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Embedding ℓ_∞^n-Cubes in Low Dimensional Schatten Classes

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We prove that for some $\alpha = \alpha(\varepsilon) > 0$, the $\ell_\infty^{\alpha n^2}$-cube $(1+\epsilon)$-embeds in the Schatten class C_E^n, for every 1-symmetric n-dimensional normed space E.

This paper deals with an instance of the general problem of Lipschitz embeddings of finite metric spaces in low dimensional normed spaces. We begin by recalling some definitions. Let (M,d) be a finite metric space and $(E, \|\cdot\|)$ a finite dimensional real normed space. Given $\varepsilon > 0$, we say that the metric space (M,d) $(1+\varepsilon)$-embeds into $(E,\|\cdot\|)$ if there is a one-to-one map f from M into E such that

$$(1-\varepsilon)d(x,y) \le \|f(x)-f(y)\| \le (1+\varepsilon)d(x,y)$$

for all $x, y \in M$.

We report here some progress on the following problem ([B-B-K] and [B-B]): Given a finite dimensional normed space E, what is the biggest n such that the ℓ_∞^n-cube is $(1+\varepsilon)$-embedded in E.

The ℓ_p^n-cubes were introduced in [B-M-W] where some embedding relations among the different ℓ_p^n-cubes are given. The ℓ_∞^n-cube is the metric space $(\{-1,1\}^n, d_\infty)$ where $d_\infty(\varepsilon,\varepsilon') = \max_{1\le i\le n}|\varepsilon_i - \varepsilon_i'|$, for any pair of elements $\varepsilon, \varepsilon'$ in $\{-1,1\}^n$.

In [B-B-K] the following result is proved:

"There exists a numerical constant $C > 0$ such that the ℓ_∞^n-cube is $(1+\varepsilon)$ embedded in any finite dimensional 1-subsymmetric space E, provided that $\dim E > \frac{C}{\varepsilon^2} n$" (the result is the best possible, asymptotically in n).

Some extensions of this result appear in [B-B], where sharp estimates are given for the case of the 1-unconditional space $\ell_p^n(\ell_q^m)$ $1 \le p, q < \infty$.

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In this paper we will study $(1+\varepsilon)$ embeddings of the ℓ_∞^n-cube in the unitary ideal C_E^n.

We are going to recall some definitions and introduce the necessary notations. Let E be a n-dimensional real normed space with a 1-symmetric basis $\{e_i\}$. We denote $\lambda(k) = \|\sum_{i=1}^k e_i\|_E$, $1 \le k \le n$.

Let $\mathcal{L}(\mathbb{R}^n)$ be the space of all linear operators on $\mathbb{R}^n$. C_E^n is the space $\mathcal{L}(\mathbb{R}^n)$ endowed with the norm $\|T\|_{C_E^n} = \|\sum_{i=1}^n s_i(u)e_i\|_E$, for $T \in C_E^n$, where $\{s_i(u)\}$ are defined as the singular values of T, that is, the eigenvalues of $\sqrt{T^*T}$. Obviously, $(C_E^n, \|\cdot\|_{C_E^n})$ is a unitary ideal. The most important examples of unitary ideals are the ideals C_p^n induced by ℓ_p^n, $1 \le p \le \infty$, the so-called Schatten classes, which can be viewed as the non-commutative version of ℓ_p^n. It is well known that $\|\cdot\|_{C_\infty^n}$ coincides with the operator norm, denoted by $\|\cdot\|_\infty$, $\|\cdot\|_{C_2^n}$ with the Hilbert-Schmidt norm and $\|\cdot\|_{C_1^n}$ with the trace class norm (see [G-K] for further information about Schatten classes).

We will denote by $\langle\cdot\rangle$ the Euclidean scalar product in $\mathbb{R}^n$, by $\|\cdot\|_2$ the Euclidean norm and by S^{n-1} the unit sphere. If X is a subspace of a Hilbert space H, we denote by P_X the orthogonal projection onto X. It is well known that $\|P_X\|_{C_E^n} = \lambda(\dim X)$. Given two subspaces X, Y of a Hilbert space H and $0 < \varepsilon < 1$ we say that X and Y are *ε-orthogonal* if $|\langle x, y\rangle| \le \varepsilon\|x\|\|y\|$, for all $x \in X$ and for all $y \in Y$.

