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By

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Abstract. Each infinite dimensional subspace of L^p $(0 < p \leqq 1)$ is shown to contain a copy of some l^q $p \leqq q < \infty$, using arguments similar to the ones that appear in Krivine and Maurey's paper concerning stable Banach spaces. Generally speaking, if X is a stable infinite dimensional p-Banach space, with $0 < p \leqq 1$, then, there exists a q $(p \leqq q < \infty)$, such that, X contains $(1 + \varepsilon)$-isomorphic copies of l^q, for all $\varepsilon > 0$. Moreover, it is possible to prove that if a stable p-Banach space, $0 < p \leqq 1$, contains an isomorphic copy of l^q, $p \leqq q < \infty$, then, it also contains $(1 + \varepsilon)$-isomorphic copies of l^q, for all $\varepsilon > 0$.

The stable Banach spaces were introduced by Krivine and Maurey in [5] to extend the theorem of Aldous concerning the existence of l^q-copies, $1 \leqq q < \infty$, in each infinite dimensional subspace of L^1, to a more general situation. Guerre and Lapresté in [1] proved that each stable Banach spaces is weakly sequentially complete and that each spreading model of a stable Banach space is stable, too. Raynaud ([9], [10]) obtained new examples of stable Banach spaces and studied the superstable Banach spaces. Haydon ([4]), proved that for a stable Banach space X, the following are equivalents: X has the Darboux property, X has the Schur property (every weakly convergent sequence is norm convergent), every spreading model of X is isomorphic to l^1. More recently Guerre and Levy, using methods of stable Banach spaces, proved that every subspace E of L^1 contains $l^{p(E)}$, where $p(E) = \sup\{p > 0;$ E is of p-Rademacher type$\}$, ([3]).

The purpose of this paper is to obtain a similar theorem as Krivine and Maurey's one in the general context of p-Banach spaces, $0 < p \leqq 1$. Moreover, using some ideas of Guerre and Lapresté, a partial solution to a classical problem of Banach spaces is found in the context of stable p-Banach spaces; exactly, if X is a p-Banach space containing a basic sequence $\{a_n\}_1^\infty$ equivalent to the l^q-basis $(p \leqq q < \infty)$, then for each $\varepsilon > 0$, there exists a normalized block basis of $\{a_n\}_1^\infty$ $(1 + \varepsilon)$-isomorphic to l^q.

All vector spaces in this paper will be real. A p-convex seminorm on a real vector space X is a map $x \to \|x\|$ of X into R_+ which verifies:

i) $\|ax\| = |a| \, \|x\|, \quad a \in R, \quad x \in X.$

ii) $\|x + y\|^p \leqq \|x\|^p + \|y\|^p, \quad x, y \in X.$

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Whenever it is $\|x\| > 0$ for all $x \neq 0$, we shall say $\|.\|$ is a *p-convex norm*. A p-convex seminorm induces a locally bounded topology on X, separate if $\|.\|$ is a p-convex norm. We shall say that $(X, \|.\|)$ is a *p-Banach space* if $\|.\|$ is a p-convex norm and X is complete.

We shall say a separable p-Banach space X is *stable* (in Krivine and Maurey's sense) if

$$\lim_{\substack{m\\ \mathscr{U}}} \lim_{\substack{n\\ \mathscr{V}}} \|a_m + b_n\| = \lim_{\substack{n\\ \mathscr{V}}} \lim_{\substack{m\\ \mathscr{U}}} \|a_m + b_n\|$$

whenever $\{a_m\}_1^\infty$, $\{b_n\}_1^\infty$ are bounded sequences in X and $\mathscr{U}$ and $\mathscr{V}$ non-trivial ultrafilters on N.

All the finite-dimensional p-Banach spaces are stable; likewise, $L^p = L^p[0, 1]$, $0 < p \leqq 1$ is also stable, because the function $\exp(-\|x - y\|_p^p)$, $x, y \in L^p$ is of positive type (see [8]) and it permits to include L^p in a Hilbert space, $U: L^p \to H$, such that $\langle U_x, U_y \rangle = \exp(-\|x - y\|_p^p)$, $x, y \in L^p$. This ensures the stability of L^p, as it is proved in [6].

