon I\!E F_A\}\}$ we therefore obtain

$$I\!P_n \times \ldots \times I\!P_n(\Delta^c) \le \binom{N}{2} I\!P_n \times I\!P_n(\Delta_{i,j}^c) \\ = \binom{N}{2} I\!P_n \times I\!P_n\{(U,V) \in SO_n \times SO_n; \\ |F_A(U,V) - I\!E F_A| > \varepsilon I\!E F_A\}$$

It is very easy to check that the function F_A and the function ψ_A defined in lemma 1 have the same distribution of probability. Thus we obtain

$$I\!P_n \times \ldots \times I\!P_n(\Delta^c) \le \binom{N}{2} \begin{cases} c_1\exp\{-c_2\varepsilon^2 n^{2-2/p}\frac{\|A\|_{C_p^n}^2}{\|A\|_\infty^2}\}, & \text{if } 1 \le p \le 2\ ; \\ c_1\exp\{-c_2 2^{-p}\varepsilon^p n \frac{\|A\|_{C_p^n}^p}{\|A\|_\infty^p}\}, & \text{if } 2 < p < \infty. \end{cases}$$

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Remark. It is clear that the optimal $(1+\varepsilon)$-embedding of a ℓ^∞-cube in C_p^n or ℓ_p^n is achieved for $A = I$ or $x = (1,\ldots,1)$ respectively, so we obtain the results appearing in [B-B-K] and [B-P-S].

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