The Grassman manifolds $G_{n,k}$, $1 \le k \le n$, consists of all k-dimensional subspaces of $\mathbb{R}^n$, with the metric being the Hausdorff distance between the unit balls of the two subspaces. Let $\mathbb{P}_{n,k}$ be the Haar measure on $G_{n,k}$, the only normalized measure which is invariant under the action of the orthogonal group $O(n)$.

The theorem we are going to prove here is the following:

Theorem 1. *Given $0 < \varepsilon < 1$, there exits a constant $C(\varepsilon) > 0$ such that for all N satisfying $\log N \le C(\varepsilon)n^2$ we can find N points $T_1, \dots, T_N$ in C_E^n, satisfying $1 - \varepsilon \le \|T_i - T_j\|_{C_E^n} \le 1$, for all $i \ne j$.*

We are going to look for the points T_i's in the set of orthogonal projections associated with a family of k-dimensional subspaces of $\mathbb{R}^n$ (for a suitable k), having large subspaces which are pairwise ε-orthogonal. We introduce some more notations.

Let $k \in \mathbb{N}$ and $0 < \varepsilon < 1$ such that $2k \le n$. Let $\mathcal{A}(\varepsilon, k)$ be the set of all the couples $(X_1, X_2) \in G_{n,k} \times G_{n,k}$, for which there exist $Y_i \subseteq X_i$, $i = 1, 2$, satisfying:

i) $\dim\ Y_i > (1-\varepsilon)k$
ii) X_1 and Y_2 are ε-orthogonal
iii) X_2 and Y_1 are ε-orthogonal.

We set $\mathcal{C}(\varepsilon, k)$ as the set of all $(X_1, X_2) \in G_{n,k} \times G_{n,k}$, such that,

$$1 - 8\varepsilon \le \left\| \frac{P_{X_1}}{\lambda(2k)} - \frac{P_{X_2}}{\lambda(2k)} \right\|_{C_E^n} \le 1.$$

The proof of the theorem is based on the following two facts:

Fact 1. *If $0 < \varepsilon < 1/4$ and $2k \le n$, then $\mathcal{A}(\varepsilon, k) \subseteq \mathcal{C}(\varepsilon, k)$.*

Fact 2. *There exits an absolute constant $C > 0$ such that if $0 < \varepsilon < 1/4$ and if $k = [C\varepsilon^3 n] \ge 1$ then*

$$\mathbb{P}_{n,k} \times \mathbb{P}_{n,k}(\mathcal{A}(\varepsilon, k)) \ge 1 - 2\exp\left(-\psi(\varepsilon)n^2\right)$$

where $\psi(\varepsilon) = \frac{C\varepsilon^6}{128}$. ($[\cdot]$ denotes the integer part.)

Proof of Theorem 1 Let N be a natural number, k and ε as in fact 2. Consider the set

$$\Delta = \{(X_1, \dots, X_N) \in G_{n,k} \times \dots \times G_{n,k};$$
$$1 - 8\varepsilon \le \left\| \frac{P_{X_i}}{\lambda(2k)} - \frac{P_{X_j}}{\lambda(2k)} \right\|_{C_E^n} \le 1,\ 1 \le i, j \le N, i \ne j\}.$$