A function σ of X, stable p-Banach space, into R^+ is a *type on X* if there exist a bounded sequence $\{a_n\}_1^\infty$ in X and a non-trivial ultrafilter $\mathscr{U}$ on N such that, for all $x \in X$, $\sigma(x) = \lim_{\substack{n\\ \mathscr{U}}} \|x + a_n\|$. Given $\sigma = \lim_{\mathscr{U}} a_n$, $\tau = \lim_{\mathscr{V}} b_m$ two types on X and $\lambda \in R$, the product of convolution $\sigma * \tau$ is a type defined by

$$\sigma * \tau(x) = \lim_{\substack{n\\ \mathscr{U}}} \lim_{\substack{m\\ \mathscr{V}}} \|x + a_n + b_m\|, \quad x \in X$$

and the product $\lambda\sigma$ is $\lambda\sigma(x) = \lim_{\substack{n\\ \mathscr{U}}} \|x + \lambda a_n\|$, $x \in X$.

These operations verify the same properties in this situation as on stable Banach spaces.

Let $\{a_n\}_1^\infty$ be a bounded sequence in X; it is possible to define a p-convex seminorm $\|.\|$ on $X \oplus R^{(N)}$ by

$$\left\| x + \sum_{i=1}^{n} \lambda_i e_i \right\| = \lambda_1 \sigma * \cdots * \lambda_n \sigma(x)$$

whenever $x \in X$, $\lambda_1, \ldots, \lambda_n \in R$, the e_i's are the vector $(0, \ldots, 0, \overset{i}{1}, 0 \ldots)$ in $R^{(N)}$ and σ is the type defined by $\{a_n\}_1^\infty$. We shall say $(X \oplus R^{(N)}, \|.\|)$ is the *spreading model of X associated to σ* when $\|.\|$ is a p-convex norm, what happens, for example, if $\{a_n\}_1^\infty$ has no Cauchy subsequences. The sequence $\{e_n\}_1^\infty$ is symmetric, i.e.

$$\left\| \sum_{i=1}^{n} \lambda_i e_i \right\| = \left\| \sum_{i=1}^{n} \lambda_i e_{m_i} \right\|,$$

for each permutation $(m_1, \ldots, m_n)$ of integers and for each finite real sequence $\lambda_1, \ldots, \lambda_n$.

A type σ on X is *non-trivial* if $\sigma(0) \neq 0$ and *symmetric* if $\sigma(x) = \sigma(-x)$ for all $x \in X$. It is clear for symmetric types that

$$\lambda_1 \sigma * \cdots * \lambda_n \sigma = |\lambda_1| \sigma * \cdots * |\lambda_n| \sigma \quad \text{for} \quad \lambda_1, \ldots, \lambda_n \in R;$$

thus, each non trivial, symmetric type on X origines a spreading model of X. In fact, if $0 = \left\| x + \sum_{i=1}^{n} \lambda_i e_i \right\|$ we have,

$$2\left\| \sum_{i=1}^{n} \lambda_i e_i \right\| \leq \left(\left\| x + \sum_{1}^{n} \lambda_i e_i \right\|^p + \left\| x - \sum_{1}^{n} \lambda_i e_i \right\|^p \right)^{1/p} = 0;$$

hence,

$$\begin{aligned} 2^p \| \lambda_n e_n \|^p &= \left\| \sum_{1}^{n} \lambda_i e_i - \sum_{1}^{n-1} \lambda_i e_i + \lambda_n e_n \right\|^p \\ &\leq \left\| \sum_{1}^{n} \lambda_i e_i \right\|^p + \left\| \sum_{1}^{n-1} \lambda_i e_i - \lambda_n e_n \right\|^p = 0 \end{aligned}$$

and, as $\| e_n \| = \sigma(0) \neq 0$, necessarily $\lambda_n = 0$.

So, for non trivial and symmetric types, the sequence $\{e_n\}_1^\infty$ is a monotone, symmetric, invariant for signs and basic sequence with constant $2^{1/p-1}$

$$\left(\text{i.e.} \left\| \sum_{1}^{n} \lambda_i e_i \right\| \leq 2^{1/p-1} \left\| \sum_{1}^{m} \lambda_i e_i \right\| \quad \text{if} \quad n \leq m \quad \text{and} \quad \lambda_1, \ldots, \lambda_m \in R \right).$$

A symmetric type σ is a *l^q-type* ($0 < q \leq \infty$) if $\alpha\sigma * \beta\sigma = (\alpha^q + \beta^q)^{1/q}\sigma$ for all $\alpha, \beta \geq 0$ ($\alpha\sigma * \beta\sigma = \max\{\alpha, \beta\}\sigma$ when $q = \infty$). Moreover, if in this case σ is non trivial, the basis $\{e_n\}_1^\infty$ is a l^q-basis (or c_0-basis, if $q = \infty$) in the complection of $X \oplus R^{(\mathbb{N})}$.

A *conical class of types* on X is a family C of types which satisfies:

i) C contains non trivial types and it is closed for the product topology of R_+^X.

ii) If σ and $\lambda \geq 0$, then $\lambda\sigma \in C$.

iii) If $\sigma, \tau \in C$, then $\sigma * \tau \in C$.

For stable p-Banach spaces it is possible to repeat the arguments of Krivine and Maurey (see [6]), to obtain:

Proposition 1. *Let X be a infinite dimensional stable p-Banach space $0 < p \leq 1$. Then, there exists a non trivial symmetric type σ on X, such that:*

(.) for each $\alpha \geq 0$ there is a unique $\beta \geq 2^{1-1/p}$ satisfying $\sigma * \alpha\sigma = \beta\sigma$.

(The type σ exists in each conical class of symmetric types on X.) □

The previous statement (.) implies, as for stable Banach spaces, that σ is a l^q-type for some q, $p \leq q \leq \infty$.

Proposition 2. *Let X be a stable p-Banach space $0 < p \leq 1$ and σ a non trivial, symmetric type on X verifying (.), then: there exists a q, $p \leq q \leq \infty$ such that, σ is a l^q-type.*

Proof. Let $X \oplus R^{(N)}$ be the spreading model of X associated to σ and without any restriction we can suppose $\sigma(0) = 1$. By homogeneity, for each $\lambda, \mu \geq 0$ there is a unique real number $F(\lambda, \mu) \geqq 0$ such that, $\lambda\sigma * \mu\sigma = F(\lambda, \mu)\,\sigma$ (exactly, $F(\lambda, \mu) = \lambda\sigma * \mu\sigma(0) = \|\lambda e_1 + \mu e_2\|$). The function $F(\lambda, \mu)$ verifies for all

$$\lambda, \mu, \nu \geqq 0 : F(\lambda, \mu) = F(\mu, \lambda), \quad F(\lambda, 0) = \lambda,$$
$$F(\lambda, F(\mu, \nu)) = F(F(\lambda, \mu), \nu), \quad F(\lambda\nu, \mu\nu) = \nu F(\lambda, \mu)$$

and $F(\lambda, 1-\lambda)^p \leqq \lambda^p + (1-\lambda)^p$ (see [5]) and furthermore, F is continuous. Hence, as it is easy to prove, either there exists a q $(p \leqq q < \infty)$ such that

$$F(\lambda, \mu) = (\lambda^q + \mu^q)^{1/q}, \quad \lambda, \mu \geqq 0$$

(what occurs if $F(1,1) > 1$) or $F(1,1) = 1$ $\Bigg($the case $F(1,1) < 1$ is not possible, because if $k \leqq n$ $\left\|\sum_1^k e_i\right\| \leqq 2^{1/p-1}\left\|\sum_1^n e_i\right\|$ and the sequence $\left\{\left\|\sum_1^k e_i\right\|\right\}_{n=1}^{\infty}$ is multiplicative$\Bigg)$. In the last situation, $F(1,1) = 1$, necessarily, σ is a c_0-type. Indeed, if $0 < \alpha < 1$, and $\beta = F(1, \alpha)$, by induction, it is easy to see that

$$\beta^n \sigma = \sigma * \alpha\sigma * \cdots * \alpha^n \sigma \qquad (n \in N)$$

so

$$\beta^{np} \leqq \sum_{i=0}^{n} |\alpha|^{pi} \|e_1\|^p \leqq \sum_{i=0}^{\infty} |\alpha|^{pi} < \infty \qquad (n \in N)$$

and $\beta \leqq 1$. But, as σ is symmetric, we have

$$\begin{aligned} 2\|e_1\| &= \|(e_1 + \alpha e_2 + \cdots + \alpha^n e_{n+1}) + (e_1 - \alpha e_2 - \cdots - \alpha^n e_{n+1})\| \\ &\leqq 2^{1/p}\beta^n \qquad n \in N, \end{aligned}$$

thus $\beta = 1$.