The problem is to find the largest N such that $\Delta \ne \emptyset$. For that, we compute the probability of Δ^c (the complementary set). Since

$$\mathbb{P}_{n,k} \times \dots \times \mathbb{P}_{n,k}(\Delta^c) \le \mathbb{P}_{n,k} \times \mathbb{P}_{n,k}\Big\{ \bigcup_{i,j\in\{1,\dots,N\}, i\ne j} \Delta_{i,j}^c \Big\}$$

where $\Delta_{i,j} = \{(X_i, X_j) \in G_{n,k} \times G_{n,k}; 1 - 8\varepsilon \le \|\frac{P_{X_i}}{\lambda(2k)} - \frac{P_{X_j}}{\lambda(2k)}\|_{C_E^n} \le 1\}$ we obtain

$$\begin{aligned}\mathbb{P}_{n,k} \times \dots \times \mathbb{P}_{n,k}(\Delta^c) &\le \binom{N}{2} \mathbb{P}_{n,k} \times \mathbb{P}_{n,k}(\mathcal{C}(\varepsilon, k)^c) \\ &\le \binom{N}{2} 2\exp\left(-\psi(\varepsilon)n^2\right) < 1\end{aligned}$$

whenever

$$\log N \le \frac{1}{2}\psi(\varepsilon)n^2.$$

□

Next we are going to prove the two facts. We begin with some preparatory lemmas. The first one is most certainly known but we include an elementary proof for sake of completeness.

Lemma 1. *Let P and Q be two orthogonal projections on a Hilbert space H, then $\|P - Q\|_{\mathcal{L}(H)} \le 1$.*

Proof: Let $x \in H$, with $\|x\| = 1$. Let $E = \text{span}\{Px, Qx\}$, ($\dim E \le 2$). If we denote by Rx the orthogonal projection of x onto E and by P' and Q' the orthogonal projection from E onto $\text{span}\{Px\}$ and onto $\text{span}\{Qx\}$, respectively, it is easy to see that $P'(Rx) = Px$ and $Q'(Rx) = Qx$. Since it is enough to prove that $\|Px - Qx\| \le \|Rx\|$, we may assume without loss of generality that H is $\mathbb{R}^2$ with the euclidean norm.

Let $(x,y)\in\mathbb{R}^2$, $P(x,y)=(x,0)$ and $Q(x,y)=(ax+by)(a,b)$, with $a^2+b^2=1$, (as we may assume without loss of generality).

$$(P-Q)(x,y)=\big((1-a^2)x-aby,-abx-b^2y\big)$$

and thus

$$\left\|(P-Q)(x,y)\right\|_2^2=(1-a^2)x^2+b^2y^2\le x^2+y^2=\left\|(x,y)\right\|_2^2 .$$

□

Lemma 2. *Let $0<\varepsilon<\frac{1}{4}$ and let X and Y two ε-orthogonal subspaces of a Hilbert space H. Then*
i) $\|P_XP_Y\|_{\mathcal{L}(H)}\le\varepsilon$.
ii) *If $H=X\oplus Y$ and $X^\perp$ is the orthogonal complement of X then* $\|P_Y-P_{X^\perp}\|_{\mathcal{L}(H)}\le 4\varepsilon$.

Proof: i) Let $z\in H$ and let $P_Y(z)=y$. Then,

$$\begin{aligned}\left\|P_XP_Y(z)\right\|^2&=\langle P_X(y),P_X(y)\rangle=\langle P_X(y),y\rangle\\&\le\varepsilon\left\|P_X(y)\right\|\|y\|\le\varepsilon\left\|P_XP_Y(z)\right\|\|z\| .\end{aligned}$$

Hence, $\|P_XP_Y\|_{\mathcal{L}(H)}\le\varepsilon$.

ii) First we should notice that the ε-orthogonality implies that $X+Y=H$ is a direct sum. Let $z=x+y\in H$, $x\in X$, $y\in Y$ with $\|z\|\le 1$. Then, by i,

$$\begin{aligned}1\ge\|P_Xz\|&\ge\|x\|-\|P_Xy\|\ge\|x\|-\varepsilon\|y\|\\&\ge\|x\|-\varepsilon-\varepsilon\|x\|\ge(1-\varepsilon)\|x\|-\varepsilon .\end{aligned}$$