(This part of the proof was remarked to the author by B. Maurey.)

The statement of above propositions prove that on each stable p-Banach space, there is a spreading model, associated to a non trivial symmetric type, such that, the sequence $\{e_n\}_1^\infty$ of this spreading model origines a space isometric to l^q for some q, $p \leqq q \leqq \infty$. Now, repeating similar arguments, with minor variations, to the ones which appear in [6], it can be obtained the following

Proposition 3. *Let X be a stable p-Banach space, and let σ be a non trivial symmetric type on X defined by a bounded sequence $\{a_n\}_1^\infty$ of X; if σ is a l^q-type for some q, $p \leqq q \leqq \infty$ for each $\varepsilon > 0$, there is a basic subsequence $\{b_n\}_1^\infty$ of $\{a_n\}_1^\infty$, $(1+\varepsilon)$-isomorphic to the basis $\{e_n\}_1^\infty$ of the spreading model of X originated by σ.* □

In the sequel we are going to obtain the main results, before mentionated in the introduction.

Corollary 4. *Let X be a infinite dimensional stable p-Banach space, $0 < p \leqq 1$, then there exists a q $(p \leqq q < \infty)$ such that, for each $\varepsilon > 0$, X contains a $(1+\varepsilon)$-isomorphic copy of l^q.*

Proof. It is a inmediate consequence of the above propositions and of the following result of Y. Raynaud (see, [11], part V): "c_0 has no translation invariant distance, uniformly equivalent to the canonical norm and stable." (This result was communicated to the author by S. Guerre and Y. Raynaud.)

Corollary 5. *If X is a infinite-dimensional subspace of $L^p = L^p[0, 1]$, $0 < p \leqq 1$, there exists a q ($p \leqq q \leqq 2$) such that, for each $\varepsilon > 0$, there is a subspace of X, $(1 + \varepsilon)$-isomorphic to l^q.*

Proof. Note that X is of cotype Rademacher 2 (see [12]) and, then, X cannot contain any isomorphic subspace to l^q if $2 < q < \infty$. □

Now, we shall find a partial solution (for stable p-Banach spaces) to a classical problem that appears, for example, in [7], problem 2.e.2. In order to get it, we shall begin with a technical lemma and after that we shall use some ideas of Guerre and Lapresté in a inverse way.

Lemma 6. *Let X be a stable p-Banach space ($0 < p \leqq 1$), and let τ and σ be two types on X with $\tau(0) = 1$ and $\sigma = \lim\limits_{\mathscr{U}} a_n$. Suppose that for each $n \in N$ there exists a finite real sequence $\lambda_1^{(n)}, \ldots, \lambda_{k(n)}^{(n)}$ such that $\sigma_n = \lambda_1^{(n)}\sigma * \cdots * \lambda_{k(n)}^{(n)}\sigma$ converges to τ, when $n \to \infty$, in the topology of the space of the types on X, then, given a compact $K \subseteqq X$, a $\nu \in N$ and $\varepsilon > 0$, it is possible to find integers $n_1, \ldots, n_k$ and real numbers $\mu_1, \ldots, \mu_k$ verifying: $\nu < n_1 < \cdots < n_k$, $\left\| \sum\limits_{i=1}^{k} \mu_i a_{n_i} \right\| = 1$ and*

$$\left| \tau(x) - \left\| x + \sum_{i=1}^{k} \mu_i a_{n_i} \right\| \right| < \varepsilon \quad \text{for all} \quad x \in K.$$