Thus $\|x\|$ (and similarly $\|y\|$) $\le\frac{1+\varepsilon}{1-\varepsilon}<2$. Now,

$$\begin{aligned}\left\|P_Y-P_{X^\perp}(z)\right\|&=\left\|P_Y(x)+y-P_{X^\perp}(y)\right\|\\&\le\left\|P_Y(x)\right\|+\left\|y-P_{X^\perp}(y)\right\|\\&=\left\|P_Y(x)\right\|+\left\|P_X(y)\right\|\\&\le 4\varepsilon .\end{aligned}$$

□

Proof of Fact 1 Let (X_1,X_2) be a couple in $\mathcal{A}(\varepsilon,k)$ and let $Y_i\subseteq X_i$, $i=1,2$ be the corresponding subspaces. Since $P_{X_1}-P_{X_2}=(P_{X_1}-P_{X_2})P_{X_1+X_2}$, we have:

$$\|P_{X_1}-P_{X_2}\|_{C_E^n}\le\left\|P_{X_1}-P_{X_2}\right\|_{\mathcal{L}(H)}\lambda\big(\dim\,(X_1+X_2)\big)\le\lambda(2k) .$$

To obtain the other inequality we use lemma 1,

$$\begin{aligned}\|P_{X_1}-P_{X_2}\|_{C_E^n}&\ge\left\|(P_{X_1}-P_{X_2})(P_{Y_1}-P_{Y_2})\right\|_{C_E^n}\\&\ge\|P_{Y_1}+P_{Y_2}\|_{C_E^n}-\|P_{X_2}P_{Y_1}\|_{C_E^n}-\|P_{X_1}P_{Y_2}\|_{C_E^n} .\end{aligned}$$

Since $P_{X_2}P_{Y_1}=P_{X_2}P_{Y_1}P_{X_1}$ we have that $\|P_{X_2}P_{Y_1}\|_{C_E^n}\le\varepsilon\lambda(k)$ and in a similar way we have $\|P_{X_1}P_{Y_2}\|_{C_E^n}\le\varepsilon\lambda(k)$.

We still need to get a lower estimate for $\|P_{Y_1}+P_{Y_2}\|_{C_E^n}$. In order to obtain it, we compare with $\|P_{Y_1}+P_{Y_1^\perp}\|_{C_E^n}$ where $Y_1^\perp$ is the orthogonal complement of Y_1 in $Y_1\oplus Y_2$. Then

$$\left|\|P_{Y_1}+P_{Y_2}\|_{C_E^n}-\|P_{Y_1}+P_{Y_1^\perp}\|_{C_E^n}\right|\le\|P_{Y_2}-P_{Y_1^\perp}\|_{C_E^n}\le 4\varepsilon\lambda(2k) .$$

The last inequality follows from Lemma 2.ii and the fact that the rank of $P_{Y_2}-P_{Y_1}^\perp$ is smaller than or equal to $2k$. Since $P_{Y_1}+P_{Y_1^\perp}=P_{Y_1+Y_1^\perp}=P_{Y_1+Y_2}$ and

$$\|P_{Y_1}+P_{Y_1^\perp}\|_{C_E^n}=\lambda(\dim\,Y_1+\dim\,Y_2)\ge\lambda\big(2([(1-\varepsilon)k]+1)\big) ,$$

we get that

$$\lambda\big(2([(1-\varepsilon)k]+1)\big)-6\varepsilon\lambda(2k)\le\|P_{X_1}-P_{X_2}\|_{C_E^n}\le\lambda(2k) .$$

Finally, a simple and well known averaging argument shows that for E, a 1-symmetric Banach space, we have $n\lambda(m)\le m\lambda(n)$ if $n,m\in\mathbb{N}$ and $n\le m$. This concludes the proof of this fact. □

Proof of Fact 2 By using a symmetrization argument and Fubini's theorem it is enough to show that for any fixed $X\in G_{n,k}$, $\mathbb{P}_{n,k}(\mathcal{B}(X,\varepsilon)\ge 1-\exp\{-\psi(\varepsilon)n^2\}$, where $\mathcal{B}(X,\varepsilon)$ denotes the set of all $Y\in G_{n,k}$ for which we can find a subspace $Y_1\subseteq Y$, with dim $Y_1>(1-\varepsilon)k$ and Y_1 ε-orthogonal to X.

In order to do that we are going to estimate the probability of the complementary set $\mathcal{B}(X,\varepsilon)^c$. Note that if $Y\in\mathcal{B}(X,\varepsilon)^c$ and $A\subset Y$ is any orthonormal set of cardinality smaller than $[\varepsilon k]$ then there exist a $y\in Y\cap A^\perp$ and an $x\in X$ with $\|y\|=\|x\|=1$ and $|\langle x,y\rangle|>\varepsilon$. Therefore

$$\begin{aligned}\mathcal{B}(X,\varepsilon)^c\subseteq\big\{Y\in G_{n,k};&\exists y_1\in Y\cap S^{n-1},x_1\in X\cap S^{n-1},|\langle x_1,y_1\rangle|>\varepsilon ,\\&\exists y_2\in Y\cap[y_1]^\perp\cap S^{n-1},x_2\in X\cap S^{n-1},|\langle x_2,y_2\rangle|>\varepsilon ,\\&\dots\\&\exists y_{[\varepsilon k]}\in Y\cap[y_1,\dots,y_{[\varepsilon k]-1}]^\perp\cap S^{n-1},x_{[\varepsilon k]}\in X\cap S^{n-1} ,\\&|\langle x_{[\varepsilon k]},y_{[\varepsilon k]}\rangle|>\varepsilon\big\}\end{aligned}$$

and, for some orthonormal sets A_i with card$(A_i)=i$, $i=0,\dots,[\varepsilon k]$,

$$\mathbb{P}_{n,k}\big(\mathcal{B}(X,\varepsilon)^c\big)\le$$
$$\prod_{l=1}^{[\varepsilon k]}\mathbb{P}_{n,k}\big(\exists y_l\in Y\cap A_{l-1}^\perp\cap S^{n-1},x_l\in X\cap S^{n-1},|\langle x_l,y_l\rangle|>\varepsilon\;/A_{l-1}\subset Y\big) . \quad (1)$$

Here $\mathbb{P}_{n,k}(\,\cdot\,/\,\cdot\,)$ denotes the conditional probability.

In the following lemma we estimate the individual terms in the product above.

Lemma 3. *Let $A \subseteq \mathbb{R}^n$ be an orthonormal set, with* card$(A) = l$. *Then*

$$\mathbb{P}_{n,k}\big(\{Y \in G_{n,k}; \exists y \in Y \cap A^{\perp} \cap S^{n-1}, \exists x \in X \cap S^{n-1}, |\langle x,y\rangle| > \varepsilon\} \;/ A \subseteq Y\big) \leq \left(\frac{4}{\varepsilon}\right)^k \left(\frac{8}{\varepsilon}\right)^{k-l} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{\varepsilon^2(n-l-2)}{32}\right).$$

Proof: First note that the invariance of the Haar measure and the fact that the lemma is supposed to hold for all k-dimensional subspace, X, of $\mathbb{R}^n$ imply that we may assume that A consists of $\{e_{n-l+1}, \dots, e_n\}$, the last l canonical basis vectors in $\mathbb{R}^n$. Next note that $\mathbb{P}_{n,k}\big(\cdot \;/\; \{e_{n-l+1}, \dots, e_n\} \subset Y\big)$ induces a measure, μ, on the set of all $k-l$ subspaces of $\mathbb{R}^{n-l} = \text{span}\{e_1, \dots, e_{n-l}\}$ by the formula

$$\mu(\mathcal{Y}) = \mathbb{P}_{n,k}\big(\{Y \in G_{n,k};\; P_{[e_1,\dots,e_{n-l}]}Y \in \mathcal{Y}\} / \;\{e_{n-l+1}, \dots, e_n\} \subset Y\big)\;.$$

μ is clearly invariant under the action of the orthogonal group of $\mathbb{R}^{n-l} = [e_1, \dots, e_{n-l}]$ and thus is equal to $\mathbb{P}_{n-l,k-l}$.