Proof. As $\tau(0) = 1$ and $\sigma_n(0) \underset{n\to\infty}{\to} \tau(0)$, we assume without loss of generality that $\sigma_n(0) = 1$ for all n. Let σ' be one of the types σ_n (we shall denote

$$\sigma' = \lambda_1 \sigma * \cdots * \lambda_k \sigma$$

while there is not confusion). For each k-tuple $\mathbb{N} = (n_1, \ldots, n_k)$ of integers, let $f_{\mathbb{N}}(x) = \left\| x + \sum\limits_{1}^{k} \lambda_i a_{n_i} \right\|$ be a function of X into R_+; when $\mathbb{N} \in N^k$, the $f_{\mathbb{N}}$'s are equicontinuous on $K \cup \{0\}$ and thus, by Ascoli's theorem, the sequence $f_{\mathbb{N}}$ converges uniformly on $K \cup \{0\}$ through the ultrafilter $\mathscr{U} * \cdots * \mathscr{U}$ of N^k associated to the reiterated limit $\lim\limits_{\substack{n_1\\ \mathscr{U}}} \ldots \lim\limits_{\substack{n_k\\ \mathscr{U}}}$ (see [12]). As $f_{\mathbb{N}}$ also converges to σ', we can choose k integers $n_1 < \cdots < n_k$ with $\nu < n_1$ satisfying

$$\left| \sigma'(x) - \left\| x + \sum_{1}^{k} \lambda_i a_{n_i} \right\| \right| < \min\{\varepsilon/3, \delta^{1/p}\}$$

for all $x \in K \cup \{0\}$, where δ is a positive number such that for some $A > 0$, $|t - t'| < \delta$ and $t, t' \in [0, A]$ imply $|t^{1/p} - t'^{1/p}| < \varepsilon/3$.

By the same reasons, $\sigma_n \underset{n\to\infty}{\to} \tau$ uniformly on $K \cup \{0\}$, hence, we can select a $n_0 \in N$ so that

$$|\sigma_{n_0}(x) - \tau(x)| < \varepsilon/3$$

for all $x \in K \cup \{0\}$. Now, applying the above to $\sigma' = \sigma_{n_0}$ and defining

$$\mu_i = \lambda_i \left\| \sum_1^k \lambda_i a_{n_i} \right\|^{-1}$$

$\left(\text{note that, as } \sigma'(0) = 1, \left\| \sum_1^k \lambda_i a_{n_i} \right\| \neq 0\right)$ we have

$$\left| \left\| x + \sum_1^k \lambda_i a_{n_i} \right\|^p - \left\| x + \sum_1^k \mu_i a_{n_i} \right\|^p \right| \leqq \left| 1 - \left\| \sum_1^k \lambda_i a_{n_i} \right\|^p \right| < \delta$$

and then

$$\begin{aligned} \left| \tau(x) - \left\| x + \sum_1^k \mu_i a_{n_i} \right\| \right| &\leqq |\tau(x) - \sigma'(x)| \\ &+ \left| \sigma'(x) - \left\| x + \sum_1^k \lambda_i a_{n_i} \right\| \right| \\ &+ \left| \left\| x + \sum_1^k \lambda_i a_{n_i} \right\| - \left\| x + \sum_1^k \mu_i a_{n_i} \right\| \right| < \varepsilon \end{aligned}$$

for all $x \in K$.

Corollary 7. *Let X be a stable p-Banach space and let $\{a_n\}_1^\infty$ be a sequence in X, equivalent to the l^q-basis ($p \leqq q < \infty$), then, for each $\varepsilon > 0$, there exists a normalized block basis of $\{a_n\}_1^\infty$, $(1 + \varepsilon)$-equivalent to the l^q-basis.*

Proof. Let $\mathcal{U}$ be a non trivial ultrafilter on N; the type $\sigma = \lim_{\mathcal{U}} a_n$ on X, origines a spreading model of X, $X \oplus [e_i]_1^\infty$ such that $\{e_i\}_1^\infty$ is a basic sequence equivalent to the l^q-basis. Now, we considere the type $\sigma * (-\sigma)$, that is non trivial and symmetric; its spreading model, $X \oplus [\xi_i]_1^\infty$, verifies that the basic sequence $\{\xi_i\}_1^\infty$ is also equivalent (with a different constant) to the l^q-basis. Repeating the arguments of Lemma 1 of [1], it can be obtained a non trivial l^q-type on X, τ, with $\tau = \lim_{n\to\infty} \sigma_n$, where $\sigma_n = \lambda_1^{(n)}[\sigma * (-\sigma)] * \cdots * \lambda_{k(n)}^{(n)}[\sigma * (-\sigma)]$. Hence, using the above lemma and the proof of Theorem III.1 of [6], for the type τ it is attained a normalized block sequence of $\{a_n\}_1^\infty$ $(1 + \varepsilon)$-equivalent to the l^q-basis and the result holds.

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