Let η be an $\frac{\varepsilon}{2}$-net in $X \cap S^{n-1}$, with card$(\eta_1) \leq (\frac{4}{\varepsilon})^k$. Then

$$\{Y \in G_{n-l,k-l}; \exists y \in Y \cap S^{n-l-1}, \exists x \in X \cap S^{n-1}, |\langle x,y\rangle| > \varepsilon\} \subseteq \bigcup_{x \in \eta} \left\{Y \in G_{n-l,k-l}; \exists y \in Y \cap S^{n-l-1}, |\langle x,y\rangle| > \frac{\varepsilon}{2}\right\}\;.$$

(We regard $\mathbb{R}^{n-l}$ as a subspace of $\mathbb{R}^n$ spanned by $e_1, \dots, e_{n-l}$ so $Y \in G_{n-l,k-l}$ is regarded also as a subspace of $\mathbb{R}^n$.) Hence

$$\mathbb{P}_{n-l,k-l}\big(\{Y \in G_{n-l,k-l}; \exists y \in Y \cap S^{n-l-1}, \exists x \in X \cap S^{n-1}, |\langle x,y\rangle| > \varepsilon\}\big) \leq \left(\frac{4}{\varepsilon}\right)^k \sup_{x \in S^{n-1}} \mathbb{P}_{n-l,k-l}\big(\{Y \in G_{n-l,k-l}; \exists y \in Y \cap S^{n-l-1}, |\langle x,y\rangle| > \frac{\varepsilon}{2}\}\big)\;. \quad (2)$$

The sup is clearly attained for $x \in [e_1, \dots, e_{n-l}]$. Fixing a subspace $Z \in G_{n-l,k-l}$, an $\frac{\varepsilon}{4}$ net in $Z \cap S^{n-l}$, and a $z_0 \in Z \cap S^{n-l}$ and denoting by $\mathbb{P}_{O_{n-l}}$ the Haar measure on the orthogonal group O_{n-l}, we get similarly that

$$\begin{aligned}\mathbb{P}_{n-l,k-l}&\big(\{Y \in G_{n-l,k-l}; \exists y \in Y \cap S^{n-l-1}, |\langle x,y\rangle| > \frac{\varepsilon}{2}\}\big)\\ &= \mathbb{P}_{O_{(n-l)}}\big(\{U \in O(n-l); \exists z \in Z \cap S^{n-l-1}, |\langle Uz,x\rangle| > \frac{\varepsilon}{2}\}\big)\\ &\leq \left(\frac{8}{\varepsilon}\right)^{k-l} \mathbb{P}_{O_{(n-l)}}\big(\{U \in O(n-l); |\langle Uz_0,x\rangle| > \frac{\varepsilon}{4}\}\big)\;. \end{aligned} \quad (3)$$

The last probability is twice the measure of an appropriate cap in S^{n-l-1} and thus is dominated by $\sqrt{\frac{\pi}{2}} \exp\left(-\frac{\varepsilon^2(n-l-2)}{32}\right)$ (see [M-S]). Combining this with (2) and (3) we get the lemma. □

We return now to the proof of Fact 2. By (1) and Lemma 3,

$$\begin{aligned}\mathbb{P}_{n,k}\left(\mathcal{B}(X,\varepsilon)^c\right) &\leq \prod_{l=0}^{[\varepsilon k]-1} \left(\frac{4}{\varepsilon}\right)^k \left(\frac{8}{\varepsilon}\right)^{k-l} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{\varepsilon^2(n-l-2)}{32}\right)\\ &\leq \exp\left(\left(2k\log\frac{32}{\varepsilon^2} - \frac{\varepsilon^2(n-[\varepsilon k]-1)}{32}\right)[\varepsilon k]\right)\\ &\leq \exp(-C\varepsilon^6 n)\end{aligned}$$

as long as $k \leq \delta\varepsilon^3 n$. □

Note In the particular case of $E = \ell_p^n$, we could obtain a better estimation for $C(\varepsilon)$. The method we can use in that situation is different. We could attack the problem by looking for the points in the orbit of one particular element under the action of a group of isometries acting on C_p^n.

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